Optimal control of two scales stochastic systems by BSDEs

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Example: two scale system of reaction-diffusion equations

We consider the following system of controlled SPDEs:

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{X}^\epsilon(t,x) = \dfrac{\partial^2}{\partial x^2}\mathcal{X}^\epsilon(t,x) + b(\mathcal{X}^\epsilon(t,x), \mathcal{Q}^\epsilon(t,x), u(t,x)) + \\ \qquad\qquad + \sigma(x, \mathcal{X}^\epsilon(t,x))\dfrac{\partial}{\partial t}\mathcal{W}^1(t,x), \\ \epsilon\dfrac{\partial}{\partial t}\mathcal{Q}^\epsilon(t,x) = (\dfrac{\partial^2}{\partial x^2} - m)\mathcal{Q}^\epsilon(t,x) + \rho(x)r(u(t,x)) + \epsilon^{1/2}\rho(x)\dfrac{\partial}{\partial t}\mathcal{W}^2(t,x), \\ \mathcal{X}^\epsilon(t,0) = \mathcal{X}^\epsilon(t,1) = \mathcal{Q}^\epsilon(t,0) = \mathcal{Q}^\epsilon(t,1) = 0, \end{cases}$$

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Together with the cost:

$$J^\epsilon(u) = \mathbb{E}\int_0^1\int_0^1 \ell(\mathcal{X}^\epsilon(t,x), \mathcal{Y}^\epsilon(t,x), u(t,x))\,dx\,dt + \mathbb{E}\int_0^1 h(\mathcal{X}^\epsilon(1,x))\,dx.$$

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We are interested into the limit, as $\epsilon \searrow 0$, of the value function

$$V^\epsilon = \inf_u J^\epsilon(u)$$

Abstract formulation

We consider a two scale system of controlled ∞-dimensional SDEs:

$$dX_t^{\epsilon,u} = (AX_t^{\epsilon,u} + b(X_t^{\epsilon,u}, Q_t^{\epsilon,u}, u_t))\, dt + R\, dW_t^1,\ X_0^{\epsilon,u} = x^0,$$

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Notice that if $\hat{Q}_s^{\epsilon,u} := Q_{\epsilon s}^{\epsilon,u}$ and $\hat{W}_s^{2,\epsilon} := \frac{1}{\sqrt{\epsilon}} W_{\epsilon s}^{2,\epsilon}$ then

$$d\hat{Q}_s^{\epsilon,u} = \left(B\hat{Q}_s^{\epsilon,u} + F(X_{\epsilon s}^{\epsilon,u}\hat{Q}_s^{\epsilon,u}) + G\rho(u_{\epsilon s})\right)\, dt + \ Gd\hat{W}_s^{2,\epsilon}$$

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- G is a bounded linear operator
- R is a bounded invertible linear operator
- u is a control adapted to the filtration generated by (W^1, W^2) it take values in a suitable topological space U

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- $B + F$ is dissipative with respect to Q e.g.

$$\langle (q - q'), B(q - q') + F(x, q - q')\rangle \leq -\eta|q - q'|^2, \quad \eta > 0.$$

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$$J^{\epsilon}(u) = \mathbb{E}\left[\int_0^1 l(X_t^{\epsilon,u}, Q_t^{\epsilon,u}, u_t)dt + h(X_1^{\epsilon,u})\right]$$

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Also see, Kabanov-Pergamenchicov, Goldys, Yang, Zhou...

BSDE reformulation of the problem

For $\epsilon > 0$ fixed we rewrite the state equation as:

$$dX_t^{\epsilon,u} = AX_t^{\epsilon,u}dt + R\left[R^{-1}b(X_t^{\epsilon,u}, Q_t^{\epsilon,u}, u_t)dt + dW_t^1\right],$$

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then

$$V(\epsilon) = Y_0^\epsilon.$$

The *parametrized* ergodic BSDE

We freeze the slow variables $X_t = x \in H$ and $Z_t = z \in H^*$ and 'stretch' time (roughly speaking we set $\widehat{Q}_s = Q_{\epsilon s}$, $\widehat{W}_s^2 = e^{-1/2} W_{\epsilon s}^2$, $s \in [0, 1/\epsilon]$).

