Convergence rate of RBSDE by penalisation method and its application to American Option

Wanqing WANG[1],
Supervised by Emmanuel GOBET[1], Mingyu XU[2]

[1]CMAP, Ecole Polytechnique

[2]Fudan University

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Introduction

American Option

Given the payoff process S and the maturity T, the fair price of the American option is

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{Q}}[e^{-r(\tau - t)} S_\tau]$$

which is equivalent to a solution of Reflected Backward Stochastic Differential Equation problem(RBSDE in short)(see [EKP+97]).

RBSDE

The equation writes as

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathbf{d}s + K_T - K_t - \int_t^T Z_s \mathbf{d}B_s, \quad \forall 0 \leq t \leq T, \\ Y_t \geq S_t, 0 \leq t \leq T, \\ \int_0^T (Y_t - S_t)\mathbf{d}K_t = 0 \end{cases}$$

where we call the triple (Y, Z, K) the solution w.r.t. (ξ, f, S) with $\xi \geq S_T$.

Motivation

Define the Penalised BSDE as following:

PBSDE:

$$Y_t^\lambda = \xi + \int_t^T f(s, Y_s^\lambda, Z_s^\lambda)\mathbf{d}s + \lambda \int_t^T (Y_s^\lambda - S_s)^- \mathbf{d}s - \int_t^T Z_s^\lambda \mathbf{d}B_s$$

The penalisation is usually used to prove the existence problem and $Y^\lambda \nearrow Y$(see [EKP+97]).

Q: What about the convergence rate?

1. Introduction
2. Penalisation
3. Example
4. Approximation

Notation and Hypothesis

For some $p \geq 2$,

- $\mathbb{L}^p = \{\zeta; \mathbb{E}[|\zeta|^p] < \infty\}$;
- $\mathscr{S}^p = \left\{\phi; \{\phi_t, 0 \leq t \leq T\} \text{ is continuous s.t. } \mathbb{E}\left[\sup_{0 \leq t \leq T} |\phi_t|^p\right] < \infty\right\}$;
- $\mathbb{H}^p = \left\{\phi; \{\phi_t, 0 \leq t \leq T\} \in \text{ is predictable s.t. } \mathbb{E}\left[\left(\int_0^T |\phi_t|^2 \mathbf{d}t\right)^{p/2}\right] < \infty\right\}$;

($\mathbf{H}_1$) Standard assumption for RBSDE;

($\mathbf{H}_2$) The barrier S

$$S_t = S_0 + \int_0^t U_s \mathbf{d}s + \int_0^t V_s \mathbf{d}B_s + A_t$$

with $U, V, A \in \mathscr{S}^p$ and A non-decreasing;

($\mathbf{H}_3$) In the financial market, the liquidation X^0 and the risky assets $X := (X^1, \cdots, X^d)^\top$ satisfy $\mathbf{d}X_t^0 = X_t^0 r_t \mathbf{d}t$ and

$$\frac{\mathbf{d}X_t^i}{X_t^i} = \mu_t^i \mathbf{d}t + \sum_{j=1}^d \sigma_t^{ij} \mathbf{d}B_t^j$$

and r interest rate and σ is invertible, μ, σ, r can be stochastic.

Background

Reflected BSDE

- El karoui *et al.* (1997): $Y^\lambda \nearrow Y$ in $\mathscr{S}^2$;
- Lepeltier (2004): $Y^\lambda \to Y$ in $\mathbb{H}^2$ for double reflected barrier;
- Jeanblanc (2004): $Y^\lambda \to Y$ at $1/\lambda$ in $\mathscr{S}^2$ for double reflected constant barrier;
- Reisinger (2013): $Y^\lambda \to Y$ at $1/\lambda$ for convex payoff using PDE;

Suppose $h := \frac{T}{N}$.

