Sannikov's Principal-Agents problem with multiple agents

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9th Colloquium on BSDE and mean field systems

Sannikov's problem: the Agent's problem

Let X be the $\mathbb{R}$-valued output process, $\mathbb{A} \subset \mathbb{R}$ and $\mathcal{A}$ be the set of $\mathbb{A}$-valued controls. For $\alpha \in \mathcal{A}$, we consider the probability $\mathbb{P}^\alpha$ such that

$$X_t = X_0 + \int_0^t \alpha_s ds + \sigma W_t^\alpha \quad \mathbb{P}^\alpha\text{−a.s.},$$

where W^α is a $\mathbb{P}^\alpha$−standard Brownian motion. Given a contract $C := (\pi, \xi, \tau)$ by the Principal, the Agent faces the problem

$$V_A := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^\alpha}\Big[u(\xi)e^{-r\tau} + \int_0^\tau re^{-rs}\big(u(\pi_s) - h(\alpha_s)\big)ds\Big],$$

with u a utility function, h a cost function and r a constant interest rate.

Sannikov's problem: the Principal's problem

Given an optimal effort $\hat{\alpha} = \hat{\alpha}(C)$ of the Agent, the Principal wants to solve the optimization problem

$$V_P := \sup_{C \in \mathcal{C}} \mathbb{E}^{\mathbb{P}^{\hat{\alpha}(C)}} \Big[-\xi e^{-r\tau} + \int_0^\tau r e^{-rs} \big(\hat{\alpha}_s(C) - \pi_s \big) ds \Big].$$

It can be showed (see Sannikov [4], Possamaï & Touzi [3]) that $V_P = v(V_A)$, where v is solution of the ODE on $\mathbb{R}_+$

$$\min \Big\{ rv(y) - yv'(y) + F^*(v'(y)) \\ + \sup_{z \in \mathbb{R}} \{ \hat{a}(z) + h(\hat{a}(z))v'(y) + \frac{1}{2}\sigma^2 z^2 v''(y) \}, v(y) - F(y) \Big\} = 0,$$

with $v(0) = 0$, where

$$F := -u^{-1} 1_{[0, u(\infty))} - \infty 1_{[u(\infty), \infty)}$$

and F^* is the concave Fenchel-Legendre transform of F.

Multi-agents case: the output process

Let $N \geq 1, X_0 \in \mathbb{R}^N$ and $\boldsymbol{X} = (X^1, \dots, X^N)^\top$ be the output process. For $\boldsymbol{\alpha} := (\alpha^1, \dots, \alpha^N) \in \mathcal{A}^N$, we consider the probability $\mathbb{P}^{\boldsymbol{\alpha}}$ such that $\boldsymbol{X}$ satisfies coordinatewise the dynamics

$$X_t^k = X_0^k + \int_0^t \alpha_s^k ds + \sigma W_t^{\boldsymbol{\alpha},k} \quad \mathbb{P}^{\boldsymbol{\alpha}}-\text{a.s. for all } k \in \{1, \dots, N\},$$

where $\boldsymbol{W}^{\boldsymbol{\alpha}} := (W^{\boldsymbol{\alpha},k})_{1 \leq k \leq N}$ is a $\mathbb{P}^{\boldsymbol{\alpha}}-$standard Brownian motion.

The agents' problem

Then, introducing

$$\mathcal{A}_k(\boldsymbol{\alpha}^{-k}) := \{\alpha \in \mathcal{A} : \alpha \otimes_k \boldsymbol{\alpha}^{-k} \in \mathcal{A}^N\} \text{ for all } k \in \{1, \ldots, N\},$$

we may state the optimization problem faced by every agent:

$$V_A^{k,N}(C^k, \boldsymbol{\alpha}^{-k}) := \sup_{\alpha \in \mathcal{A}_k(\boldsymbol{\alpha}^{-k})} \mathbb{E}^{\mathbb{P}^{\alpha \otimes_k \boldsymbol{\alpha}^{-k}}} \left[u(\xi^k) e^{-r\tau^k} + \int_0^{\tau^k} r e^{-rs} \big(u(\pi_s^k) - h(\alpha_s, \boldsymbol{\alpha}_s^{-k})\big) ds \right],$$

with $C^k := (\pi^k, \xi^k, \tau^k)$ the contract proposed by the Principal to the Agent k, u and h the Agents' utility and cost functions, $r > 0$ a constant interest rate.

Nash equilibrium

Assumption

For all $z \in \mathcal{M}_N(\mathbb{R})$, there exists $\hat{\boldsymbol{a}} := (\hat{a}^1, \dots, \hat{a}^N) \in \mathbb{A}^N$ such that

$$\hat{a}^k \in \operatorname*{argmax}_{a\in\mathbb{A}}\{a z^{k,k} - h(a, \hat{\boldsymbol{a}}^{-k})\} \text{ for all } k \in \{1, \dots, N\}.$$

We shall write $\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}(z)$.