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Theorem (Fuhrman, Hu, T. '07)

$\forall x \in H, z \in H^*$, $\exists!$ *solution* $(Y^{x,z}, \Xi^{x,z}, \lambda(x,z))$ *of the infinite horizon ergodic BSDE*

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$\forall x \in H, z \in H^*$, $\exists!$ solution $(Y^{x,z}, \Xi^{x,z}, \lambda(x,z))$ of the infinite horizon ergodic BSDE

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Moreover $|\check{Y}^{x,q_0,p}_t| \leq c(1 + |\widehat{Q}^{x,q_0}_t|)$ where $c > 0$ only depends on the Lipschitz constants of ψ with respect to q and on the dissipativity constant of $B + F(x, \cdot)$.

Moreover $\lambda(x,z)$ is the value function of a control problem with state equation

$$d\widehat{Q}_s^{x,u} = \left(B\widehat{Q}_s^{x,u} + F(x, \widehat{Q}_s^{x,u})\right) ds + G\rho(u_s)ds + Gd\widehat{W}_s^2, \quad \widehat{Q}_0^u = q_0$$

and ergodic cost

$$J(x,z,u) = \liminf_{T\to 0} \frac{1}{T}\mathbb{E}\int_0^T \left[zb(x, \widehat{Q}_s^{x,u}, u_s) + l(x, \widehat{Q}_s^{x,u}, u_s)\right] ds$$

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For further results on Ergodic BSDEs see [Richou '08] [Debussche, Hu, T. '11], [Hu, Madec, Richou '15], [Hu, Tang '18], [Hu, Lemonnier '19], [Hu Cohen], [Guatteri, Cosso, T. '18], [Guatteri T. ']

Reduced system - Main result in the non-degenerate case

We can now introduce the limit forward-backward system:

$$\begin{cases} dX_t &= AX_t\,dt + dW_t^1, \quad X_0 = x_0 \\ d\bar{Y}_t &= -\lambda(X_t, R^{-*}\bar{Z}_t)\,dt + \bar{Z}\,dW_t^1, \quad t \in [0,1), \quad \bar{Y}_1 = h(X_1), \end{cases}$$

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Recall the f.b. system for the original, two scales problem:

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Theorem (Main result)

$$\lim_{\epsilon \to 0} |Y^\epsilon_0 - \bar{Y}_0| = 0$$

Proof (Sketch): The idea is to freeze the slow equation to give time to the fast equation to behave as the optimal ergodic state.

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For notational simplicity we set $R = I_H$. We have to estimate:

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Y_0^\epsilon - \bar{Y}_0 \quad &= \quad \int_0^1 (\psi(X_t, Q_t^\epsilon, Z_t^\epsilon, \Xi_t^\epsilon/\sqrt{\epsilon}) - \lambda(X_t, \bar{Z}_t))\, dt \\
&\quad + \int_0^1 (Z_t^\epsilon - \bar{Z}_t)\, dW_t^1 + \int_0^1 \Xi_t^\epsilon\, dW_t^2.
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Since the difference $\int_0^1 (\psi(X_t, Q_t^\epsilon, \bar{Z}_t, \Xi_t^\epsilon/\sqrt{\epsilon}) - \psi(X_t, Q_t^\epsilon, Z_t^\epsilon, \Xi_t^\epsilon/\sqrt{\epsilon})dt$ can be easily treated by a change of probability we are left with

$$\int_0^1 (\psi(X_t, Q_t^\epsilon, \bar{Z}_t, \Xi_t^\epsilon/\sqrt{\epsilon}) - \lambda(X_t, \bar{Z}_t))\, dt + \int_0^1 (Z_t^\epsilon - \bar{Z}_t)\, dW_t^1 + \int_0^1 \Xi_t^\epsilon\, dW_t^2.$$

We start a discretization procedure. Let $t_k = k2^{-N}$, $k = 0, 1, \ldots, 2^N - 1$ and define for $t_k \le t < t_{k+1}$:

$$X_t^N = X_{t_k}, \quad Z^N(t) = 2^N \int_{t_{k-1}}^{t_k} \bar{Z}_s \, ds.$$

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Fixed k we consider the system (with stretched time) for $s \geq t_k/\epsilon$:

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$$-d\check{Y}_s^{N,k} = [\psi(X_{t_k}, \widehat{Q}_s^{N,k}, Z_{t_k}^N, \check{\Xi}_s^{N,k}) - \lambda(X_{t_k}, Z_{t_k}^N)]\, ds - \check{\Xi}_t^{N,k} d\widehat{W}_t^2,$$

