

# Necessary and Sufficient Conditions for Optimal Control of Semilinear SPDEs

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# Outline

- 1 Peng's Maximum Principle
- 2 Relationship Between Adjoint States and Value Function
- 3 Verification Theorem

# Setting

Minimize

$$J(u) := \mathbb{E} \left[ \int_0^T \int_{\Lambda} l(x_t^u(\lambda), u_t) d\lambda dt + \int_{\Lambda} h(x_T^u(\lambda)) d\lambda \right]$$

subject to

$$\begin{cases} dx_t^u = [\Delta x_t^u + b(x_t^u, u_t)]dt + \sigma(x_t^u, u_t)dW_t, & t \in [0, T] \\ x_0^u = x \in L^2(\Lambda), \end{cases}$$

where

- $b, \sigma$  Nemytskii operators, Lipschitz;
- $(W_t)_{t \geq 0}$  cylindrical Wiener process;
- $\Lambda \subset \mathbb{R}$  bounded interval;
- $l, h$  Nemytskii operators of (at most) quadratic growth;
- control domain  $U$  non-convex.

Goal: Derive necessary conditions for optimality.

# Setting

Minimize

$$J(u) := \mathbb{E} \left[ \int_0^T \int_{\Lambda} l(x_t^u(\lambda), u_t) d\lambda dt + \int_{\Lambda} h(x_T^u(\lambda)) d\lambda \right]$$

subject to

$$\begin{cases} dx_t^u = [\Delta x_t^u + b(x_t^u, u_t)]dt + \sigma(x_t^u, u_t)dW_t, & t \in [0, T] \\ x_0^u = x \in L^2(\Lambda). \end{cases}$$

- Pontryagin's maximum principle (1956): Necessary optimality condition in finite-dimensional, deterministic case ( $\sigma \equiv 0$ ).
- Peng's maximum principle (1990): Generalization to finite-dimensional stochastic case.
- Since then: Many generalizations to infinite-dimensional, stochastic case by Du, Fuhrman, Frankowska, Guatteri, Hu, Li, Lü, Meng, Tang, Tessitore, Zhang, ...  
Major drawback in previous works: Strong assumptions on coefficients  $l$  and  $h$ , in particular excluding quadratic costs.

# Spike Variation

Let  $\bar{u}$  be optimal. Fix  $\tau \in [0, T)$ ,  $\varepsilon > 0$ ,  $v \in U$ , and set

$$u_t^\varepsilon := \begin{cases} v, & \tau \leq t \leq \tau + \varepsilon, \\ \bar{u}_t & \text{otherwise.} \end{cases}$$

Then

$$0 \leq J(u^\varepsilon) - J(\bar{u}) = \mathbb{E} \left[ \int_0^T \int_{\Lambda} I(x_t^\varepsilon, u_t^\varepsilon) - I(\bar{x}_t, \bar{u}_t) d\lambda dt + \int_{\Lambda} h(x_T^\varepsilon) - h(\bar{x}_T) d\lambda \right].$$

- Roadmap: 1) Taylor expand integrands;  
2) Divide by  $\varepsilon$ , send  $\varepsilon \rightarrow 0$ ;  
3) Identify remaining terms.

Because of stochastic calculus, we have to Taylor expand up to second order.

## Quadratic Terms

Taylor expanding the cost functional to second order leads to the quadratic terms

$$\mathbb{E} \left[ \int_0^T \int_{\Lambda} l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) d\lambda dt + \int_{\Lambda} h_{xx}(\bar{x}_T(\lambda)) y_T^\varepsilon(\lambda) y_T^\varepsilon(\lambda) d\lambda \right].$$

Idea: Linearize using tensor product. In finite dimensions, Peng derived equation for  $y_t^\varepsilon \otimes y_t^\varepsilon$  on

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \mathbb{R}^{n \times n}.$$

In existing literature, this is generalized in infinite dimensions to

$$H \otimes H \cong L_2(H).$$

Problem: In order to perform duality analysis, we need to solve equation in  $L_1(H)$ .

# Explicit Tensor Product

Instead, we use explicit representation

$$L^2(\Lambda) \otimes L^2(\Lambda) \cong L^2(\Lambda^2),$$

where a simple tensor  $y \otimes z$  is identified with the function  $(\lambda, \mu) \mapsto y(\lambda)z(\mu)$ . Thus, we can rewrite quadratic terms as

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T \int_{\Lambda} l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) y_t^\varepsilon(\lambda) y_t^\varepsilon(\lambda) d\lambda dt \right] \\ &= \mathbb{E} \left[ \int_0^T \int_{\Lambda} l_{xx}(\bar{x}_t(\lambda), \bar{u}_t) \delta(y_t^\varepsilon \otimes y_t^\varepsilon)(\lambda) d\lambda dt \right], \end{aligned}$$

where  $\delta : H_0^1(\Lambda^2) \rightarrow L^2(\Lambda)$  is defined by  $\delta(w)(\lambda) := w(\lambda, \lambda)$ . This expression is linear in  $y_t^\varepsilon \otimes y_t^\varepsilon$ .

