The Law of Large Numbers under Sublinear Expectations

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Based on a joint work with Shige Peng.

Outline

- Basic notions of sublinear expectations
- Some preliminaries
- Main results
- Sketch of proofs

Sublinear Expectations

- $(\Omega, \mathcal{H}, \hat{\mathrm{E}})$ is called a sublinear expectation space if $\hat{\mathrm{E}} : \mathcal{H} \to \mathbb{R}$ satisfies
  1) $\hat{\mathrm{E}}[\xi] \geq \hat{\mathrm{E}}[\eta]$ for $\xi \geq \eta$;
  2) $\hat{\mathrm{E}}[c] = c$;
  3) $\hat{\mathrm{E}}[\lambda\xi] = \lambda\hat{\mathrm{E}}[\xi]$, $\lambda \geq 0$;
  4) $\hat{\mathrm{E}}[\xi + \eta] \leq \hat{\mathrm{E}}[\xi] + \hat{\mathrm{E}}[\eta]$.
- For $X \in \mathcal{H}$, $\mathcal{N}_X : C_{b,Lip}(\mathbb{R}) \to \mathbb{R}$ defined below is called the distribution of X:

$$\mathcal{N}_X[\phi] \overset{def}{=} \hat{\mathrm{E}}[\phi(X)].$$

- We say Y is independent of X if

$$\hat{\mathrm{E}}[\varphi(X, Y)] = \hat{\mathrm{E}}[\hat{\mathrm{E}}[\varphi(x, Y)]|_{x=X}].$$

The Law of Large Numbers under Sublinear Expectations

Theorem 1 (Peng (2007))

Let $(\xi_k)_{k\geq 1}$ be a sequence of i.i.d random variables under a sublinear expectation $\mathbb{E}$ with the assumption $\mathbb{E}[|\xi_1|^{1+\beta}] < \infty$ for some $\beta > 0$. Set $\overline{\xi}_n = \frac{\xi_1+\cdots+\xi_n}{n}$, $\underline{\mu} = -\mathbb{E}[-\xi_1]$, $\overline{\mu} = \mathbb{E}[\xi_1]$. Then we have

$$\lim_n \mathbb{E}[\phi(\overline{\xi}_n)] = \sup_{y\in[\underline{\mu},\overline{\mu}]} \phi(y), \tag{1}$$

for any $\phi \in C_{b,Lip}(\mathbb{R})$, the collection of bounded and Lipschitz continuous functions on $\mathbb{R}$.

This is called the (weak) law of large numbers under sublinear expectations (wLLN*).

Convergence Rate of wLLN*

S. (2021) gives a convergence rate of wLLN* by Stein's method under sublinear expectations.

Theorem 2

[1] *Under the same conditions with Theorem 1, we have*

$$\sup_{|\phi|_{Lip}\leq 1}\left|\mathbb{E}[\phi(\overline{\xi}_n)] - \sup_{y\in[\underline{\mu},\overline{\mu}]}\phi(y)\right| \leq Cn^{-\frac{\beta}{1+\beta}}, \tag{2}$$

where C is a constant depending only on $\mathbb{E}[|\xi_1|^{1+\beta}]$.

[1] Hu, Li, Li (2021) gives a different proof.

Some Remarks on wLLN*

Let $\mathbb{E}[\cdot] = \sup_{P\in\Theta} E_P[\cdot]$.

- $\nexists\ \xi$ such that $\mathbb{E}[|\overline{\xi}_n - \xi|] \to 0$.
  Otherwise, ξ is independent of $\xi_1, \cdots, \xi_n$, $n \in \mathbb{N}$, and ξ is maximally distributed.
  Then $\mathbb{E}\big[\max_{y\in[\underline{\mu},\overline{\mu}]} |\overline{\xi}_n - y|\big] \geq \frac{1}{2}(\overline{\mu} - \underline{\mu})$.
- For any $P \in \Theta$, $\underline{\mu} \leq \liminf_{n\to\infty} \overline{\xi}_n \leq \limsup_{n\to\infty} \overline{\xi}_n \leq \overline{\mu}$, P-a.s.
  Chen (2016), Chen et al (2019)
- The collection of the cluster points of empirical averages $\overline{\xi}_n$ coincides with the interval $[\underline{\mu}, \overline{\mu}]$. (not always true)

A Counterexample

Example 3 (Terán (2018))

Let $\Omega = \mathbb{N}$, the set of positive integers. For $\omega \in \Omega$, define $\xi(\omega) := (\xi_1(\omega), \xi_2(\omega), \cdots) \in \{0,1\}^{\mathbb{N}}$ satisfying $\omega = \Sigma_{n=1}^{\infty} 2^{n-1}\xi_n(\omega)$. Let $\Theta = \{\delta_\omega \mid \omega \in \Omega\}$ and set $\mathbb{E}[\phi(\xi_1, \cdots, \xi_m)] = \sup\limits_{P \in \Theta} E_P[\phi(\xi_1, \cdots, \xi_m)]$.