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The above system is composed by a

- a forward-dissipative equation (for $\hat{Q}$) with initial time t_k/ϵ
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- a backward-ergodic equation (for $(\check{Y}, \check{\Xi}, \lambda)$)

It admits a unique solution $(\widehat{Y}^{N,k}, \widehat{\Xi}^{N,k}, \lambda(X_{t_k}^N, Z_{t_k}^N))$ with $|\check{Y}_s^{N,k}| \leq c(1 + |\widehat{Q}_s^{N,k}|)$

If join the processes setting $\widehat{Q}_s^N = \widehat{Q}_s^{N,k}$, $\check{\underline{\Xi}}_s^N = \check{\underline{\Xi}}_s^{N,k}$ for $s \in [t_k/\epsilon, t_{k+1}/\epsilon[$.

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$$\check{Y}^{N,k}_{t_{k+1}/\epsilon} - \check{Y}^{N,k}_{t_k/\epsilon} = \int_{t_k/\epsilon}^{t_{k+1}/\epsilon} [\psi(X^N_{\epsilon s}, \widehat{Q}^N_s, Z^N_{\epsilon s}, \check{\Xi}^N_s) - \lambda(X^N_{\epsilon s}, Z^N_{\epsilon s})]\, ds + \int_{t_k/\epsilon}^{t_{k+1}/\epsilon} \check{\Xi}^N_s \, d\widehat{W}^2_s.$$

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Therefore, summing up:

$$0 = \sum_{k=1}^{2^N} (\check{Y}^{N,k}_{t_k/\epsilon} - \check{Y}^{N,k}_{t_{k+1}/\epsilon}) + \int_0^{1/\epsilon} \check{\Xi}^N_s \, d\widehat{W}^2_s + - \int_0^{1/\epsilon} \psi(X^N_{\epsilon s}, \widehat{Q}^N_s, Z^N_{\epsilon s}, \check{\Xi}^N_s) ds + \int_0^{1/\epsilon} \lambda(X^N_{\epsilon s}, Z^N_{\epsilon s})\, ds$$

Recall that we had to estimate (after stretching of time, that is for: $\widehat{Q}^{\epsilon}_s := Q^{\epsilon}_{\epsilon s}$, $\widehat{\Xi}^{\epsilon}_s := \Xi^{\epsilon}_{\epsilon s}/\sqrt{\epsilon}$)

$$
\begin{aligned}
Y^{\epsilon}_0 - \bar{Y}_0 = \epsilon \int_0^{1/\epsilon} &(\psi(X_{\epsilon s}, \widehat{Q}^{\epsilon}_s, \bar{Z}_{\epsilon s}, \widehat{\Xi}^{\epsilon}_s) - \lambda(X_{\epsilon s}, \bar{Z}_{\epsilon s}))\, ds \\
&+\sqrt{\epsilon} \int_0^{1/\epsilon} (Z^{\epsilon}_{\epsilon s} - \bar{Z}_{\epsilon s})\, d\widehat{W}^1_t + \epsilon \int_0^{1/\epsilon} \widehat{\Xi}^{\epsilon}_s\, d\widehat{W}^2_s.
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Adding (ϵ times) the above null term we get:

$$\begin{aligned} Y^\epsilon_0 - \bar{Y}_0 = \quad & \epsilon \int_0^{1/\epsilon} \mathcal{R}^{\epsilon,N}_s\, ds + \epsilon \sum_{k=1}^{N} (\check{Y}^{N,k}_{t_k/\epsilon} - \check{Y}^{N,k}_{t_{k+1}/\epsilon}) \\ & + \epsilon \int_0^{1/\epsilon} (\check{\Xi}^N_s - \widehat{\Xi}^\epsilon_s)\, d\widehat{W}^2_s + \epsilon^{\frac{1}{2}} \int_0^{1/\epsilon} (Z^\epsilon_{\epsilon s} - \bar{Z}_{\epsilon s})\, dW^1_t \\ & + \epsilon \int_0^{1/\epsilon} [\psi(X^N_{\epsilon t}, \widehat{Q}^N_t, Z^N_{\epsilon t}, \widehat{\Xi}^\epsilon_s) - \psi(X^N_{\epsilon t}, \widehat{Q}^N_s, Z^N_{\epsilon s}, \check{\Xi}^N_s)]\, ds \\ & + \epsilon \int_0^{1/\epsilon} [\psi(X_{\epsilon s}, \widehat{Q}^\epsilon_s, Z^\epsilon_{\epsilon s}, \widehat{\Xi}^\epsilon_s) - \psi(X_{\epsilon s}, \widehat{Q}^\epsilon_s, \bar{Z}_{\epsilon s}, \widehat{\Xi}^\epsilon_s)]\, ds \end{aligned}$$

