A $C^{0,1}$–functional Itô formula and regularity of solutions to PPDEs

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Outline

1. Introduction : functional Itô and PPDE
2. A functional $C^{0,1}$–Itô formula and its applications
   - Itô calculus via regularization
   - A functional $C^{0,1}$–Itô formulas
   - Applications in finance
3. Approximate solution to PPDE and its regularity

The Itô formula

• Itô formula : let $f : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be a $C^{1,2}$ smooth function, and X be a continuous semimartingale, then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t Df(s, X_s) dX_s \\ &+ \int_0^t \partial_t f(s, X_s) ds + \int_0^t \frac{1}{2} D^2 f(s, X_s) d\langle X \rangle_s. \end{aligned}$$

• Parabolic PDEs :

$$\partial_t u + F(u, Du, D^2 u) = 0.$$

• For example, let B be a Brownian motion, and $u(t,x) := \mathbb{E}\big[g(B_T)\big| B_t = x\big]$. Then u satisfies the heat equation :

$$\partial_t u + \frac{1}{2} D^2 u = 0, \quad u(T, \cdot) = g(\cdot).$$

Path-dependent functional and its derivative

• Let $D([0,T])$ denote the Skorokhod space of all càdlàg paths on $[0,T]$, $u : [0,T] \times D([0,T]) \longrightarrow \mathbb{R}$ is non-anticipative if

$$u(t, \mathrm{x}) = u(t, \mathrm{x}_{t\wedge\cdot}), \quad \text{for all } (t, \mathrm{x}).$$

• Dupire : The horizontable derivative of u :

$$\partial_t u(t, \mathrm{x}) := \lim_{h \searrow 0} \frac{u(t+h, \mathrm{x}_{t\wedge\cdot}) - u(t, \mathrm{x}_{t\wedge\cdot})}{h};$$

the vertical derivative of u :

$$\nabla_{\mathrm{x}} u(t, \mathrm{x}) := \lim_{y \to 0} \frac{u(t, \mathrm{x} \oplus_t y) - u(t, \mathrm{x})}{y}.$$

Functional Itô formula

• Functional Itô formula (Cont and Fournié) : Let $u : [0, T] \times D([0, T]) \longrightarrow \mathbb{R}$ belong to $C^{1,2}([0, T] \times D([0, T]))$, and X be a continuous semimartingale, then

$$
\begin{aligned}
u(t, X_\cdot) &= u(0, X_0) + \int_0^t \nabla_{\mathrm{x}} u(s, X_\cdot) dX_s \\
&\quad + \int_0^t \partial_t u(s, X_\cdot) ds + \int_0^t \frac{1}{2} \nabla_{\mathrm{x}}^2 u(s, X_\cdot) d\langle X \rangle_s .
\end{aligned}
$$

Path-dependent PDEs

- Example : let B be a Brownian motion, and

$$u(t, \mathrm{x}) := \mathbb{E}\big[g(B_\cdot)\big|B_{t\wedge} = \mathrm{x}_{t\wedge}\big].$$

Assume that $u \in C_b^{1,2}([0,T] \times D([0,T]))$, then it solves the path-dependent PDE (PPDE, heat equation) :

$$\partial_t u + \frac{1}{2}\nabla_{\mathrm{x}}^2 u = 0, \quad u(T, \cdot) = g(\cdot).$$

Path-dependent PDEs

• In practice, it is not easy to obtain smooth (path-dependent) value function $u : [0, T] \times D([0, T]) \to \mathbb{R}$, even for the above simple heat equation.

• Viscosity solution of (nonlinear) PPDE (Ekren, Keller, Peng, Ren, Tang, Touzi, Zhang, Cosso, Russo, Zhou, etc.)

$$\partial_t u + F(u, \nabla_{\mathrm{x}} u, \nabla_{\mathrm{x}}^2 u) = 0.$$

Outline

Itô calculus via regularization (Russo, Vallois, etc.)