Reflected FSDE on convex domain

- Liu (1993): $Y^{\lambda,h} \to Y$ at $h^{1/4-a}$ in $\mathscr{S}^2$ if $\lambda \sim O(h^{-1/2})$;
- Petterson (1997): $Y^{\lambda,h} \to Y$ at $\sqrt[4]{h\log(\frac{1}{h})}$ in $\mathscr{S}^2$ if $\lambda \leq O(h^{-1})$;

Main result

Theorem 1

Suppose $\mathbf{H}_2$ is fulfilled by S and besides $\mathbf{H}_1$, suppose $f(t, y, z)$ is decreasing on y, assume

$$\alpha := \operatorname*{ess\,sup}_{t \in [0,T]} (f(t, S_t, V_t) + U_t)^- < \infty$$

then the penalisation term is dominated by

$$(Y_t^\lambda - S_t)^- \leq \frac{\alpha}{\lambda}, \quad \forall 0 \leq t \leq T, \quad a.s.$$

Remark

Decreasing: if $y_1 < y_2$, $f(t, y_1) > f(t, y_2)$. If not, one can apply Itô formula on $e^{\nu t} Y_t$ with choosing $\nu > C_{Lip}^f$, then

$$\alpha^\nu := \operatorname*{ess\,sup}_{t \in [0,T]} e^{\nu t} (f(t, S_t, V_t) + U_t)^- .$$

Main result

Theorem 2

Under $\mathbf{H}_1$ and $\mathbf{H}_2$, if α is well defined as before, then by the comparison theorem

$$0 \leq Y_t - Y_t^{\lambda} \leq \frac{\alpha}{\lambda}, \quad \forall 0 \leq t \leq T, a.s..$$

Example of Call/Put in Linear Market

Assume there is no dividend, in the linear market,

$$f(t, y, z) = -r_t y - z\sigma_t^{-1}(\mu_t - r_t).$$

For the Call option,

$$\mathbf{d}S_t = \mathbf{d}(X_t - K)^+ = \mu_t X_t \mathbf{1}_{S_t > K}\mathbf{d}t + \sigma_t X_t \mathbf{1}_{S_t > K}\mathbf{d}B_t + \frac{1}{2}\mathbf{d}L_t,$$

so

$$\begin{aligned}
&(f(t, S_t, V_t) + U_t)^- \\
&= (-r_t(X_t - K)^+ - \mathbf{1}_{\{X_t > K\}} X_t(\mu_t - r_t) + \mathbf{1}_{\{X_t > K\}}\mu_t X_t)^- \\
&= (r_t K \mathbf{1}_{\{X_t > K\}})^- = |r_t^-| K \mathbf{1}_{\{X_t > K\}} \leq |r^-|_\infty K.
\end{aligned}$$

then $\alpha := |r^-|_\infty K$.

Example of Call/Put in Linear Market

1. This α is true for all stochastic parameters if r is bounded from below!

2. What happens if $\alpha = 0$? Two explanations:
   - from Theorem 1 & 2, $S_t \leq Y_t^\lambda = Y_t, \forall t$;
   - from the density of K_t (see [EKP+97]), $\frac{\mathbf{d}K_t}{\mathbf{d}t} \leq \alpha, \forall t; \Longrightarrow K = 0$.

The RBSDE problem $\rightarrow$ a BSDE problem!
The price of American Option $=$ the European one!

Example of Call/Put in Linear Market

The similar conclusion for Put, suppose there is no dividend,

$$\mathbf{d}S_t = \mathbf{d}(K - X_t)^+ = -\mu_t X_t \mathbf{1}_{S_t \leq K} \mathbf{d}t - \sigma_t X_t 1_{X_t \leq K} \mathbf{d}B_t + \frac{1}{2}\mathbf{d}L_t.$$

So

$$\begin{aligned}
&(f(t, S_t, V_t) + U_t)^- \\
&= (-r_t(K - X_t)^+ + \mathbf{1}_{\{X_t \leq K\}} X_t(\mu_t - r_t) - \mathbf{1}_{\{X_t \leq K\}} \mu_t X_t)^- \\
&= (-r_t K \mathbf{1}_{\{X_t \leq K\}})^- = |r_t^+| K \mathbf{1}_{X_t \leq K} \leq |r^+|_\infty K, \forall t \in [0, T].
\end{aligned}$$

Then $\alpha := |r^+|_\infty K$.

