An exit contract optimization problem

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Outline

The Principal-Agent Problem

- Sannikov (2008), Cvitanic-Possamaï-Touzi (2018), etc.
- Agent's problem : given a contract $\xi : C([0,T],\mathbb{R}) \longrightarrow \mathbb{R}$,

$$\max_{\alpha} \ \mathbb{E}\Big[\int_0^T L(\alpha_t)dt + \xi(X^{\alpha})\Big], \quad \text{subject to } dX_t^{\alpha} = \alpha_t dt + dW_t.$$

- Principal's problem :

$$\max_{\xi} \ \mathbb{E}\Big[U(\hat{\alpha}, X^{\hat{\alpha}}, \xi)\Big].$$

The exit contract problem : multiple agents

- Multiple agents share a universal exit contract.

For each agent $i = 1, \cdots, n$:

$$\max_{\tau_i \in \mathcal{T}} \mathbb{E}\Big[\int_0^{\tau_i} f_i(t, X_{t\wedge\cdot})dt + Y_{\tau_i}\Big].$$

- Principal's problem :

$$\max_{Y} \mathbb{E}\Big[\sum_{i=1}^{n} \int_0^{\hat{\tau}_i} g_i(t, X_{t\wedge\cdot})dt - Y_{\hat{\tau}_i}\Big], \text{ subject to } Y_T \geq \xi,$$

for some random variable $\xi \in \mathcal{F}_T$.

- Difference : we study the case with multi-agents and use stopping instead of control to describe the action of the agents and Moral hazard disappears since the contract depends directly on the action of each agent.

The exit contract problem : multiple agents

- Motivations/Applications :
  - Layoff problem : because of the Labour Union or law constraint, the company can not choose the employees to fire, but needs to suggest a universal plan and ask the employees to take volunteer leave.
  - An exit contract can be considered as the price process of some public service (e.g. electricity price), which can be time-dependent, but needs to be universal to everyone in a group.
  - Retirement system design : people can choose to retire in some range of ages, and the retirement pension depends on the retirement age.

Outline

Bank-El Karoui's representation of stochastic processes

Theorem (Bank-El Karoui)

Let $\ell \mapsto f(t,x,\ell)$ is strictly increasing and $f(t,x,\pm\infty) = \pm\infty$. Then, for any optional process Y (u.s.c. in expectation), there exists an optional process L such that, for all $t \in [0,T]$,

$$Y_t = \mathbb{E}\Big[\int_t^T f\Big(s, X, \sup_{r\in[t,s)} L_r\Big)ds + \xi\Big|\mathcal{F}_t\Big], \text{ with } \xi := Y_T.$$

As one of the applications, for each $\ell \in \mathbb{R}$, the hitting time

$$\tau_\ell := \inf\{t \geq 0 \ : L_t \geq \ell\}$$

is (the smallest) solution to the optimal stopping problem :

$$\sup_\tau \Big[\int_0^\tau f(t,X,\ell)dt + Y_\tau\Big].$$

Bank-El Karoui's representation of stochastic processes

• For intuition, it can be seen as an extension of "convex envelope" to the stochastic case.

• Other applications of the representation in optimal consumption problems (Bank-Riedel, 2001), singular control problem (Chiarolla-Ferrari, 2014), etc. see also the review paper of (Bank-Föllmer, 2003).

• Extensions to the nonlinear case by a variant of the reflected BSDE (Ma-Wang, 2009), the case where the measure μ has atom and Meyer-σ-field (Bank-Besslich, 2018), etc.

Idea of the reduction

• Key assumption (monotonicity assumption) : for any $(t, x) \in [0, T] \times \mathbb{R}$,

$$f_1(\cdot) < f_2(\cdot) < \cdots < f_n(\cdot).$$

• By interpolation, setting $\ell \longrightarrow i$ in Bank-El Karoui's theorem, the process L provides

- the principal's contract Y by

$$Y_t = \mathbb{E}\Big[\int_t^T f\Big(t, X, \sup_{r\in[t,s)} L_r\Big) ds + \xi \Big| \mathcal{F}_t\Big], \quad t \geq 0,$$

- the agents' optimal stopping time $\hat{\tau}_i$ by

$$\hat{\tau}_i \;:=\; \inf\{t \geq 0 \;: L_t \geq i\}.$$

• The Principal's problem now becomes an optimization problem over all admissible L.

Main theorem

- Let $\mathcal{Y}$ denote the class of all optional processes in class (D), u.s.c. in expectation and s.t. $Y_T \geq \xi$, $\mathcal{L}^+$ the class of all increasing optional processes taking value in $[0, n]$.