• Let X be a càdlàg process, $H \in L^1([0,T])$, the forward integral of H w.r.t. X is defined by

$$\int_0^t H_s d^- X_s := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t H_s (X_{(s+\varepsilon)\wedge t} - X_s) ds, \quad t \geq 0.$$

• Let X and Y be two càdlàg processes, the co-quadratic variation $[X,Y]$ is defined by

$$[X,Y]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds.$$

• The limits are defined in sense of "uniformly on compacts in probability" (u.c.p.).

When X and Y are càdlàg semimartingales and H is càdlàg and adapted, they are well defined and coincide with the usual Itô integral.

Itô calculus via regularization (Russo, Vallois, etc.)

- Weak Dirichlet process :
  - A càdlàg process A is called is called orthogonal (with zero weak energy), if $[A, N] = 0$ for all continuous martingale N.
  - A càdlàg process X is called a weak Dirichlet process if it has the decomposition

$$X_t = X_0 + M_t + A_t,$$

    where M is a local martingale, A is orthogonal.
  - A weak Dirichlet process X is called a special weak Dirichlet process if it has a decomposition with a predictable and orthogonal process A.

A $C^{0,1}$-Itô formula (Russo, Vallois, etc.)

Theorem (e.g. Gozzi and Russo (2006), or Bandini and Russo (2017))

Let $f \in C^{0,1}([0,T] \times \mathbb{R}^d)$, $X = M + A$ be a continuous weak Dirichlet process, then $f(t, X_t)$ is also a (continuous) weak Dirichlet process with the decomposition

$$f(t, X_t) = f(0, X_0) + \int_0^t \nabla_x f(s, X_s) \cdot dM_s + \Gamma_t.$$

A $C^{0,1}$-Itô formula (Russo, Vallois, etc.)

- Proof : Step 1 : define

$$\Gamma_t := f(t, X_t) - f(0, X_0) - \int_0^t \nabla_x f(s, X_{s-}) \cdot dM_s.$$

Step 2 : check that, for any continuous martingale N,

$$[\Gamma, N]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (\Gamma_{s+\varepsilon} - \Gamma_s)(N_{s+\varepsilon} - N_s) ds = 0.$$

A functional $C^{0,1}$–Itô formula

Theorem (Bouchard, Loeper and Tan (2021))

Let $F \in C^{0,1}([0,T] \times D([0,T]))$ and $X = M + A$ be a weak Dirichlet process, under a technical condition, $F(t, X_{t\wedge\cdot})$ is a also a weak Dirichlet process with the decomposition

$$F(t, X_{t\wedge}) = F(0, X) + \int_0^t \nabla_x F(s, X^{s-}_{s\wedge\cdot}) \cdot dM_s + \Gamma_t.$$

Further, if X is a special weak Dirichlet process, then under some technical conditions, $F(t, X_{t\wedge\cdot})$ is also a special weak Dirichlet process.

- Extension of the Itô formula in two senses :
  - $C^{0,1}$–Itô formula of Russo, Vallois, etc.
  - functional $C^{1,2}$–Itô formula of Cont and Fournié.

An option replication problem

• In the above financial market with underlying B, which is a Brownian motion, for a path-dependent option with payoff $g(B_\cdot)$, assume that

$$u(t, \mathrm{x}_{t\wedge\cdot}) := \mathbb{E}\big[g(B_\cdot)\big|B_{t\wedge\cdot} = \mathrm{x}_{t\wedge\cdot}\big] \;\in\; C^{0,1}([0,T]\times D([0,T])),$$

then both B and $u(t, B_{t\wedge\cdot})$ are continuous martingale, and hence

$$g(B_\cdot) = u(t, B_{t\wedge\cdot}) + \int_t^T \nabla_{\mathrm{x}} u(s, B_{s\wedge\cdot}) dB_s.$$

- In this linear context, when u is Fréchet differentiable, one can also use Clark-Haussmann-Ocone formula to obtain the replication strategy.