# Second Order Adjoint Equation

Theorem (Stannat, W., SICON 2021)

*The equation*

$$\begin{cases} dP_t(\lambda, \mu) = -[\Delta P_t(\lambda, \mu) + (b_x(\bar{x}_t(\lambda), \bar{u}_t) + b_x(\bar{x}_t(\mu), \bar{u}_t))P_t(\lambda, \mu) \\ \quad + \langle \sigma_x(\bar{x}_t(\lambda), \bar{u}_t), \sigma_x(\bar{x}_t(\mu), \bar{u}_t) \rangle_{L_2(\Xi, \mathbb{R})}P_t(\lambda, \mu) \\ \quad + \langle \sigma_x(\bar{x}_t(\lambda), \bar{u}_t) + \sigma_x(\bar{x}_t(\mu), \bar{u}_t), Q_t(\lambda, \mu) \rangle_{L_2(\Xi, \mathbb{R})} \\ \quad + \delta^*(I_{xx}(\bar{x}_t(\lambda), \bar{u}_t)) + \delta^*(b_{xx}(\bar{x}_t(\lambda), \bar{u}_t)p_t(\lambda)) \\ \quad + \delta^*(\langle \sigma_{xx}(\bar{x}_t(\lambda), \bar{u}_t), q_t \rangle_{L_2(\Xi, \mathbb{R})})]dt + Q_t(\lambda, \mu)dW_t \\ \\ P_T(\lambda, \mu) = \delta^*(h_{xx}(\bar{x}_T(\lambda))) \end{cases}$$

has a unique adapted solution  $(P, Q)$ , where

$$P \in L^2([0, T] \times \Omega; L^2(\Lambda^2)) \cap L^2(\Omega; C([0, T]; H^{-1}(\Lambda^2))),$$

and

$$Q \in L^2([0, T] \times \Omega; L_2(\Xi; H^{-1}(\Lambda^2))).$$

# Peng's Maximum Principle for SPDEs

Theorem (Stannat, W., SICON 2021)

Let  $(\bar{x}, \bar{u})$  be an optimal pair. Then there exist adapted processes

$$(p, q) \in L^2([0, T] \times \Omega; H_0^1(\Lambda)) \times L^2([0, T] \times \Omega; L_2(\Xi, L^2(\Lambda)))$$

satisfying the first order adjoint equation and adapted processes

$$(P, Q) \in L^2([0, T] \times \Omega; L^2(\Lambda^2)) \times L^2([0, T] \times \Omega; L_2(\Xi, H^{-1}(\Lambda^2)))$$

satisfying the second order adjoint equation such that

$$\inf_{u \in U} \mathcal{G}(t, \bar{x}_t, u) = \mathcal{G}(t, \bar{x}_t, \bar{u}_t),$$

for almost all  $(t, \omega) \in [0, T] \times \Omega$ , where  $\mathcal{G} : [0, T] \times L^2(\Lambda) \times U \rightarrow \mathbb{R}$

$$\begin{aligned} \mathcal{G}(t, x, u) := & \int_{\Lambda} l(x(\lambda), u) d\lambda + \langle p_t, b(x, u) \rangle_{L^2(\Lambda)} + \langle q_t, \sigma(x, u) \rangle_{L_2(\Xi, L^2(\Lambda))} \\ & + \text{tr}(\sigma(x, u)^* [q_t - P_t \sigma(\bar{x}_t, \bar{u}_t)]) . \end{aligned}$$

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# Dynamic Programming Approach

Minimize

$$J(s, x; u) := \mathbb{E} \left[ \int_s^T \int_{\Lambda} l(x_t^u(\lambda), u_t) d\lambda dt + \int_{\Lambda} h(x_T^u(\lambda)) d\lambda \right]$$

over  $u \in \mathcal{U}_s$  subject to

$$\begin{cases} dx_t^u = [\Delta x_t^u + b(x_t^u, u_t)]dt + \sigma(x_t^u, u_t)dW_t, & t \in [s, T] \\ x_s^u = x \in L^2(\Lambda). \end{cases}$$

Introduce value function  $V : [0, T] \times L^2(\Lambda) \rightarrow \mathbb{R}$ ,

$$V(s, x) := \inf_{u \in \mathcal{U}_s} J(s, x; u).$$

Satisfies dynamic programming principle

$$V(s, x) = \inf_{u \in \mathcal{U}_s} \mathbb{E} \left[ \int_s^{\tau} l(x_t^u, u_t) dt + V(\tau, x_{\tau}^u) \right], \quad \forall \tau \in [s, T].$$

Can be used to derive optimality conditions.