where $|\mathcal{R}^{\epsilon,N}_s| \leq L(|X^\epsilon_{\epsilon s} - X^N_{\epsilon s}| + |\widehat{Q}^\epsilon_s - \widehat{Q}^N_s| + |\bar{Z}_{\epsilon s} - Z^N_{\epsilon s}|)$

We can get rid of the last two terms by Girsanov change of probability. Namely we prove that

$$Y_0^\epsilon - \bar{Y}_0 = \widetilde{\mathbb{E}}^\epsilon \int_0^1 \mathcal{R}_{t/\epsilon}^{\epsilon,N}\,dt + \epsilon \widetilde{\mathbb{E}}^\epsilon \sum_{k=1}^{N} (\check{Y}_{t_k/\epsilon}^{N,k} - \check{Y}_{t_{k+1}/\epsilon}^{N,k}) \tag{1}$$

where we denote by $\widetilde{\mathbb{E}}^\epsilon$ the expectation with respect to the Girsanov probability $\widetilde{\mathbb{P}}^\epsilon$ that we obtain when absorbing the last two term in the stochastic integrals
It is crucial to notice that

$$d\widetilde{\mathbb{P}}^\epsilon = \mathcal{E}(\delta^{\epsilon,N}(.))_1 d\mathbb{P}$$

where the perturbations $\delta^{\epsilon,N}$ are bbd, uniformly in ϵ and N.

Since $|\mathcal{R}^{\epsilon,N}_s| \leq L(|X^\epsilon_{\epsilon s} - X^N_{\epsilon s}| + |\widehat{Q}^\epsilon_s - \widehat{Q}^N_s| + |\bar{Z}_{\epsilon s} - Z^N_{\epsilon s}|)$ and $\delta^{\epsilon,N}$ is bdd. unif. in ϵ, N we can estimate the 'error' in the new probability. Namely

$$\widetilde{\mathbb{E}}^\epsilon \int_0^1 \mathcal{R}^{\epsilon,N}_{t/\epsilon}\, dt \to 0 \text{ as } N \to \infty \text{ uniformly with respect to } \epsilon$$

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Coming to the last term $\epsilon \widetilde{\mathbb{E}}^{\epsilon} \sum_{k=1}^{N} (\check{Y}_{t_k/\epsilon}^{N,k} - \check{Y}_{t_{k+1}/\epsilon}^{N,k})$ we recall that

- $|\check{Y}_s^{N,k}| \leq c(1 + |\widehat{Q}_s^N|)$
- $\widetilde{\mathbb{E}}^{\epsilon} \sup_{s \geq 0} |\widehat{Q}_s^N|^2 \leq C$ (by dissipativity of the fast equation).

Since $|\mathcal{R}_s^{\epsilon,N}| \leq L(|X_{\epsilon s}^{\epsilon} - X_{\epsilon s}^N| + |\widehat{Q}_s^{\epsilon} - \widehat{Q}_s^N| + |\bar{Z}_{\epsilon s} - Z_{\epsilon s}^N|)$ and $\delta^{\epsilon,N}$ is bdd. unif. in ϵ, N we can estimate the 'error' in the new probability. Namely

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$$|\epsilon\widetilde{\mathbb{E}}^{\epsilon} \sum_{k=1}^N (\widehat{Y}_{t_k/\epsilon}^{N,k} - \widehat{Y}_{t_{k+1}/\epsilon}^{N,k})| \leq \epsilon \sum_{k=1}^N \widetilde{\mathbb{E}}^{\epsilon}(1 + |\widehat{Q}_{t_k/\epsilon}^N| + |\widehat{Q}_{t_{k+1}/\epsilon}^N|) \leq \widetilde{C}\epsilon N.$$