A replication problem under market impact

• In a setting with market impact, the dynamic trading strategy H can impact the dynamic of the underlying process X and that of the portfolio V :

$$dH_t = \gamma_t dW_t + b_t dt, \qquad dX_t = \sigma(X_t, \gamma_t) dW_t + \mu(X_t, \gamma_t) dt,$$

$$dV_t = H_t dX_t + \frac{1}{2}\gamma_t^2 f(X_t) dt.$$

We study the replication problem, i.e. for a given path-dependent option $\Phi(\cdot)$, find a strategy (H, γ) so that $V_T = \Phi(X_\cdot)$.

• The same structure has been studied in the Markovian context by Bouchard, Loeper, Soner, Zhou, etc. by the PDE approach.

- B. Bouchard, X. Tan, Understanding the dual formulation for the hedging of path-dependent options with price impact, arXiv :1912.03946.

A super-replication problem

• We consider a market with uncertain volatility : let $\Omega := C([0, T])$ be the canonical space of continuous paths on $[0, T]$, and X be the canonical process. We consider a family of probability measure $(\mathcal{P}(t,\omega))_{(t,\omega)}$ given by

$$\mathcal{P}(t,\omega) := \Big\{\mathbb{P} \ : \mathbb{P}[X_{t\wedge} = \omega_{t\wedge}] = 1, \ dX_t = \sigma_t dW_t^{\mathbb{P}}, \ \sigma_t \in [\underline{\sigma}, \overline{\sigma}]\Big\}.$$

• The super-replication cost of a path-dependent option $\Phi(X_\cdot)$:

$$D_0 := \inf\Big\{x \ : x + \int_0^T H_t dX_t \geq \Phi(X_\cdot), \ \mathbb{P}\text{-a.s.}, \ \forall \mathbb{P} \in \mathcal{P}(0, X_0)\Big\}.$$

• The pricing-hedging duality (Denis-Martini) :

$$D_0 = V_0 := \sup_{\mathbb{P}\in\mathcal{P}(0,X_0)} \mathbb{E}^{\mathbb{P}}\big[\Phi(X_\cdot)\big].$$

A super-replication problem, main result

Theorem

Assume that $V(t,\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\big[\Phi(X_\cdot)\big]$ *satisfies the technical conditions for the functional* $C^{0,1}$*-Itô formula. Then* $H^*_t := \nabla_x V(t, X_{t\wedge\cdot})$ *is the optimal superhedging strategy, i.e.*

$$V_0 + \int_0^T H^*_t \, dX_t \ge \Phi(X_\cdot), \ \mathbb{P}\text{-a.s.}, \ \forall \mathbb{P} \in \mathcal{P}(0, X_0).$$

A super-replication problem, proof

• Step 1. By dynamic programming principle, one has

$$V(t, X_{t\wedge\cdot}) = \sup_{\mathbb{P}\in\mathcal{P}(t,X)} \mathbb{E}^{\mathbb{P}}\big[V(t+h, X_{t+h\wedge\cdot})\big] \geq \mathbb{E}^{\mathbb{P}}\big[V(t+h, X_{t+h\wedge\cdot})\big|\mathcal{F}_t\big].$$

Then, the process $(V(t, X_{t\wedge\cdot}))_{t\in[0,T]}$ is a supermartingale under any $\mathbb{P}$.

• Step 2. Under a fixed $\mathbb{P}$, one has the Doob-Meyer decomposition

$V(t, X_{t\wedge\cdot}) = V_0 + M_t - K_t$, for some martingale M and increasing process K.

• Step 3. Under a fixed $\mathbb{P}$, the $C^{0,1}$-Itô formula gives

$V(t, X_{t\wedge\cdot}) = V_0 + \int_0^t \nabla_x V(t, X_{t\wedge\cdot}) dX_t + A_t$, for some orthogonal process A.

• By uniqueness of the decomposition of the (continuou) weak Dirichlet process $(V(t, X_{t\wedge\cdot}))_{t\in[0,T]}$, it follows that $K = -A$.

Remarque on the applications

• More applications with supermartingale or semimartingale structure :

- American option pricing,
- Option hedging under constraints,
- BSDE.