Other type of option in Linear Market

type	pay-off	α	order $\left(\frac{1}{\lambda}\right)$
Call	$(X_t - K)^+$	$\lvert r^- \rvert_\infty K$	$1\ (r < 0)$
Put	$(K - X_t)^+$	$\lvert r^+ \rvert_\infty K$	$1\ (r > 0)$
Call on spread	$(X_t^1 - X_t^2 - K)^+$	$\lvert r^- \rvert_\infty K$	1
Put on spread	$(K - (X_t^1 - X_t^2))^+$	$\lvert r^+ \rvert_\infty K$	1
Call on max	$(X_t^1 \vee X_t^2 - K)^+$	$\lvert r^- \rvert_\infty K$	1
Put on min	$(K - X_t^1 \wedge X_t^2)^+$	$\lvert r^+ \rvert_\infty K$	1
Call on basket	$(\sum_{i=1}^d X_t^i - K)^+$	$\lvert r^- \rvert_\infty K$	1
Call on min	$(X_t^1 \wedge X_t^2 - K)^+$		$\frac{1}{2}$

Example of Call/Put in Non-Linear Market

Suppose the lending rate r and the borrowing rate R with $r_t < R_t, 0 \leq t \leq T$. The generator f is defined as following

$$f(t, y, z) = -r_t y - z\sigma_t^{-1}(\mu_t - r_t) + (R_t - r_t)(y - z\sigma^{-1})^-.$$

We can also find,

$$\text{for Call}, \alpha := |R^-|_\infty K;$$

$$\text{for Put}, \alpha := |r^+|_\infty K.$$

Main result

Theorem 3

Suppose $\mathbf{H}_1, \mathbf{H}_2$ are satisfied by the data (ξ, f, S), and if one supposes $U, V, f(\cdot, 0, 0) \in \mathscr{S}^p$, then

$$\mathbb{E}\left[\sup_{0\leq t\leq T}(Y_t - Y_t^\lambda)^p + \left(\int_0^T |Z_t - Z_t^\lambda|^2 \mathbf{d}t\right)^{p/2} + \sup_{0\leq t\leq T}(K_t - K_t^\lambda)^p\right] \leq \frac{C_{T,f,S}}{\lambda^{p/2}}$$

where $C_{T,f,S}$ is a constant depending on T, f, S.

Implicit Schemes

Suppose f does not depend on z.

Implicit Schemes

Define the implicit discrete solution as

$$Y_{t_i}^{h,\lambda} = \mathbb{E}_{t_i}\left[Y_{t_{i+1}}^{h,\lambda} + f^{\lambda}(t, Y_{t_i}^{h,\lambda})h\right]$$

with $f^{\lambda}(t,y) = f(t,y) + \lambda(y - S_t)^-$.

Why not the explicit one as below?

$$Y_{t_i}^{h,\lambda,e} = \mathbb{E}_{t_i}\left[Y_{t_{i+1}}^{h,\lambda,e} + f^{\lambda}(t_{i+1}, Y_{t_{i+1}}^{h,\lambda,e})h\right].$$

When λ is big, we need h much smaller to achieve the same precision.

Approximation-Main result

Theorem 4

Suppose $\mathbf{H}_1, \mathbf{H}_2$ are satisfied. If

- the driver $f(t,y,z) = f(t,y)$ and $f(t,\cdot) \in \mathbb{C}^1$;
- $|V_t| \leq C, a.s. 0 \leq t \leq T$;
- α is well defined.

Then we have

$$\sup_{0 \leq i \leq N-1} \mathbb{E}\left[(Y_{t_i}^{\lambda,h} - Y_{t_i}^{\lambda})^2\right] \leq O(\lambda^2 h^{5/4}).$$