Let $(\boldsymbol{Y}, \boldsymbol{Z})$ be a solution to the $N-$dimensional BSDE

$$\boldsymbol{Y}_\tau = U(\boldsymbol{\xi}),\ d\boldsymbol{Y}_t = r\mathrm{Diag}(\boldsymbol{I}_t)(\boldsymbol{Y}_t + H(\hat{\boldsymbol{a}}(\tilde{\boldsymbol{Z}}_t)) - U(\boldsymbol{\pi}_t))dt + \sigma\tilde{\boldsymbol{Z}}_t dW_t^{\hat{\boldsymbol{a}}(\boldsymbol{Z})},$$

with $\boldsymbol{Y}_\tau := (Y^1_{\tau^1}, \dots, Y^N_{\tau^N})$, $U(\boldsymbol{x}) := \{u(x^k)\}_{1\le k\le N}$ for all $\boldsymbol{x} \in \mathbb{R}^N$, $H(\boldsymbol{a}) := \{h(a^k, \boldsymbol{a}^{-k})\}_{1\le k\le N}$ for all $\boldsymbol{a} \in \mathbb{A}^N$, $\boldsymbol{I}_t := \{I^k_{0-}1_{t<\tau^k}\}_{1\le k\le N}$ and $\tilde{\boldsymbol{Z}}_t := \{Z^{kl}_t I^k_t I^l_t\}_{1\le k,l\le N}$. Then

$$Y^k_0 = V^{k,N}_A(C^k, \hat{\boldsymbol{a}}^{-k}(\boldsymbol{Z})) \text{ for all } k \in \{1, \dots, N\}.$$

The Principal's problem

The Principal faces the problem:

$$V_P^N := \sup_{\boldsymbol{C}\in\mathcal{C}^N} \mathbb{E}^{\mathbb{P}^{\hat{\boldsymbol{a}}(Z)}}\Big[\sum_{k=1}^N \Big(-\xi^k e^{-r\tau^k} + \int_0^{\tau^k} re^{-rs}\big(\hat{a}^k(\tilde{\boldsymbol{Z}}_s) - \pi_s^k\big)ds\Big)\Big],$$

with $\boldsymbol{C} = \{(\xi^k, \pi^k, \tau^k)\}_{1\le k\le N}$. Then we may write (see Elie & Possamaï [2]) $V_P^N = \sup_{\boldsymbol{Y}_0\in[R,\infty)^N} V^N(\boldsymbol{Y}_0)$, with $V^N(\boldsymbol{Y}_0)$ equal to

$$\sup_{(\boldsymbol{\eta},\boldsymbol{Z},\boldsymbol{\tau})\in\mathcal{Z}(\boldsymbol{Y}_0)} \mathbb{E}^{\mathbb{P}^{\hat{\boldsymbol{a}}(\boldsymbol{Z})}}\Big[\sum_{k=1}^N \Big(F(Y_{\tau^k}^{k,\boldsymbol{Y}_0})e^{-r\tau^k} + \int_0^{\tau^k} re^{-rs}\big(\hat{a}^k(\tilde{\boldsymbol{Z}}_s) + F(\eta_s^k)\big)ds\Big)\Big],$$

with $\eta^k := u(\pi^k)$ for all k and $\boldsymbol{Y}^{\boldsymbol{Y}_0}$ the forward version of the above BSDE.

Dynamic programming equation

Denote $\boldsymbol{S}_+ := \mathbb{R}_+ \times \{0,1\}$ and, for all $(\boldsymbol{y}, \boldsymbol{i}) \in \boldsymbol{S}_+^N$:

$$V^N(\boldsymbol{y}, \boldsymbol{i}) := \sup_{(\boldsymbol{Z},\boldsymbol{\eta},\boldsymbol{\tau}) \in \mathcal{Z}(\boldsymbol{Y}_0)} \mathbb{E}^{\mathbb{P}^{\hat{\boldsymbol{a}}(\boldsymbol{Z})}}\Big[\sum_{k=1}^N \Big(F(Y_{\tau^k}^{k,\boldsymbol{Y}_0}) e^{-r\tau^k} + \int_0^{\tau^k} r e^{-rs}(\hat{a}^k(\tilde{\boldsymbol{Z}}_s) + F(\eta_s^k)) ds \Big) \Big| (Y_0, I_{0-}) = (\boldsymbol{y}, \boldsymbol{i}) \Big].$$

