Optimal Consumption with Loss Aversion and Reference to Past Spending Maximum

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Outline

Outline

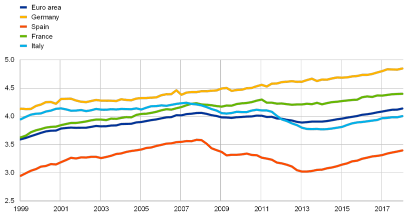
Merton Problem

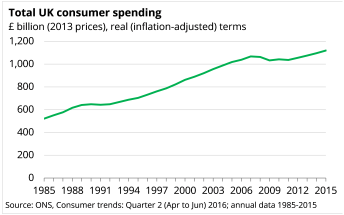
- The standard Merton problem on optimal consumption:

$$u(x) = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-rt} U(c_t)dt\right],$$

where $\mathcal{A}$ is the admissible set of portfolio-consumption strategies (π, c).

- However, some empirical and psychological studies argued that the consumer's satisfaction level and risk tolerance sometimes rely more on recent changes instead of absolute values. Moreover, some observed aggregate consumption is rather smooth.





Path-Dependent Consumption

- One partial and feasible answer: the utility function can also depend on the history of the whole consumption path.

- **Model 1**: The habit formation preference is defined by (see *Constantinides*, JPE 1990, *Detemple and Zapatero*, MF 1992)

$$u(x) = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-rt} U(c_t - Z_t)dt\right],$$

  where the accumulative process Z is called the habit formation process that takes the form $dZ_t = (\delta c_t - \alpha Z_t)dt$, $Z_0 = z \geq 0$.

- **Model 2**: Utility with the reference to the past consumption maximum: $\sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-rt} U(c_t - \lambda H_t)dt\right]$, where $H_t = \max(h, \sup_{s\leq t} c_s)$, $H_0 = h \geq 0$ and the constant $\lambda \in [0,1]$ stands for the reference intensity, see *Deng et al, FS 2021*.

Reference to Past Spending Maximum

- Some variant problems have been considered as "consumption ratcheting problem" (Dybvig, RES 1995) and "consumption with drawdown constraint" (Arun, 2012) focusing on the conventional utility maximization $\sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-rt}\frac{(c_t)^p}{p}dt\right]$ with the control constraint $c_t \geq \lambda H_t$.
- Another interesting problem with reference to past spending maximum is formulated as $\sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty \frac{(c_t/H_t^\alpha)^p}{p}dt\right]$ in *Guasoni et al, MF 2020*.
- We are also interested in **Model 2** when the utility is generated by the difference between consumption and a fraction of past spending maximum, but the investor is allowed to strategically suppress the consumption below the reference level from time to time.

Preference

- Preference:

$$u(x,h) = \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t - \lambda H_t)dt\right],$$

- Discount factor $\rho > 0$
- Past spending maximum:

$$H_t = \max\left\{h,\ \sup_{s\le t} c_s\right\},\quad H_0 = h \ge 0,\quad \text{and}\quad 0 < \lambda < 1.$$

- Non-negativity constraint on consumption: $c_t \ge 0$
- Integrability: $\int_0^T (c_t + \pi_t^2)dt < \infty$ for any $T > 0$
- Admissible: $(\pi, c) \in \mathcal{A}$ satisfies the wealth process without bankruptcy, non-negativity constraint and is integrable

Canonical Two-part Power Utility and Loss Aversion

- The canonical two-part power utility is defined by (see *Kahneman and Tversky, JRU 1992*)

$$U(x) = \begin{cases} \frac{1}{\beta_1} x^{\beta_1}, & \text{if } x \geq 0, \\ -\frac{k}{\beta_2}(-x)^{\beta_2}, & \text{if } x < 0, \end{cases}$$

  where $0 < \beta_1, \beta_2 < 1$, $k > 0$.
- Non-concave utility function: non-differentiable at $x = 0$.
- Commonly used to study loss-averse agent's behavior (for instance, *Bilsen et al. MS 2020, He and Yang MF 2019, He and Zhou MS 2011, Jin and Zhou MF 2008*)