Then (ξ_n) is a sequence of i.i.d random variables under $\mathbb{E}$ with $\mathbb{E}[\xi_1] = 1$, $-\mathbb{E}[-\xi_1] = 0$. But for each ω,

$$\frac{\xi_1(\omega) + \cdots + \xi_n(\omega)}{n} \to 0$$

since $\xi_n(\omega) = 0$ except finite n.

Main Results

Let Ω be a Polish space. For $\Theta \subset \mathcal{M}_1(\Omega)$ which is weakly compact, the associated sublinear expectation is defined by

$$\mathbb{E}[\xi] = \sup_{P \in \Theta} E_P[\xi], \ \xi \in C_b(\Omega).$$

Theorem 4

Let $\{\xi_i\} \subset L_{\mathbb{E}}^{1+\beta}(\Omega)$, $\beta > 0$, be a sequence of independent and identically distributed random variables under $\mathbb{E}$. Set $\underline{\mu} = -\mathbb{E}[-\xi_1]$, $\overline{\mu} = \mathbb{E}[\xi_1]$ and $\overline{\xi}_n = \frac{\xi_1 + \cdots + \xi_n}{n}$. Then, for any $\mu \in [\underline{\mu}, \overline{\mu}]$, there exists $P_\mu \in \Theta$ such that,

$$\overline{\xi}_n \to \mu, \ \ P_\mu\text{-a.s.}$$

as n goes to $+\infty$.

Main Results

Let $\Omega = \mathbb{R}^{\mathbb{N}}$ endowed with the metric $d(x, y) := \sum_{k=1}^{\infty} \frac{1}{2^n}(|x(k) - y(k)| \wedge 1)$, for $x, y \in \Omega$. For $n \in \mathbb{N}$, set $\xi_n(\omega) = \omega(n)$, $\omega \in \Omega$. Assume that (ξ_n) is i.i.d under a regular sublinear expectation $\mathbb{E} = \sup_{P \in \Theta} E_P$ with Θ weakly compact.

Theorem 5

Assume Θ is convex. For any $\mathcal{F}_d$-measurable random variable Π with values in $[\underline{\mu}, \overline{\mu}]$ and $d \in \mathbb{N}$, and any $P \in \Theta$, there exists a probability $P^{\Pi} \in \Theta$ such that $P^{\Pi} = P$ on $\mathcal{F}_d$, and

$$\lim_{n \to \infty} \overline{\xi}_n = \Pi, \quad P^{\Pi}\text{-a.s.} \tag{3}$$

Furthermore, if $P_{1,d} \in \text{ext}\, \Xi_d$, P^{Π} can also be chosen from ext Θ. Here $P_{1,d} = P \circ (\xi_1, \cdots, \xi_d)^{-1}$ and $\Xi_d = \{\tilde{P}_{1,d} \mid P \in \Theta\}$.

Main Results

Corollary 6

Let Θ_0 be a weakly compact subset of $\mathcal{M}_1(\Omega)$ such that

$$\mathbb{E}[X] = \sup_{P \in \Theta_0} E_P[X] \text{ for } X \in C_b(\Omega).$$

Then, for any $\mathcal{F}_d$-measurable random variable Π with values in $[\underline{\mu}, \overline{\mu}]$ and $d \in \mathbb{N}$, there exists a probability $P^\Pi \in \Theta_0$ such that

$$\lim_{n\to\infty} \bar{\xi}_n = \Pi, \;\; P^\Pi\text{-a.s.}$$

Proof.

Since Θ_0 represents $\mathbb{E}$, it follows from Hahn-Banach theorem that $\overline{\text{co}}(\Theta_0) = \Theta$. By Krein-Milman Theorem, we have ext $\Theta \subset \Theta_0$. For any $\mu \in$ ext Ξ_d, it follows from Theorem 5 that there exists $P^\Pi \in$ ext $\Theta \subset \Theta_0$ such that $P^\Pi_{1,d} = \mu$ and

$$\lim_{n\to\infty} \bar{\xi}_n = \Pi, \;\; P^\Pi\text{-a.s.}$$

□

Triviality of Tail σ-algebra

• As is known, the tail σ-algebra of an i.i.d sequence of random variables is trivial.
• The last result shows that it is generally not true under a probability $P \in \text{ext}\ \Theta$ for the nonlinear case.
• But, we have the following result.
Let θ_n be the n-shift on $\Omega = \mathbb{R}^{\mathbb{N}}$. A probability P on Ω is called stationary if $P = P \circ \theta_n^{-1}$ for some $n \in \mathbb{N}$. We denote Θ^s the subset of *ext* Θ, the probabilities in which is stationary.