Then, if $V^N \in C^2(\boldsymbol{S}_+^N)$, it satisfies the dynamic programming equation

$$\begin{cases} \min\{-\mathcal{L}(V^N - \bar{F})(\boldsymbol{y}, \boldsymbol{i}), V^N(\boldsymbol{y}, \boldsymbol{i}) - \max_{\boldsymbol{i}' < \boldsymbol{i}} V^N(\boldsymbol{y}, \boldsymbol{i}')\} = 0, \\ V^N(\boldsymbol{y}, 0) = \bar{F}(\boldsymbol{y}, 0), \end{cases}$$

Dynamic programming equation

where $\bar{F}(\boldsymbol{y}, \boldsymbol{i}) := \sum_{k=1}^{N} F(y_k)(1 - i_k)$ and

$$\mathcal{L}\varphi(m) := \sum_{k=1}^{N} \left[y_k \partial_{y_k}\varphi(\boldsymbol{y}, \boldsymbol{i}) - F^*(\partial_{y_k}\varphi(\boldsymbol{y}, \boldsymbol{i})) \right] i_k - \varphi(\boldsymbol{y}, \boldsymbol{i})$$

$$+ \sup_{\boldsymbol{z} \in \mathcal{M}_N(\mathbb{R})} \Bigg\{ \sum_{k=1}^{N} \left[\hat{a}^k(\boldsymbol{z}) + h(\hat{a}^k(\boldsymbol{z}), \hat{\boldsymbol{a}}^{-k}(\boldsymbol{z})) \partial_{y_k}\varphi(\boldsymbol{y}, \boldsymbol{i}) \right] i_k$$

$$+ \frac{\sigma^2}{2} \sum_{k,l=1}^{N} z^{k,l} z^{l,k} i_k i_l \partial^2_{y_k y_l} \varphi(\boldsymbol{y}) \Bigg\}$$

If $h(0, \cdot) = 0$, then the above equation may be reduced to

$$-\mathcal{L}(u - \bar{F})(\boldsymbol{y}, \boldsymbol{i}) = 0, \;\; u(\boldsymbol{y}, 0) = \bar{F}(\boldsymbol{y}, 0),$$

and therefore the Principal's problem is a standard control one.

Mean field approximation

Now the output process X is a $\mathbb{R}$-valued process, and $\mathbb{P}^{\alpha}$ is s.t.

$$X_t = X_0 + \int_0^t \alpha_s ds + \sigma dW_s^{\alpha},\ \mathbb{P}^{\alpha}\text{-a.s.}$$

Given a contract C, the problem faced by a typical agent may be written

$$V_A(C, \boldsymbol{q}) := \sup_{\alpha \in \mathcal{A}} \mathbb{E}^{\mathbb{P}^{\alpha}} \Big[u(\xi) e^{-r\tau} + \int_0^{\tau} r e^{-rs} (u(\pi_s) - h(\alpha_s, \bar{q}_s)) ds \Big],$$

where $\boldsymbol{q} := \{q_t\}_{t \geq 0}$ is a flow of measures on $\mathbb{A}$, and $\bar{q}_t := \int_{\mathbb{A}} x q_t(dx)$.

Mean field Nash equilibrium

Assumption

For all $z \in \mathbb{R}$, $\rho \in \mathbb{A}$, there exists $\hat{a} := \hat{a}(z,\rho)$ s.t.

$$\hat{a} \in \operatorname*{argmax}_{a \in \mathbb{A}} \{az - h(a,\rho)\}.$$

Moreover, we assume that the BSDE

$$Y_\tau = u(\xi),\ dY_t = r(Y_t + h(\hat{a}(\tilde{Z}_t, \bar{q}_t), \bar{q}_t) - u(\pi_t))I_t dt + \sigma \tilde{Z}_t dW_t^{\hat{a}(Z,\mathbf{q})},$$

has a solution such that $\mathbb{E}^{\mathbb{P}^{\hat{a}(\tilde{Z},\mathbf{q})}}\left[\hat{a}(\tilde{Z}_t, \bar{q}_t)\right] = \bar{q}_t$ for all $t \geq 0$, with $\tilde{Z} = ZI$. In this case, we simply write $\hat{a}(\tilde{Z}_t, \hat{q}_t) = \hat{a}(\tilde{Z}_t)$ and $\hat{q}_t := \mathbb{E}^{\mathbb{P}^{\hat{a}(\tilde{Z},\mathbf{q})}}\left[\hat{a}(\tilde{Z}_t)\right]$.