Market Model

- One riskless asset $(B_t)_{t \geq 0}$: $\quad dB_t = rB_t dt$
  - $r > 0$: risk-free rate
- One risky asset $(S_t)_{t \geq 0}$: $\quad dS_t = S_t \mu dt + S_t \sigma dW_t$
  - $\mu > r$ is the expected return rate, $\sigma > 0$ is the volatility
  - W: one-dimensional Brownian motion
- Consumption rate c_t
- Investment amount π_t
- Wealth process (state system):

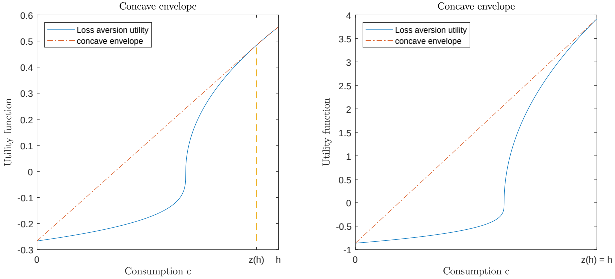
$$dX_t = rX_t dt + \pi_t(\mu - r)dt + \pi_t \sigma dW_t - c_t dt, \quad X_0 = x, \quad t \geq 0.$$

- No bankruptcy: $X_t > 0$ all the time

Concave Envelope

- The concave envelope $\tilde{f}$ of f is defined by the minimum concave function that is larger than f on the same domain everywhere.
- Concave envelope $\tilde{U}(c,h)$ of $U(c-\lambda h)$ for any fixed h:

$$\tilde{U}(c,h) = \begin{cases} U(-\lambda h) + \frac{U(z(h)-\lambda h)-U(-\lambda h)}{z(h)} c, & \text{if } 0 \le c < z(h), \\ U(c-\lambda h), & \text{if } z(h) \le c \le h. \end{cases}$$



Concave Envelope

Equivalent Problem

- Equivalent preference based on concave envelope:

$$\tilde{u}(x,h) = \sup_{(\pi,c)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\tilde{U}(c_t, H_t)dt\right].$$

- $\tilde{U}(c,h)$: the concave envelope of $U(c-\lambda h)$ in $c \in [0,h]$ for fixed h

Lemma

The equivalent problem has the same value function $\tilde{u}(x,h) = u(x,h)$ *with the original problem for any* $(x,h) \in \mathbb{R}^2_+$. *Moreover, the two problems have the same optimal consumption and portfolio choices.*

Outline

The HJB equation

- Special case $\rho = r$
- The HJB variational inequality

$$\sup_{c\in[0,h],\pi\in\mathbb{R}} \left[-r\tilde{u} + \tilde{u}_x(rx + \pi(\mu - r) - c) + \frac{1}{2}\sigma^2\pi^2\tilde{u}_{xx} + \tilde{U}(c,h) \right] = 0,$$
$$\tilde{u}_h(x,h) \leq 0,$$

for $x \geq 0$ and $h \geq 0$ and $\tilde{u}_h(x,h) = 0$ on some set to be determined by martingale optimality condition.

- If $u(x,\cdot)$ is C^2 in x, the first order condition in π gives $\pi^*(x,h) = -\frac{\mu-r}{\sigma^2}\frac{\tilde{u}_x}{\tilde{u}_{xx}}$. The HJB variational inequality can be written as

$$\sup_{c\in[0,h]} \left[\tilde{U}(c,h) - c\tilde{u}_x \right] - r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2}{2}\frac{\tilde{u}_x^2}{\tilde{u}_{xx}} = 0,$$

and $\tilde{u}_h \leq 0, \quad \forall x \geq 0, h \geq 0.$

Auxiliary curves and consumption

- Three curves

$$
\begin{aligned}
y_1(h) &:= \frac{k(\lambda h)^{\beta_2}}{\beta_2 z(h)} + \frac{w(h)^{\beta_1}}{\beta_1 z(h)}, \\
y_2(h) &:= \min\left(y_1(h), \left((1-\lambda)h\right)^{\beta_1 - 1}\right), \\
y_3(h) &:= (1-\lambda)^{\beta_1} h^{\beta_1 - 1},
\end{aligned}
$$

where $w(h) := z(h) - \lambda h \in (0, (1-\lambda)h]$.