Theorem 7

1) *For any $P \in \Theta^s$ and $A \in \mathcal{T}$, we have*

$$\textit{either } P(A) = 0, \ \textit{ or } P(A) = 1;$$

2) *For any $P \in \Theta^s$, $\lim_n \overline{\xi}_n$ exists and equals to some constant m_P; Moreover, $\{m_P\}_{P\in\Theta^s}$ is dense in $\left[\underline{\mu},\ \overline{\mu}\right]$.*
3) *Θ^s is a set that represents $\mathbb{E}$:*

$$\sup_{P\in\Theta^s} E_P[X] = \mathbb{E}[X], \ \textit{for } X \in C_b(\Omega).$$

Proof to Theorem 4

Case 1. $\mu = \overline{\mu}$, $\underline{\mu}$
For any $n \in \mathbb{N}$, there exists $P_{\overline{\mu},n} \in \Theta$ such that
$E_{P_{\overline{\mu},n}}[\xi_1 + \cdots + \xi_n] = \mathbb{E}[\xi_1 + \cdots + \xi_n] = n\overline{\mu}$, which implies that

$$E_{P_{\overline{\mu},n}}[\xi_k] = \overline{\mu},\ \ k \leq n,$$

noting that $E_{P_{\overline{\mu},n}}[\xi_k] \leq \overline{\mu}$.
Since Θ is weakly compact, there is $P_{\overline{\mu}} \in \Theta$ such that for any $n \in \mathbb{N}$,

$$E_{P_{\overline{\mu}}}[\xi_n] = \lim_{k\to\infty} E_{P_{\overline{\mu},n_k}}[\xi_n] = \overline{\mu}.^{[2]}$$

[2] ①$P_{\overline{\mu}} \in \Theta$; ②$E_{P_{\overline{\mu}}}[\xi_n] = \overline{\mu}$.

Proof to Theorem 4

Note that

$$\begin{aligned} P_{\overline{\mu}}[\bar{\xi}_n - \overline{\mu} > \varepsilon] &\leq \frac{1}{\varepsilon} E_{P_{\overline{\mu}}}[(\bar{\xi}_n - \overline{\mu})^+] \leq \frac{1}{\varepsilon} \mathbb{E}[(\bar{\xi}_n - \overline{\mu})^+], \\ P_{\overline{\mu}}[\bar{\xi}_n - \overline{\mu} < -\varepsilon] &\leq \frac{1}{\varepsilon} E_{P_{\overline{\mu}}}[(\bar{\xi}_n - \overline{\mu})^-] = \frac{1}{\varepsilon} E_{P_{\overline{\mu}}}[(\bar{\xi}_n - \overline{\mu})^+] \leq \frac{1}{\varepsilon} \mathbb{E}[(\bar{\xi}_n - \overline{\mu})^+]. \end{aligned}$$

Therefore, we have

$$P_{\overline{\mu}}[|\bar{\xi}_n - \overline{\mu}| > \varepsilon] \leq \frac{2}{\varepsilon} \mathbb{E}[(\bar{\xi}_n - \overline{\mu})^+],$$

which converges to 0 as n goes to infinity by Peng's wLLN*.

Proof to Theorem 4

Case 2. $\mu \in (\underline{\mu}, \overline{\mu})$
Set

$$\kappa_1(\mu) = \begin{cases} \overline{\mu} & \text{if } \mu \geq \frac{\overline{\mu}-\underline{\mu}}{2}, \\ \underline{\mu} & \text{if } \mu < \frac{\overline{\mu}-\underline{\mu}}{2}. \end{cases}$$

$$\mu^n = (\kappa_1(\mu) + \cdots + \kappa_n(\mu))/n,$$

$$\kappa_{n+1}(\mu) = \begin{cases} \overline{\mu} & \text{if } \mu \geq \mu^n, \\ \underline{\mu} & \text{if } \mu < \mu^n. \end{cases}$$