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Approximation of the PPDE

- We consider the PPDE, on $[0,T] \times D([0,T])$,

$$\partial_t u + F(t, \mathrm{x}, u, \nabla_{\mathrm{x}} u, \nabla^2_{\mathrm{x}} u) = 0, \quad u(T,\cdot) = g(\cdot).$$

- Let $\pi_n = (t_i^n)_{i=0,\cdots,n}$ be a discrete time grid of $[0,T]$, we define

$$[\mathrm{x}]_k^n := (\mathrm{x}_{t_i^n})_{0\le i\le k}, \quad F^n(t,[\mathrm{x}]_k^n, y, z, \gamma) := F(t, \mathrm{x}, y, z, \gamma),$$

and u_k^n be the viscosity solution of (classical) PDE, on $[t_k^n, t_{k+1}^n)$,

$$\partial_t u_k^n + F^n\big(t, [\mathrm{x}]_k^n, x, u_k^n, Du_k^n, D^2 u_k^n\big) = 0,$$

with terminal condition

$$\lim_{t\nearrow t_{k+1}^n} u_k^n(t,[\mathrm{x}]_k^n, x, \cdot) = u_{k+1}^n(t_{k+1}^n, [\mathrm{x}]_k^n, x, x, \cdot).$$

Approximate viscosity solution of the PPDE

- Given $(u^n_k)_{k=0,\cdots,n-1}$, we define $u^n : [0,T] \times D([0,T]) \longrightarrow \mathbb{R}$ by

$$u^n(t,\mathrm{x}) \;:=\; u^n_k(t, [\mathrm{x}]^n_k, \mathrm{x}_t), \text{ when } t \in [t^n_k, t^n_{k+1}),$$

so that

$$\partial_t u^n(t,\mathrm{x}) = \partial_t u^n_k(t, [\mathrm{x}]^n_k, \mathrm{x}_t), \quad \nabla_{\mathrm{x}} u^n(t,\mathrm{x}) = D u^n_k(t, [\mathrm{x}]^n_k, \mathrm{x}_t).$$

- We call u is an approximate viscosity solution of the PPDE if

$$u^n \longrightarrow u \text{ pointwisely on } [0,T] \times D([0,T]).$$

We then study the (existence, uniqueness, comparison principle, stability, etc.)

Approximate viscosity solution of the PPDE

• Strong viscosity solution of Cosso and Russo (2019) :

$$\partial_t u^n + F^n(\cdot, u^n, \nabla_{\mathrm{x}} u^n, \nabla^2_{\mathrm{x}} u^n) = 0, \quad F^n \longrightarrow F, \quad u^n \longrightarrow u.$$

• Pseudo-Markovian viscosity solution of Ekren-Zhang (2016) : Approximate the PPDE by discretization of both time $[0, T]$ and space $D([0, T])$.

• Difficulty : the existence. We are able to deal with a general case

$$F(t, \mathrm{x}, y, z, \gamma) = H(t, \mathrm{x}, y, z, \gamma) + r(t, \mathrm{x})y + \mu(t, \mathrm{x}) \cdot z + \frac{1}{2}\sigma\sigma^\top(t, \mathrm{x}) : \gamma,$$

where H is only uniformly continuous in (y, z, γ).

Approximate viscosity solution to the PPDE

• A first key technical result : with technical conditions, there exists a constant C and a continuous modulus w independent of n, such that

$$\left|u^n(t',\mathrm{x}') - u^n(t,\mathrm{x})\right| \;\leq\; Cw\Big(|t'-t|^{1/2} + \rho(\mathrm{x}_{t\wedge\cdot}, \mathrm{x}'_{t'\wedge\cdot})\Big),$$

where ρ is the Skorokhod metric on $D([0,T])$.

Regularity of solution to the PPDE

• A first key technical result : Under additional technical conditions, the derivative $\nabla_{\mathrm{x}} u^n$ exists and is uniformly continuous in (t, x), uniformly in n. Consequently,

$$u \in C^{0,1}.$$