- Auxiliary consumption

$$
\hat{c}(x,h) = \arg\max_{c}[\tilde{U}(c,h) - c\tilde{u}_x] \begin{cases} < 0, & \text{if } \tilde{u}_x > y_1(h), \\ = \lambda h + \tilde{u}_x^{\frac{1}{\beta_1 - 1}}, & \text{if } y_2(h) \le \tilde{u}_x \le y_1(h), \\ > h, & \text{if } \tilde{u}_x < y_2(h). \end{cases}
$$

Separated Regions

- *Region I*: $\mathcal{R}_1 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) > y_1(h)\}$
  - $\hat{c}(x,h) < 0$, optimal consumption $c^*(x,h) = 0$
  - HJB variational inequalities:

$$-\frac{k}{\beta_2}(\lambda h)^{\beta_2} - r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2 \tilde{u}_x^2}{2\tilde{u}_{xx}} = 0, \text{ and } \quad u_h \leq 0.$$

- *Region II*: $\mathcal{R}_2 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : y_2(h) \leq \tilde{u}_x(x,h) \leq y_1(h)\}$
  - $\lambda h < \hat{c}(x,h) \leq h$, optimal consumption $c^* = \lambda h + \tilde{u}_x^{\frac{1}{\beta_1 - 1}}$
  - HJB variational inequalities:

$$\frac{1-\beta_1}{\beta_1}\tilde{u}_x^{\frac{\beta_1}{\beta_1-1}} - \lambda h \tilde{u}_x - r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2 \tilde{u}_x^2}{2\tilde{u}_{xx}} = 0, \text{ and } \tilde{u}_h \leq 0.$$

Separated Regions

- *Region III*: $\mathcal{R}_3 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) < y_2(h)\}$
- $\hat{c}(x,h) > h$, optimal consumption $c^*(x,h) = h$
- The HJB variational inequalities:

$$\frac{1}{\beta_1}((1-\lambda)h)^{\beta_1} - h\tilde{u}_x - r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2\tilde{u}_x^2}{2\tilde{u}_{xx}} = 0, \text{ and } \ u_h \leq 0.$$

- c_t^* coincides with the running maximum process H_t^*
- Question: will $c^*(x,h) = h$ really be useful?

Separated Regions

- Substitute $h = c$ into HJB inequalities with auxiliary control $\hat{c} := \tilde{u}_x^{\frac{1}{\beta_1 - 1}}(1-\lambda)^{-\frac{\beta_1}{\beta_1 - 1}}$
- $\mathcal{D}_1 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : y_3(h) < \tilde{u}_x(x,h) \le y_2(h)\}$
  - $\hat{c}(x) < h$ and $c^*(x,h) = h$ does not update past spending maximum
- $\mathcal{D}_2 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) = y_3(h)\}$
  - $\hat{c} = h$ and $c^*(x,h) = \hat{c} = h$ attains/creates the (new) peak $H_t^* = c_t^*$
  - Free boundary condition $\tilde{u}_h(x,h) = 0$
- $\mathcal{D}_3 := \{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) < y_3(h)\}$
  - $\hat{c} > h$ and $c^*(x,h) = \hat{c} > h$ creates a new peak $H_t^* = c_t^* > H_{t-}^*$
  - $(X_t, H_{t-}^*) \in \mathcal{D}_3$ and $(X_t, H_t^*) \in \mathcal{D}_2$

Effective Domain

- Effective domain

$$\mathcal{C} := \left\{(x,h) \in \mathbb{R}_+ \times \mathbb{R}_+ : \tilde{u}_x(x,h) \geq y_3(h)\right\},$$

where $\mathcal{C} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \subset \mathbb{R}_+^2$.

- The only possibility for $(X_t^*, H_t^*) \in \mathcal{D}_3$: initial time $t = 0$, and $t = 0$ is the only possible jump time of H_t^*.