Moreover, if S is a Itô process ($A = 0$), then we have

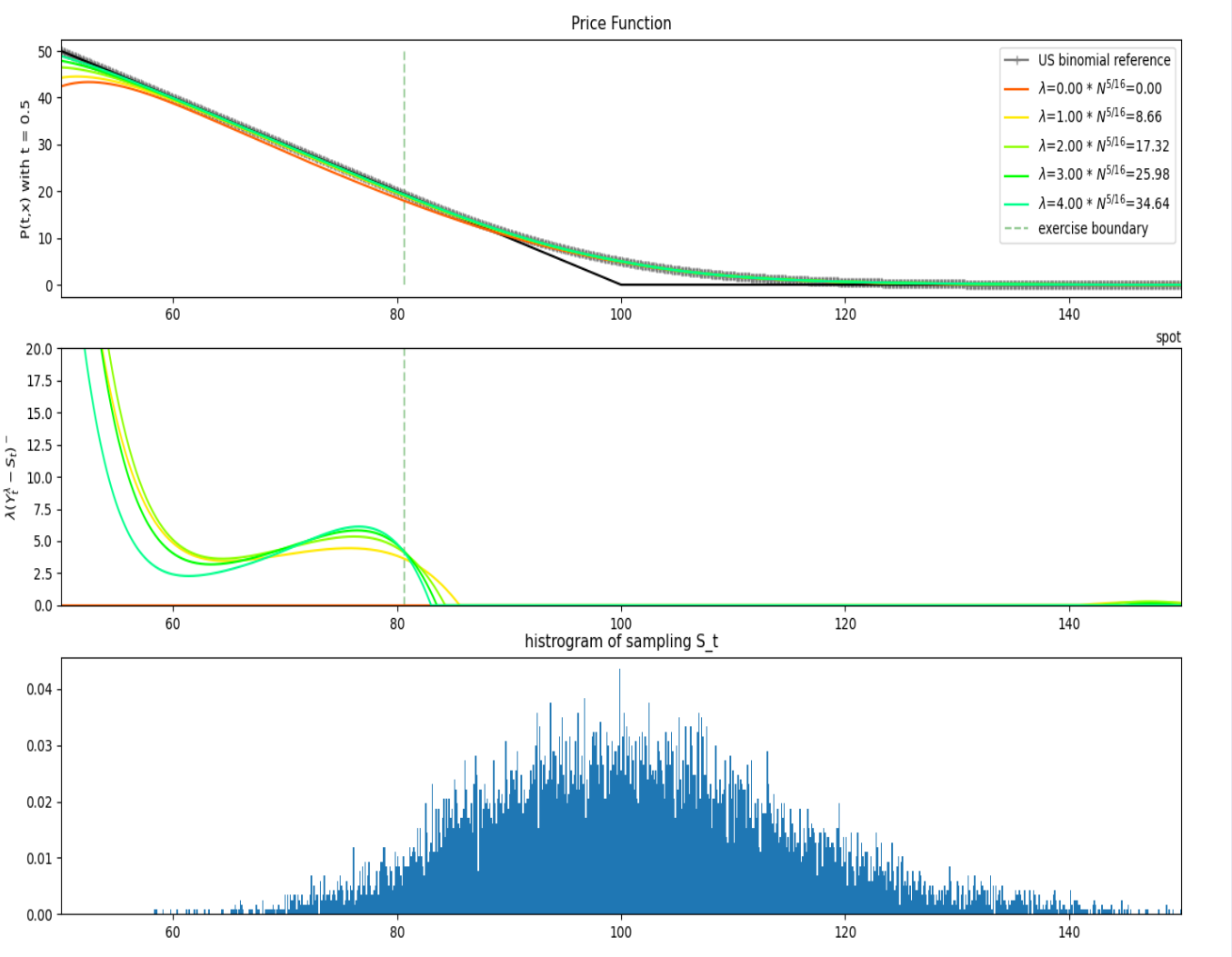
$$\sup_{0 \leq i \leq N-1} \mathbb{E}\left[(Y_{t_i}^{h,\lambda} - Y_{t_i}^{\lambda})^2\right] \leq O(\lambda^2 h^{3/2}).$$

Corollary 5

Under $\mathbf{H}_1$, $\mathbf{H}_2$ and suppose α is defined as before. Let $\lambda = h^{-5/16}$, we get the global error

$$\sup_{0 \leq i \leq N-1} \mathbb{E}\left[(Y_{t_i}^{h,\lambda} - Y_{t_i})^2\right]^{1/2} \leq O(h^{5/16}).$$

Price Function at $t = 0.5$



λ	RMSE
0	1.35279
8.66	0.85609
17.32	0.50866
25.98	0.29885
34.65	0.15876

Parameters: $r = 0.03$, $\sigma = 0.2$, $T = 1$, $K = X_0 = 100$, $M = 10e3$, $N = 10e2$, $n = 6$ (see [GLW05]) The reference Y_t is given by binomial tree with 1000 time steps.