The Principal's problem: a mean field control-and-stopping problem

Assuming furthermore the Nash equilibrium is unique, we have

$$V_P = \sup_{Y_0 \in [R,\infty)} V(Y_0),$$

with (similarly to Elie, Mastrolia & Possamaï [1])

$$V(Y_0) := \sup_{(Z,\eta,\tau)\in\mathcal{Z}(Y_0)} \mathbb{E}^{\mathbb{P}^{\hat{a}(\tilde{Z})}}\left[\left(F(Y_T^{Y_0,Z,\eta,\tau})e^{-r\tau} + \int_0^\tau re^{-rs}(\hat{a}(\tilde{Z}_s) + F(\eta_s))ds\right)\right],$$

with $Y := Y^{Y_0,Z,\eta,\tau}$ the forward version of the above BSDE given by

$$Y_t = Y_0 + r\int_0^t (Y_s + h(\hat{a}(Z_s),\hat{q}_s) - \eta_s)I_s ds + \sigma\int_0^t \tilde{Z}_s dW_s^{\hat{a}(Z,\hat{q})}.$$

Differentiability on $\mathcal{P}_2(\boldsymbol{S}_+)$

We say $u : \mathcal{P}_2(\boldsymbol{S}_+) \longrightarrow \mathbb{R}$ has a functional linear derivative if there exists a function

$$\delta_m u : \mathcal{P}_2(\boldsymbol{S}_+) \times \boldsymbol{S}_+ \to \mathbb{R}$$

such that for all $m, m' \in \mathcal{P}_2(\boldsymbol{S}_+)$, it holds:

$$u(m') - u(m) = \int_0^1 \int_{\boldsymbol{S}_+} \delta_m u(\lambda m' + (1-\lambda)m)(x,i) d(m'-m)(x,i) d\lambda.$$

We introduce $\delta_m u_i(m,x) := \delta_m u(m,x,i)$ for all $(m,x,i) \in \mathcal{P}_2(\boldsymbol{S}_+) \times \boldsymbol{S}_+$.
We say that $u : [0,T] \times \mathcal{P}_2(\boldsymbol{S}_+) \longrightarrow \mathbb{R}$ belongs to $C^2(\mathcal{P}_2(\boldsymbol{S}_+))$ if u has functional linear derivative such that $\delta_m u_1$ is C^2 with respect to x and $\delta_m u_1$, $\partial_x \delta_m u_1$ and $\partial^2_{xx} \delta_m u_1$ are continuous in m and x.
We also define

$$D_I u := \delta_m u_1 - \delta_m u_0.$$

Dynamic programming equation

Assume $V \in C^2(\mathcal{P}_2(\boldsymbol{S}_+))$ and Z is a feedback control. Then V satisfies the equation (similarly to T., Touzi & Zhang [5])

$$\min_{m' \in C_u(m)} [-\boldsymbol{L}(u - \bar{F})(m')] = 0, \; D_I u(m,.) \geq 0, \; u(m^{\emptyset}) = F(m^{\emptyset}),$$

where $C_u(m) := \{m' \preceq m : u(m') = u(m)\}$,
$\bar{F}(m) := \int_{\boldsymbol{S}_+} F(y)(1-i)m(dy,di)$ and

$$\boldsymbol{L}\varphi(m) := \int_{\boldsymbol{S}_+} \big[y\partial_y \delta_m \varphi_1(m,y) - F^*(\partial_y \delta_m \varphi_1(m,y)) - r\varphi(m)\big] i \; m(dy,di)$$

$$+ \sup_{Z \in \mathbb{L}^2} \int_{\boldsymbol{S}_+} \Big[\hat{a}(Z(y)) + h(\hat{a}(Z(y)), \hat{q}(m))\partial_y \delta_m \varphi_1(m,y) + \frac{\sigma^2}{2} Z(y)^2 \partial^2_{yy} \delta_m \varphi_1(m,y)\Big] i \; m(dy,di),$$

$$\hat{q}(m) := \int_{\boldsymbol{S}_+} \hat{a}(Z(y)) i \; m(dy,di).$$

Reduction to a mean field control problem

Under the assumption $h(0,\cdot)=0$, V is the unique solution of

$$-\boldsymbol{L}(u-\bar{F})=0,\ u(m^{\emptyset})=F(m^{\emptyset}).$$

Then the Principal faces a mean field control problem.

From the N agents to the mean field problem

Assume that:
• $\mathbb{A}$ (the arrival set of the agents' efforts) is bounded;
• the value function V of the mean field Principal's problem is in $C^2(\mathcal{P}_2(\boldsymbol{S}_+))$;
• comparison principle holds for classical super/subsolutions of the finite-dimensional Principal's PDE;
then we have

$$V^N(\boldsymbol{y}^N, \boldsymbol{i}^N) \underset{\substack{N \to \infty \\ m^N(\boldsymbol{y}^N, \boldsymbol{i}^N) \xrightarrow{\mathcal{W}_2} m}}{\longrightarrow} V(m),$$

where $m^N(\boldsymbol{y}^N, \boldsymbol{i}^N) := \frac{1}{N} \sum_{k=1}^N \delta_{(y_k^N, i_k^N)}$.

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