Since $|\mathcal{R}^{\epsilon,N}_s| \leq L(|X^\epsilon_{\epsilon s} - X^N_{\epsilon s}| + |\widehat{Q}^\epsilon_s - \widehat{Q}^N_s| + |\bar{Z}_{\epsilon s} - Z^N_{\epsilon s}|)$ and $\delta^{\epsilon,N}$ is bdd. unif. in ϵ, N we can estimate the 'error' in the new probability. Namely

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At last we sum up all results to get

$$|Y^\epsilon_0 - \bar{Y}_0| \leq \widetilde{\mathbb{E}}^\epsilon \int_0^1 |\mathcal{R}^{\epsilon,N}_{t/\epsilon}|\, dt + \epsilon N(1 + C)$$

So our claim follow choosing N large and then ϵ close to 0,

Degenerate case - small noise regularization

Let us come back to the original problem

$$dX_t^{\epsilon,u} = (AX_t^{\epsilon,u} + b(X_t^{\epsilon,u}, Q_t^{\epsilon,u}, u_t))\, dt \,+ R(X_t^{\epsilon,u})\, dW_t^1,\; X_0^{\epsilon,u} = x^0,$$

$$dQ_t^{\epsilon,u} = \frac{1}{\epsilon}\,(BQ_t^{\epsilon,u} + F(X_t^{\epsilon,u}, Q_t^{\epsilon,u}) + G\rho(u_t))\, dt + \frac{1}{\sqrt{\epsilon}}\, GdW_t^2,\; Q_0^{\epsilon} = q_0,$$

We allow R to depend on X and **to be degenerate**

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We allow R to depend on X and **to be degenerate**

Given a H-valued cylindrical Wiener process (B_t) and a small constant η we introduce the following small-noise regularization of the problem

$$dX_t^{\epsilon,\eta,u} = (AX_t^{\epsilon,\eta,u} + b(X_t^{\epsilon,\eta,u}, Q_t^{\epsilon,\eta,u}, u_t))\,dt + R(X_t^{\epsilon,\eta,u})dW_t^1 + \eta dB_t,$$

$$dQ_t^{\epsilon,\eta,u} = \frac{1}{\epsilon}(BQ_t^{\epsilon,\eta,u} + F(X_t^{\epsilon,\eta,u}, Q_t^{\epsilon,\eta,u}) + G\rho(u_t))\,dt + \frac{1}{\sqrt{\epsilon}}GdW_t^2.$$

By direct estimates, using in a crucial way the dissipativity of $B + F(x, \cdot)$ and the boundedness of b and ρ, we have:

$$\mathbb{E} \int_0^1 \left[|Q_t^{\epsilon,u,\eta} - Q_t^{\epsilon,u}| + |X_t^{\epsilon,u,\eta} - X_t^{\epsilon,u}| \right] dt \to 0$$

as $\eta \to 0$ uniformly with respect to $\epsilon > 0$ and to the control u.

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as $\eta \to 0$ uniformly with respect to $\epsilon > 0$ and to the control u.

Therefore if we introduce again the cost J and the its value function:

$$J^{\epsilon,\eta}(u) = \mathbb{E} \left[\int_0^T l(X_t^{\epsilon,\eta,u}, Q_t^{\epsilon,\eta,u}, u_t) dt + h(X_1^{\epsilon,\eta,u}) \right], \; V^{\epsilon,\eta} = \inf_u J^{\epsilon,\eta}(u).$$

By direct estimates, using in a crucial way the dissipativity of $B + F(x, \cdot)$ and the boundedness of b and ρ, we have:

$$\mathbb{E} \int_0^1 \left[|Q_t^{\epsilon,u,\eta} - Q_t^{\epsilon,u}| + |X_t^{\epsilon,u,\eta} - X_t^{\epsilon,u}| \right] dt \to 0$$

as $\eta \to 0$ uniformly with respect to $\epsilon > 0$ and to the control u.