Theorem (He-T.-Zou)

- *If the monotonicity assumption and some integrability condition hold true, one has*

$$\max_{Y \in \mathcal{Y},\ Y_T \geq \xi} \mathbb{E}\Big[\sum_{i=1}^{n} \int_0^{\hat{\tau}_i(Y)} g_i(t, X_{t\wedge\cdot})dt - Y_{\hat{\tau}_i(Y)}\Big]$$

$$= \max_{L \in \mathcal{L}^+} \mathbb{E}\Big[\sum_{i=1}^{n} \int_0^{T} \Big(g_i(t, X)1_{\{L_t < i\}} - f_i(t, X)1_{\{L_t \geq i\}}\Big)dt - \xi\Big].$$

- *The existence is obtained.*

Main theorem

Theorem (He-T.-Zou)

• *Given $Y \in \mathcal{Y}$, there exists an optional process L such that*

$$Y_t = \mathbb{E}\Big[\int_t^T f\Big(t, X, \sup_{r\in[t,s)} L_r\Big)ds + \xi\Big|\mathcal{F}_t\Big], \ a.s.$$

and Y^+ such that the corresponding $L^+ \in \mathcal{L}^+$ and it induces the same optimal stopping times $\hat{\tau}_i$ for the agents and

$$Y_{\hat{\tau}_i} = Y^+_{\hat{\tau}_i}, \ i = 1, \cdots, n.$$

• *Given an optional process L, the corresponding process Y may not be u.s.c. in expectation. When L is in $\mathcal{L}^+$ (or even in $\{0, 1, \cdots, n\}$), the process $Y \in \mathcal{Y}$.*

Outline

Convergence of the discrete-time Principal's value function

- Given a sequence of partitions $(\pi_m)_m$ of $[0, T]$,

we define corresponding X^{π_m}, $(\mathcal{F}_t^{\pi_m})$, $\mathcal{T}^{\pi_m}$, $\mathcal{Y}^{\pi_m}$, $\mathcal{L}^{\pi_m,+}$, μ^{π_m}, V^{P,π_m} and replace integration with summation w.r.t. π_m.

Theorem (Convergence result)

Under some proper assumptions on f_i and g_i, we have that if $lim_{m\to\infty}|\pi_m| = 0$, then

$$\lim_{m\to\infty} V^{P,\pi_m} = V^P.$$

Markovian and/or continuous contract in the discrete time

• Based on some partition π of $[0, T]$ and corresponding settings X^π, $(\mathcal{F}_t^\pi)$, $\mathcal{T}^\pi$, $\mathcal{Y}^\pi$, $\mathcal{L}^{\pi,+}$, μ^π, we consider two special subsets of $\mathcal{Y}^\pi$. In particular, we require in addition that for all $t \in [0, T]$, Y_t is a measurable/continuous function of X_t^π, denoted by $\mathcal{Y}_m^\pi$ / $\mathcal{Y}_c^\pi$ and consider the corresponding value functions V_m^π/ V_c^π of the Principal, corresponding subsets $\mathcal{L}_m^{\pi,+}$ $\mathcal{L}_c^{\pi,+}$ of $\mathcal{L}^{\pi,+}$.

• We further require that X^π is markovian, then there exists a family of transition probability measures $\{\mathbb{P}_x^j\}_{j,x}$, for the continuous contract, we will further assume the continuity of f_i and $x \mapsto \mathbb{P}_x^j$.

Markovian and/or continuous contract in the discrete time

Theorem

- *One has*

$$\max_{Y\in\mathcal{Y}^\pi_m,\ Y_T\geq\xi} \mathbb{E}\Big[\sum_{i=1}^n \int_0^{\hat{\tau}_i(Y)} g_i(t, X_{t\wedge\cdot})\mu^\pi(dt) - Y_{\hat{\tau}_i(Y)}\Big]$$

$$= \max_{L\in\mathcal{L}^{\pi,+}_m} \mathbb{E}\Big[\sum_{i=1}^n \int_0^T \Big(g_i(t,X)1_{\{L_t<i\}} - f_i(t,X)1_{\{L_t\geq i\}}\Big)\mu^\pi(dt) - \xi\Big].$$

$$\max_{Y\in\mathcal{Y}^\pi_c,\ Y_T\geq\xi} \mathbb{E}\Big[\sum_{i=1}^n \int_0^{\hat{\tau}_i(Y)} g_i(t, X_{t\wedge\cdot})\mu^\pi(dt) - Y_{\hat{\tau}_i(Y)}\Big]$$

$$= \max_{L\in\mathcal{L}^{\pi,+}_c} \mathbb{E}\Big[\sum_{i=1}^n \int_0^T \Big(g_i(t,X)1_{\{L_t<i\}} - f_i(t,X)1_{\{L_t\geq i\}}\Big)\mu^\pi(dt) - \xi\Big].$$

Markovian and/or continuous contract in the discrete time

• The Snell envelop approach is not applicable any more since it requires that the optimal stopping time $\hat{\tau}_i$ of agent i would satisfy $\{\hat{\tau}_i = t\}$ should be in $\sigma(X_t)$.

Further questions

- Mean field game setting, interaction between agents.
- What if we consider control and stopping at the same time.
- More concrete applications