Boundary Conditions

- Smooth-fit conditions
- Boundary conditions when x approaches to 0
  - Optimal portfolio $\pi^*(x,h) \to 0$
  - Optimal consumption $c_t^*(x,h) \to 0$ for all $t > 0$

$$\lim_{x\to 0} \frac{\tilde{u}_x(x,h)}{\tilde{u}_{xx}(x,h)} = 0, \quad \lim_{x\to 0} \tilde{u}(x,h) = -\frac{k}{r\beta_2}(\lambda h)^{\beta_2}.$$

- Boundary conditions when x approaches to infinity
  - Value function tends to be infinity
  - Negligible effect on value function for small fluctuation of wealth
  - Existence of the limit ratio for consumption to wealth

$$\lim_{x\to+\infty} \tilde{u}(x,h) = +\infty, \quad \lim_{x\to+\infty} \tilde{u}_x(x,h) = 0,$$

$$\lim_{\substack{x\to+\infty \\ (x,h)\in\mathcal{D}_2}} \frac{\tilde{u}_x(x,h)^{\frac{1}{\beta_1-1}}}{x} = c_\infty,$$ where $c_\infty > 0$ is a positive constant.

Solving the HJB Equation

- Recall the HJB equation

$$-r\tilde{u} + rx\tilde{u}_x - \frac{\kappa^2}{2}\frac{\tilde{u}_x^2}{\tilde{u}_{xx}} + V(\tilde{u}_x, h) = 0,$$

where

$$\begin{aligned} V(q,h) &:= \sup_{c\in[0,h]} (\tilde{U}(c,h) - cq) \\ &= \begin{cases} -\frac{k}{\beta_2}(\lambda h)^{\beta_2}, & \text{if } q > y_1(h), \\ -\frac{\beta_1 - 1}{\beta_1} q^{\frac{\beta_1}{\beta_1 - 1}} - \lambda h q, & \text{if } y_2(h) \le q \le y_1(h), \\ \frac{1}{\beta_1}((1-\lambda)h)^{\beta_1} - hq, & \text{if } y_3(h) \le q < y_2(h). \end{cases} \end{aligned}$$

- Question: How to tackle the nonlinearity of the PDE?

Dual Transform

- Dual transform $v(y,h)$ for $\tilde{u}(x,h)$ with fixed h:

$$v(y,h) := \sup_{\substack{(\tilde{x},h)\in\mathcal{C} \\ \tilde{x}\geq 0}} [\tilde{u}(\tilde{x},h) - \tilde{x}y], \quad y \geq y_3(h).$$

- $x = \arg\max_{\substack{(\tilde{x},h)\in\mathcal{C} \\ \tilde{x}\geq 0}} [\tilde{u}(\tilde{x},h) - \tilde{x}y], \quad y \geq y_3(h)$
- Bijection and some properties:
  - $y := \tilde{u}_x(x,h)$
  - $x = -v_y(y,h)$
  - $\tilde{u}(x,h) = v(y,h) + yv_y(y,h)$
  - $\tilde{u}_{xx}(x,h) = -\frac{1}{v_{yy}(y,h)}$
  - $\tilde{u}_h(x,h) = v_h(y,h)$

Dual HJB Equation with boundary conditions

- Dual HJB equation:

$$\frac{\kappa^2}{2} y^2 v_{yy}(y,h) - rv(y,h) + V(y,h) = 0.$$

- Boundary conditions as $y \to 0$

$$\lim_{y\to 0} v_y(y,h) = -\infty, \ \ \lim_{y\to 0}(v(y,h) - yv_y(y,h)) = +\infty,$$

$$\lim_{y\to 0} \frac{y^{\frac{1}{\beta_1 - 1}}}{v_y(y,h)} = -c_\infty.$$

- Boundary conditions as $v_y(y,h) \to 0$

$$yv_{yy}(y,h) \to 0, \ \ v(y,h) - yv_y(y,h) \to -\frac{k}{r\beta_2}(\lambda h)^{\beta_2}.$$

- Free boundary condition $v_h(y_3(h), h) = 0$

Solution to the Dual HJB equation

- Semi-analytically solution to the dual HJB equation:

$$
v(y,h) = \begin{cases} C_2(h)y^{r_2} - \frac{k}{r\beta_2}(\lambda h)^{\beta_2}, & \text{if } y > y_1(h), \\ C_3(h)y^{r_1} + C_4(h)y^{r_2} & \\ \quad + \dfrac{2}{\kappa^2\gamma_1(\gamma_1 - r_1)(\gamma_1 - r_2)} y^{\gamma_1} - \dfrac{\lambda h}{r} y, & \text{if } y_2(h) \le y \le y_1(h), \\ C_5(h)y^{r_1} + C_6(h)y^{r_2} & \\ \quad + \dfrac{1}{r\beta_1}((1-\lambda)h)^{\beta_1} - \dfrac{h}{r} y, & \text{if } y_3(h) \le y < y_2(h), \end{cases}
$$

where $\gamma_1 = \frac{\beta_1}{\beta_1 - 1}$.