# Relationship under Smoothness Assumptions

Theorem (Bismut (1978), Bensoussan (1982))

If  $V \in C^{1,3}([0, T] \times \mathbb{R}^d)$  and  $V_{sx}$  is continuous, then for almost all  $t \in [0, T]$

$$\textcircled{1} \quad \begin{cases} V_x(t, \bar{x}_t) = p_t \\ V_{xx}(t, \bar{x}_t)\sigma(\bar{x}_t, \bar{u}_t) = q_t \end{cases}$$

$$\textcircled{2} \quad -V_s(t, \bar{x}_t) = \inf_{u \in U} \mathcal{H}(\bar{x}_t, u, V_x(t, \bar{x}_t), V_{xx}(t, \bar{x}_t))$$

$$\textcircled{3} \quad \inf_{u \in U} \mathcal{H}(\bar{x}_t, u, V_x(t, \bar{x}_t), V_{xx}(t, \bar{x}_t)) = \mathcal{H}(\bar{x}_t, \bar{u}_t, V_x(t, \bar{x}_t), V_{xx}(t, \bar{x}_t))$$

where

$$\mathcal{H}(x, u, p, P) := \int_{\Lambda} l(x(\lambda), u) d\lambda + \langle p, b(x, u) \rangle_{L^2(\Lambda)} + \frac{1}{2} \operatorname{tr}(\sigma(x, u)^* P \sigma(x, u)).$$

Problem: In general,  $V$  is not differentiable!

# Parabolic Viscosity Superdifferential

If  $V \in C^{1,2}([0, T] \times L^2(\Lambda))$ , it holds

$$\lim_{\tau \downarrow t, z \rightarrow x} \frac{1}{|\tau - t| + \|z - x\|^2} \left[ V(\tau, z) - V(t, x) - \partial_t V(t, x)(\tau - t) - \langle DV(t, x), z - x \rangle_{L^2(\Lambda)} - \frac{1}{2} \langle z - x, D^2 V(t, x)(z - x) \rangle_{L^2(\Lambda)} \right] = 0.$$

Weaker notion of differentiability: We say  $(G, p, P) \in D_{t+,x}^{1,2,+} V(t, x)$  if

$$\limsup_{\tau \downarrow t, z \rightarrow x} \frac{1}{|\tau - t| + \|z - x\|^2} \left[ V(\tau, z) - V(t, x) - G(\tau - t) - \langle p, z - x \rangle_{L^2(\Lambda)} - \frac{1}{2} \langle z - x, P(z - x) \rangle_{L^2(\Lambda)} \right] \leq 0.$$

Theorem (Stannat, W. (2022+))

For almost every  $t \in [0, T]$ , it holds that

$$[-\langle \Delta \bar{x}_t, p_t \rangle_{H^{-1}(\Lambda) \times H_0^1(\Lambda)} - \mathcal{G}(t, \bar{x}_t, \bar{u}_t), \infty) \times \{p_t\} \times [P_t, \infty) \subset D_{t+,x}^{1,2,+} V(t, \bar{x}_t)$$

$\mathbb{P}$ -almost surely.

# Viscosity Solutions

Value function formally satisfies HJB equation

$$\begin{cases} V_s + \langle \Delta x, DV \rangle_{L^2(\Lambda)} + \inf_{u \in U} \mathcal{H}(x, u, DV, D^2V) = 0, & (s, x) \in [0, T] \times L^2(\Lambda) \\ V(T, x) = \int_{\Lambda} h(x(\lambda)) d\lambda, & x \in L^2(\Lambda) \end{cases}$$

where

$$\mathcal{H}(x, u, p, P) := \int_{\Lambda} l(x(\lambda), u) d\lambda + \langle p, b(x, u) \rangle_{L^2(\Lambda)} + \frac{1}{2} \text{tr} (\sigma(x, u)^* P \sigma(x, u)).$$