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It holds:

$$V^{\epsilon,\eta} \to V^{\epsilon}$$

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For the regularized problem we can use the above results. Let:

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dX_t^{\epsilon,\eta} &= AX_t^{\epsilon,\eta}\,dt + R(X_t^{\epsilon,\eta})dW_t^1 + \eta\,dB_t,\ X_0^{\epsilon,\eta} = x_0,\\
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Finally interchanging the limits (since $V^{\epsilon,\eta} \to V^{\epsilon}$ uniformly in ϵ)

$$\lim_{\epsilon\to 0} V^{\epsilon} = \lim_{\eta\to 0} Y_0^{\eta}$$

Limit control problem

We concentrate on the convergence of the reduced f. b. system

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Recall that λ is the optimal value of a parametrized ergodic control problem

$$d\widehat{Q}_s^u = \left(B\widehat{Q}_s^u + F(x, \widehat{Q}_s^u)\right) ds + G\rho(u_s)ds + Gd\widehat{W}_s^2, \quad \widehat{Q}_0^u = q_0$$

$$J(x,z,u) = \liminf_{T\to 0} \frac{1}{T}\mathbb{E}\int_0^T [zb(x, Q_s^u, u) + l(x, Q_s^u, u)]\, ds$$

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- $\tilde{\lambda}$ coincides with λ on a ball and points 1. and 2. still hold,
- $\tilde{\lambda}$ is Lipschitz in x uniformly with respect to z,
- $\tilde{\lambda}(x, z) \approx \kappa_1 - \kappa_2 |z|$ for $|z|$ large.

Let $\tilde{\lambda}_*$ the Legendre transform of $\tilde{\lambda}$ (recall that $\tilde{\lambda}$ is concave, this justifies the negative signs):

$$\tilde{\lambda}_*(x,a) := \inf_{z \in H^*} \{-za - \tilde{\lambda}(x,z)\}, \qquad x, a \in H$$

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It turns out that $\tilde{\lambda}_*$ is Lipschitz continuous with respect to x. Indeed:

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That yields the following simplification in the Fenchel duality:

$$\tilde{\lambda}(x,z) := \inf_{a \in H : |\alpha| \leq L} \{-za - \tilde{\lambda}_*(x,a)\}$$

Resuming we have

$$\begin{aligned} dX_t^\eta &= AX_t^\eta\, dt + R(X_t^\eta)dW_t^1 + \eta\, dB_t, \quad X_0^\eta = x_0, \\ -dY_t^\eta &= \tilde{\lambda}(X_t^\eta, \eta^{-1} Z_t^{2,\eta})dt - Z_t^{1,\eta} dW_t^1 - Z_t^{2,\eta} dB_t, \quad Y_1^{\epsilon,\eta} = h(X_1^{\epsilon,\eta}). \end{aligned}$$

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where $X^{\eta,\alpha}$ solves:

$$dX_s^{\eta,\alpha} = AX_s^{\eta,\alpha}ds - \alpha_s ds + R(X_s^{\eta,\alpha})dW_s^1 + \eta\,dB_t, \quad X_0 = x_0.$$

and α is a (B, W^1) adapted H-valued (bounded) control.

Passing to the limit as $\eta \to 0$ we have the final characterization

Theorem (Guatteri, T. 2022)

$$\lim_{\epsilon \to 0} V^\epsilon = \lim_{\eta \to 0} Y_0^\eta = \inf_{|\alpha| \le L} \bar{\mathbb{E}} \left(h(X_1^\alpha) - \int_t^1 \tilde{\lambda}_*(X_\ell^\alpha, \alpha_\ell) d\ell \right)$$

where X^α solves:

$$dX_s^\alpha = AX_s^{\eta,\alpha} ds - \alpha_s ds + R(X_s^{\eta,\alpha}) dW_s^1, \quad X_0 = x_0.$$

BSDEs with 'reflection' in the martingale term

From the control interpretation of the limit we may go back to BSDEs. The control problem is singular we have to use randomization technique (see [Kharroubi-Pham '15], and also Bandini, Cosso, Guatteri, Fuhrman and many others) .

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and $(\mathcal{Y}, \mathcal{Z}, \mathcal{K})$ be the maximal solution of the constrained BSDE:

$$-d\mathcal{Y}_t = \tilde{\lambda}_*(\mathcal{X}_t, \mathcal{W}_t)dt - d\mathcal{K}_t + \mathcal{Z}_t dW_t^1$$

where $(\mathcal{K}_t)$ is non decreasing. Notice that the solution is adapted to the filtration generated by $(\mathcal{W}, W^1)$.

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We can conclude:

$$\lim_{\epsilon \to 0} V^\epsilon = \mathcal{Y}_0$$

Thank you for your attention!