- $C_2(h), C_3(h), C_4(h), C_5(h), C_6(h)$ will be introduced in the next page.

Solution to the Dual HJB equation

- The coefficients $C_2(h), C_3(h), C_4(h), C_5(h), C_6(h)$ are defined by

$$
\begin{aligned}
C_2(h) &= C_4(h) + \frac{y_1(h)^{-r_2}}{r(r_1-r_2)}\left(\frac{kr_1}{\beta_2}(\lambda h)^{\beta_2} + \frac{r_1 r_2}{\gamma_1(\gamma_1-r_2)}y_1(h)^{\gamma_1} + \lambda h r_2 y_1(h)\right),\\
C_3(h) &= \frac{y_1(h)^{-r_1}}{r(r_1-r_2)}\left(\frac{kr_2}{\beta_2}(\lambda h)^{\beta_2} + \frac{r_1 r_2}{\gamma_1(\gamma_1-r_1)}y_1(h)^{\gamma_1} + \lambda h r_1 y_1(h)\right),\\
C_4(h) &= C_6(h) + \frac{y_2(h)^{-r_2}}{r(r_1-r_2)}\left(\frac{r_1}{\beta_1}((1-\lambda)h)^{\beta_1} - \frac{r_1 r_2}{\gamma_1(\gamma_1-r_2)}y_2(h)^{\gamma_1} + (1-\lambda)h r_2 y_2(h)\right),\\
C_5(h) &= C_3(h) + \frac{y_2(h)^{-r_1}}{r(r_1-r_2)}\left(\frac{r_2}{\beta_1}((1-\lambda)h)^{\beta_1} - \frac{r_1 r_2}{\gamma_1(\gamma_1-r_1)}y_2(h)^{\gamma_1} + (1-\lambda)h r_1 y_2(h)\right),\\
C_6(h) &= \int_h^{+\infty}(1-\lambda)^{(r_1-r_2)\beta_1}C_5'(s)s^{(r_1-r_2)(\beta_1-1)}ds,
\end{aligned}
$$

where $r_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1+\frac{8r}{\kappa^2}}\right)$.

Inverse Lengendre Transform

Lemma

In all regions, $v_{yy}(y,h) > 0$, $\forall h \geq 0$*. Moreover, the inverse Lengendre transform* $\tilde{u}(x,h) = \inf_{y \geq y_3(h)}[v(y,h) + xy]$ *is well defined.*

- $f(\cdot,h) := \tilde{u}_x(\cdot,h)$ with form $f_1(\cdot,h), f_2(\cdot,h)$ or $f_3(\cdot,h)$:
  (i) If $f_1(x,h) > y_1(h)$, $f_1(x,h)$ can be determined by
$$x = -C_2(h)r_2(f_1(x,h))^{r_2-1}.$$
  (ii) If $y_2(h) \leq f_2(x,h) \leq y_1(h)$, $f_2(x,h)$ can be uniquely determined by
$$\begin{aligned} x = & -C_3(h)r_1(f_2(x,h))^{r_1-1} - C_4(h)r_2(f_2(x,h))^{r_2-1} \\ & - \frac{2}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)}(f_2(x,h))^{\gamma_1 - 1} + \frac{\lambda h}{r}. \end{aligned}$$
  (iii) If $y_3(h) \leq f_3(x,h) < y_2(h)$, $f_3(x,h)$ can be uniquely determined by
$$x = -C_5(h)r_1(f_3(x,h))^{r_1-1} - C_6(h)r_2(f_3(x,h))^{r_2-1} + \frac{h}{r}.$$