Monte-Carlo Simulation

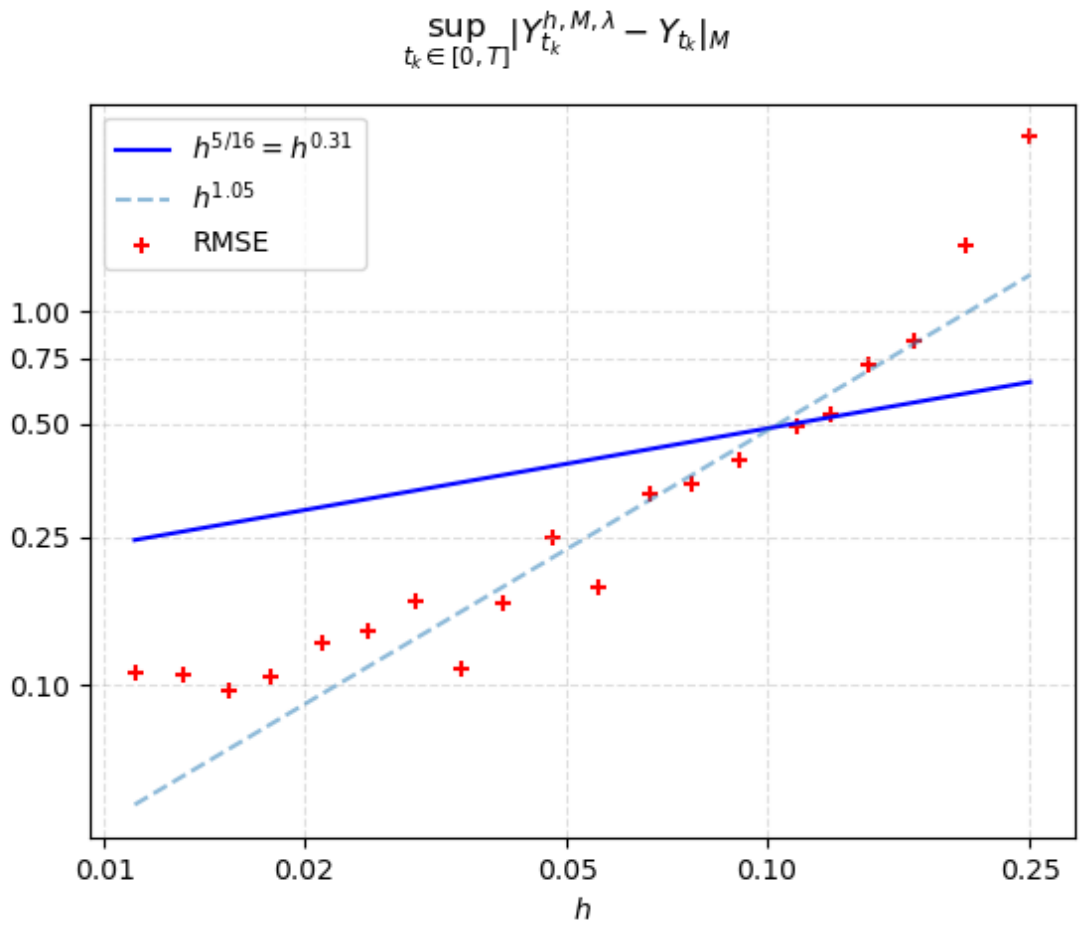


Figure: Define the statistical error ε as $\varepsilon(h) := \sup_{t_k \in [0,T]} \| Y_{t_k}^{h,M,\lambda} - Y_{t_k} \|_M$ with $\|x\|_M := \sqrt{\frac{\sum_{m=1}^{M} (x^m)^2}{M}}$ and $\lambda := h^{-5/16}$. Parameters: $r = 0.03$, $\sigma = 0.2$, $T = 1$, $K = X_0 = 100$, $M = 10e3$, $n = 6$, the reference Y_{t_k} is given by binomial with 1000 time steps.

Conclusion and Perspective

Conclusion

- $Y^\lambda \nearrow Y$ at least at $1/\sqrt{\lambda}$, at $1/\lambda$ if α;
- $Y^{\lambda,h} \to Y$ at $\lambda^2 h^{5/4}$;
- The global error $Y^{\lambda,h} \to Y$ at $h^{5/16}$ if $\lambda = h^{-5/16}$, the numerical experiment shows a better order;

Perspective

- The order 1 rate holds for the market with dividend?
- Complete the scheme for Z motivated by non-linear market?
- More experiments for the stochastic parameters and also for some type of option without α?

Thanks for your attention!

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