## Definition (Viscosity Solution, Bounded Case)

$V$  is viscosity subsolution, if

- $V(T, x) \leq \int_{\Lambda} h(x(\lambda)) d\lambda, \quad x \in L^2(\Lambda);$
- for every  $(G, p, P) \in D_{t,x}^{1,2,+} V(t, x)$

$$G + \langle Ax, p \rangle + \inf_{u \in U} \mathcal{H}(x, u, p, P) \leq 0.$$

# Viscosity Solutions II

It holds:

$$(G, p, P) \in D_{t,x}^{1,2,+} v(t, x)$$
$$\iff$$

$\exists \phi \in C^{1,2}((s, T) \times L^2(\Lambda))$  such that:

- ①  $v - \phi$  attains maximum at  $(t, x)$ ,
- ②  $(\phi(t, x), \partial_t \phi(t, x), D\phi(t, x), D^2\phi(t, x)) = (v(t, x), G, p, P)$ .

Equivalent definition of viscosity solution in the bounded case (!):

## Definition (Viscosity Solution, Bounded Case)

$V$  is a viscosity subsolution, if

- $V(T, x) \leq \int_{\Lambda} h(x(\lambda)) d\lambda, \quad x \in L^2(\Lambda);$
- $\forall \phi \in C^{1,2}((s, T) \times L^2(\Lambda))$  such that  $V - \phi$  attains maximum at  $(t, x)$ , it holds

$$\phi_s(t, x) + \langle Ax, D\phi(t, x) \rangle + \inf_{u \in U} \mathcal{H}(x, u, D\phi(t, x), D^2\phi(t, x)) \leq 0.$$

# Generalized Hamiltonian vs. Hamiltonian

Corollary (Stannat, W. (2022+))

*It holds for almost all  $t \in [s, T]$*

$$\mathcal{G}(t, \bar{x}_t, \bar{u}_t) \leq \mathcal{H}(t, \bar{x}_t, \bar{u}_t, p_t, P_t),$$

$\mathbb{P}$ -almost surely, i.e.,

$$tr(\sigma(\bar{x}_t, \bar{u}_t)(q_t - P_t\sigma(\bar{x}_t, \bar{u}_t))) \leq 0.$$

In the unbounded case, we need to make sense of

$$\langle \Delta x, D\phi(t, x) \rangle_{L^2(\Lambda)}, \quad x \in L^2(\Lambda).$$

~~ Need to restrict class of test functions.

To circumvent this issue, we use higher regularity of  $\bar{x}_t \in H_0^1(\Lambda)$ ,  $dt \otimes \mathbb{P}$ -a.s.

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# Verification Theorem

Minimize

$$J(s, x; u) := \mathbb{E} \left[ \int_s^T L(x_t^u, u_t) dt + H(x_T^u) \right]$$

over  $u \in \mathcal{U}_s$  subject to

$$\begin{cases} dx_t^u = [\Delta x_t^u + B(x_t^u, u_t)]dt + \Sigma(x_t^u, u_t)dW_t, & t \in [s, T] \\ x_s^u = x \in L^2(\Lambda), \end{cases}$$

where

- $B, \Sigma$  are Lipschitz, linear growth;
- $(W_t)_{t \geq 0}$  cylindrical Wiener process;
- $\Lambda \subset \mathbb{R}$  bounded interval;
- $L, H$  of (at most) quadratic growth.

# Verification Theorem

Theorem (Stannat, W. (2022+))

Assume

- $\|\Sigma(x, u)\|_{L_2(\Xi, H_0^1(\Lambda))} \leq C(1 + \|x\|_{H_0^1(\Lambda)})$
- $V(t + \tau, x) - V(t, x) \leq C(1 + \|x\|_{H_0^1(\Lambda)}^2)\tau$
- $V(t, \cdot) - C\|\cdot\|_{L^2(\Lambda)}^2$  is concave.

Let  $(x^*, u^*)$  be an admissible pair. Suppose there are adapted processes  $(G, p, P)$  taking values in  $\mathbb{R}$ ,  $H_0^1(\Lambda)$  and  $L_2(L^2(\Lambda))$ , such that for almost all  $t \in [s, T]$ :

$$(G_t, p_t, P_t) \in D_{t+, x}^{1, 2, +} V(t, x_t^*)$$

$\mathbb{P}$ -almost surely, and

$$\mathbb{E} \left[ \int_s^T G_t + \langle \Delta x_t^*, p_t \rangle_{H^{-1}(\Lambda) \times H_0^1(\Lambda)} + \mathcal{H}(x_t^*, u_t^*, p_t, P_t) dt \right] \geq 0.$$

Then  $(x^*, u^*)$  is an optimal pair.

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