Separated Regions through Boundary Curves

- Three boundary curves: $x_{\mathrm{zero}}(h) \le x_{\mathrm{aggr}}(h) < x_{\mathrm{lavs}}(h)$
- $\mathcal{R}_1 = \{(x,h) \in \mathbb{R}^2_+ : x < x_{\mathrm{zero}}(h)\}$

$$x_{\mathrm{zero}}(h) := -y_1(h)^{r_2-1} C_2(h) r_2.$$

- $\mathcal{R}_2 = \{(x,h) \in \mathbb{R}^2_+ : x_{\mathrm{zero}}(h) \le x \le x_{\mathrm{aggr}}(h)\}$

$$\begin{aligned} x_{\mathrm{aggr}}(h) :=& - C_3(h) r_1 y_2(h)^{r_1-1} - C_4(h) r_2 y_2(h)^{r_2-1} \\ & - \frac{2}{\kappa^2(\gamma_1 - r_1)(\gamma_1 - r_2)} y_2(h)^{\gamma_1 - 1} + \frac{\lambda h}{r}. \end{aligned}$$

- $\mathcal{D}_1 \cup \mathcal{D}_2 = \{(x,h) \in \mathbb{R}^2_+ : x_{\mathrm{aggr}}(h) < x \le x_{\mathrm{lavs}}(h)\}$

$$x_{\mathrm{lavs}}(h) := -C_5(h) r_1 y_3(h)^{r_1-1} - C_6(h) r_2 y_3(h)^{r_2-1} + \frac{h}{r}.$$

Verification Theorem

- For $(x, h) \in \mathcal{C}$, value function

$$\tilde{u}(x,h) = \begin{cases} C_2(h)(f(x,h))^{r_2} - \frac{k}{r\beta_2}(\lambda h)^{\beta_2} + xf(x,h), & \text{if } x < x_{\text{zero}}(h), \\ C_3(h)(f(x,h))^{r_1} + C_4(h)(f(x,h))^{r_2} - \dfrac{\lambda h}{r} f(x,h) \\ + \dfrac{2(f(x,h))^{\gamma_1}}{\kappa^2 \gamma_1(\gamma_1 - r_1)(\gamma_1 - r_2)} + xf(x,h), & \text{if } x_{\text{zero}}(h) \le x \le x_{\text{aggr}}(h), \\ C_5(h)(f(x,h))^{r_1} + C_6(h)(f(x,h))^{r_2} \\ + \dfrac{1}{r\beta_1}((1-\lambda)h)^{\beta_1} - \dfrac{h}{r} f(x,h) + xf(x,h), & \text{if } x_{\text{aggr}}(h) < x \le x_{\text{lavs}}(h). \end{cases}$$

- The optimal consumption

$$c^*(x,h) = \begin{cases} 0, & \text{if } x < x_{\text{zero}}(h), \\ \lambda h + (f(x,h))^{\frac{1}{\beta_1 - 1}}, & \text{if } x_{\text{zero}}(h) \le x \le x_{\text{aggr}}(h), \\ h, & \text{if } x_{\text{aggr}}(h) < x < x_{\text{lavs}}(h), \\ (1-\lambda)^{-\frac{\beta_1}{\beta_1 - 1}} f(x, \tilde{h}(x))^{-\frac{1}{\beta_1 - 1}}, & \text{if } x = x_{\text{lavs}}(h), \end{cases}$$

where $\tilde{h}(x) := (x_{\text{lavs}})^{-1}(x)$.