Then

$$|\mu^n - \mu| = O(1/n).$$

Proof to Theorem 4

We can find $P_\mu \in \Theta$ such that $E_{P_\mu}[\xi_n] = \kappa_n(\mu)$. [3] Then, we have

$$\begin{aligned}
& P_\mu[|\bar{\xi}_n - \mu_n| > \varepsilon] \\
= \quad & P_\mu[|\frac{k_n}{n}(\frac{\sum_{i\in N_+(n)}\xi_i}{k_n} - \overline{\mu}) + \frac{l_n}{n}(\frac{\sum_{i\in N_-(n)}\xi_i}{l_n} - \underline{\mu})| > \varepsilon] \\
\leq \quad & P_\mu[|\frac{\sum_{i\in N_+(n)}\xi_i}{k_n} - \overline{\mu}| > \varepsilon] + P_\mu[|\frac{\sum_{i\in N_-(n)}\xi_i}{l_n} - \underline{\mu}| > \varepsilon] \\
\leq \quad & \frac{2}{\varepsilon}\mathbb{E}[(\frac{\sum_{i\in N_+(n)}\xi_i}{k_n} - \overline{\mu})^+] + \frac{2}{\varepsilon}\mathbb{E}[(\frac{\sum_{i\in N_-(n)}\xi_i}{l_n} - \underline{\mu})^-],
\end{aligned}$$

which converges to 0 as n goes to infinity by Peng's wLLN*.

By the convergence rate of wLLN* given in S. (2021), we can prove the a.s. convergence.

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[3] ① $P_\mu \in \Theta$; ② $E_{P_\mu}[\xi_n] = \kappa_n(\mu)$

Proof to Theorem 5

Lemma 8

Under the same conditions as those in Theorem 4 , we can find a Borel mapping $P_\mu : [\underline{\mu}, \overline{\mu}] \to \Theta$ such that $E_{P_\mu}[\xi_n] = \kappa_n(\mu)$.

Sketch of the proof to Theorem 5.

STEP 1. For any $\phi \in C_{b,Lip}(\mathbb{R}^n)$, $n \in \mathbb{N}$, set

$$E_{P^\Pi}[\phi(\xi_1, \cdots, \xi_n)] = E_P\left[E_{P_{\pi(x_1,\cdots,x_d)}}[\phi(x_1,\cdots,x_d,\xi_1,\cdots,\xi_{n-d})]|^{x_i=\xi_i}_{i=1,\cdots,d}\right].$$

Clearly, $P^\Pi \in \Theta$, $P^\Pi = P$ on $\mathcal{F}_d$, and
$P^\Pi(\xi_{n+d}|\mathcal{F}_d)(\omega) = P_{\Pi(\omega)}(\xi_n) = \kappa_n(\Pi(\omega))$ P-a.s.

Proof to Theorem 5

STEP 2. We denote by Θ_P^Π the set of probabilities satisfying $\tilde{P}^\Pi \in \Theta$, $\tilde{P}^\Pi = P$ on $\mathcal{F}_d$, and $\tilde{P}^\Pi(\xi_{n+d}|\mathcal{F}_d)(\omega) = \kappa_n(\Pi(\omega))$ P-a.s.

For any $\tilde{P}^\Pi \in \Theta_P^\Pi$,

$$\begin{aligned} & \tilde{P}^\Pi(\lim_{n\to\infty} \bar{\xi}_n = \Pi) \\ = \quad & E_P[\tilde{P}^\Pi(\lim_{n\to\infty} \bar{\xi}_n = \Pi|\mathcal{F}_d)] \\ = \quad & \int_\Omega \tilde{P}^\Pi(\lim_{n\to\infty} \bar{\xi}_n = \Pi(\omega))(\omega)P(d\omega) = 1. \end{aligned}$$

Proof to Theorem 5

Θ_P^Π: the set of probabilities satisfying $\tilde{P}^\Pi \in \Theta$, $\tilde{P}^\Pi = P$ on $\mathcal{F}_d$, and $\tilde{P}^\Pi(\xi_{n+d}|\mathcal{F}_d)(\omega) = \kappa_n(\Pi(\omega))$ P-a.s.

STEP 3. Θ_P^Π is a nonempty convex **closed** subset of Θ.

By Krein-Milman Theorem, ext Θ_P^Π is nonempty.

STEP 4. ext $\Theta_P^\Pi \subset$ ext Θ .

The proof is completed: $\exists P^\Pi \in$ ext Θ such that

$$P^\Pi(\lim_{n\to\infty} \bar{\xi}_n = \Pi) = 1.$$

THANK YOU.