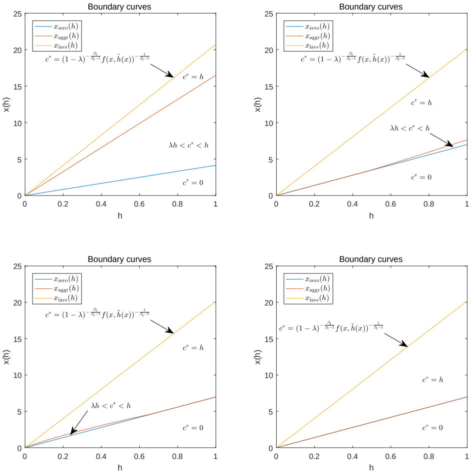
Verification Theorem

- The optimal portfolio

$$
\begin{aligned}
&\pi^*(x,h)\\
&=\frac{\mu-r}{\sigma^2}\begin{cases}
(1-r_2)x, & \text{if } x < x_{\mathrm{zero}}(h),\\
\begin{aligned}&\frac{2r}{\kappa^2}C_3(h)f^{r_1-1}(x,h)+\frac{2r}{\kappa^2}C_4(h)f^{r_2-1}(x,h)\\ &+\frac{2(\gamma_1-1)}{\kappa^2(\gamma_1-r_1)(\gamma_1-r_2)}f^{\gamma_1-1}(x,h),\end{aligned} & \text{if } x_{\mathrm{zero}}(h)\le x\le x_{\mathrm{aggr}}(h),\\
\tfrac{2r}{\kappa^2}C_5(h)f^{r_1-1}(x,h)+\tfrac{2r}{\kappa^2}C_6(h)f^{r_2-1}(x,h), & \text{if } x_{\mathrm{aggr}}(h)< x\le x_{\mathrm{lavs}}(h).
\end{cases}
\end{aligned}
$$

Outline

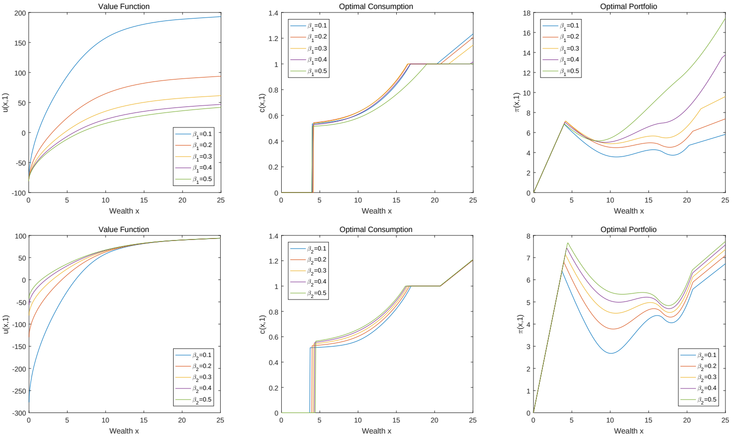
Boundary Curves: Four Cases



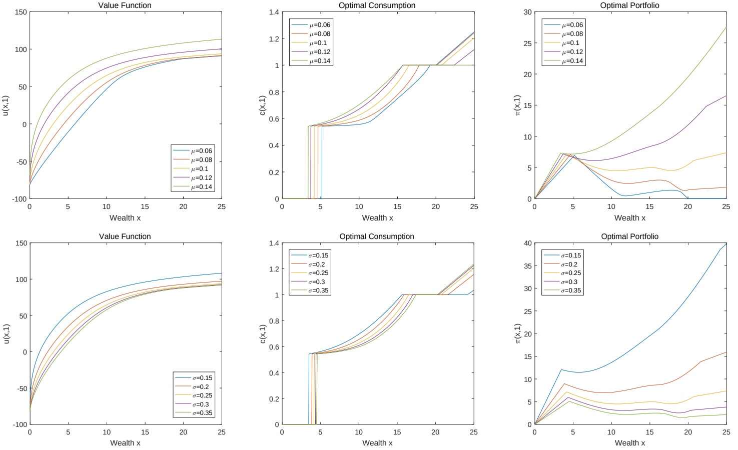
Value Function and Optimal Controls

- Basic setting
  - Market: $\mu = 0.1$, $\sigma = 0.25$, $r = 0.05$
  - Preference: $\beta_1 = 0.3$, $\beta_2 = 0.2$, $k = 1.5$, $\rho = 0.05$
  - Historical peak: $h = 1$
- Sensitivity analysis
  - $\beta_1 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$
  - $\beta_2 \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$
  - $\mu \in \{0.06, 0.08, 0.1, 0.12, 0.14\}$
  - $\sigma \in \{0.15, 0.2, 0.25, 0.3, 0.35\}$

Value function and Optimal Controls



Value function and Optimal Controls



Outline

Conclusion

- Optimal consumption and investment problem: loss aversion with reference to past spending maximum
- Dynamic programming and associated HJB equation
- Linearising PDE by dual transform
- Nonlinear structure of boundary curves
- x_{zero} and x_{aggr} may coincide in some regions
- Loss-aversion agent has a jump in the optimal consumption
- No investment in risky-asset if its expected rate is closed to risk-free rate

Future Work

- Incomplete market models: stochastic factors/ regime switching/ jump diffusion models
- Various economic/financial/insurance models: (optimal retirement; demand function; tax evasion; "catch up with peers": N agents and MFG; ......)

Thank you!

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