Progressive enlargement of filtration and control problems for step processes

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2. Progressive enlargement by a random time
3. Applications to stochastic control theory

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Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$.

The jump measure of an $\mathbb{R}^d$-valued $\mathbb{F}$-semimartingale X is the integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^d$

$$\mu^X(\omega, \mathrm{d}t, \mathrm{d}x) = \sum_{s>0} 1_{\{\Delta X_s(\omega)\neq 0\}} \delta_{(s, \Delta X_s(\omega))}(\mathrm{d}t, \mathrm{d}x).$$

X is a *step process with respect to* $\mathbb{F}$ if

$$X = \sum_{n=1}^{\infty} \xi_n 1_{[\tau_n, +\infty)},$$

where

- $(\tau_n)_n$ is a sequence of $\mathbb{F}$-stopping times s.t. $\tau_n \uparrow +\infty$, $\tau_n < \tau_{n+1}$ on $\{\tau_n < +\infty\}$
- $(\xi_n)_{n\geq 1}$ is a sequence of $\mathbb{R}^d$-valued random variables s.t. ξ_n is $\mathscr{F}_{\tau_n}$-measurable and $\xi_n \neq 0$ if and only if $\tau_n < +\infty$.

We use the notation $(X, \mathbb{F})$ to mean that X is $\mathbb{F}$-adapted, while $\mathbb{F}^X$ denotes the smallest right-continuous filtration such that X is adapted.

Progressive enlargement of filtrations

We consider two step processes $(X, \mathbb{F}^X)$ and $(H, \mathbb{F}^H)$.

Given a σ-field $\mathscr{R}^X$, we denote by $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ the filtration $\mathbb{F}^X$ initially enlarged by $\mathscr{R}^X$, i.e.

$$\mathscr{F}_t := \mathscr{R}^X \vee \mathscr{F}_t^X \quad t \geq 0.$$

Analogously, given a σ-field $\mathscr{R}^H$, we denote by $\mathbb{H} = (\mathscr{H}_t)_{t \geq 0}$ the filtration $\mathbb{F}^H$ initially enlarged by $\mathscr{R}^H$, i.e.

$$\mathscr{H}_t := \mathscr{R}^H \vee \mathscr{F}_t^H, \quad t \geq 0.$$

We denote by $\mathbb{G} = (\mathscr{G}_t)_{t \geq 0}$ the *progressive enlargement* of $\mathbb{F}$ by $\mathbb{H}$:

$$\mathscr{G}_t := \bigcap_{s>t} \mathscr{F}_s \vee \mathscr{H}_s \quad t \geq 0.$$

$\mathbb{G}$ is the smallest right-continuous filtration containing $\mathbb{F}^X$, $\mathbb{F}^H$, $\mathscr{R}^X$ and $\mathscr{R}^H$ (i.e., $\mathbb{F}$ and $\mathbb{H}$).

MRT in the enlargement of a step process filtration

We introduce the $\mathbb{R}^d \times \mathbb{R}^\ell$-valued $\mathbb{G}$-semimartingale $Z = (X, H)$.

Theorem

$(Z, \mathbb{G})$ is a step process and $\mathbb{G}$ is the smallest right-continuous filtration containing $\mathscr{R}^X \vee \mathscr{R}^H$ and such that μ^Z is optional.

Theorem

If $\mathscr{F} = \mathscr{G}_\infty$, every $Y \in \mathscr{H}^1_{\mathrm{loc}}(\mathbb{G})$ can be represented as

$$Y = Y_0 + W * \mu^Z - W * \nu^{\mathbb{G},Z}$$

where $(\omega, t, x_1, x_2) \mapsto W(\omega, t, x_1, x_2)$ is a $\mathscr{P}(\mathbb{G}) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{B}(\mathbb{R}^\ell)$-measurable function such that

$$|W| * \mu^Z \in \mathscr{A}^+_{\mathrm{loc}}(\mathbb{G})$$

and $\nu^{\mathbb{G},Z}$ denotes the $\mathbb{G}$-dual predictable projection of μ^Z.

Outline

Progressive enlargement by a random time

From here on we shall concentrate on the special case

$$H = 1_{[\tau,+\infty)},$$

where τ denotes a random time. We denote by $H^{p,\mathbb{F}}$ the $\mathbb{F}$-dual predictable projection of H. We also set

$$A_t = \mathbb{P}[\tau > t | \mathscr{F}_t] \text{ a.s.,} \quad \text{for every } t \geq 0.$$

A is a càdlàg $\mathbb{F}$-supermartingale, called the Azéma supermartingale.

The $\mathbb{G}$-dual predictable projection of H is

$$\Lambda^{\mathbb{G}} = \int_0^{\tau\wedge\cdot} \frac{1}{A_{s-}} \mathrm{d}H_s^{p,\mathbb{F}}.$$

Quasi-left continuity in the enlarged filtration

We are interested in the following question:
if μ^X is an $\mathbb{F}$-quasi left continuous random measure, is it $\mathbb{G}$-quasi left continuous?

In general, no: intuitively, the larger filtration $\mathbb{G}$ supports more predictable stopping times than $\mathbb{F}$.

Example

Let X be a homogeneous Poisson process with respect to $\mathbb{F}^X$ and let $(\tau_n)_{n\geq 1}$ be the sequence of the jump-times of $\mathbb{F}^X$.
The process X is ***not quasi-left continuous in the filtration*** $\mathbb{G}$ ***obtained enlarging*** $\mathbb{F}^X$ ***progressively by*** $\tau = \frac{1}{2}(\tau_1 + \tau_2)$.

Indeed, the jump time τ_2 of X is announced in $\mathbb{G}$ by $(\vartheta_n)_{n\geq 1}$, $\vartheta_n := \frac{1}{n}\tau + (1 - \frac{1}{n})\tau_2$, and $\vartheta_n > \tau$ is a $\mathbb{G}$-stopping time for every $n \geq 1$. Hence, τ_2 is a $\mathbb{G}$-predictable jump-time of X.

Sufficient conditions on τ s.t. μ^X is $\mathbb{G}$-quasi-left continuous?

Avoidance of $\mathbb{F}$-stopping times and immersion property

Definition (Assumption ($\mathscr{A}$))

*A random time τ satisfies the **avoidance of $\mathbb{F}$-stopping times** if $\mathbb{P}[\tau = \sigma < +\infty] = 0$ for every $\mathbb{F}$-stopping time σ.*

Remark. If τ satisfies ($\mathscr{A}$), then

- the process H is $\mathbb{G}$-quasi-left continuous;
- $\Delta X \Delta H = 0$.

τ carries an information which is completely exogenous (nothing about τ can be inferred from the information contained in $\mathbb{F}$).

Definition (Assumption ($\mathscr{H}$))

*A random time τ satisfies the **immersion property** if $\mathbb{F}$-martingales remain $\mathbb{G}$-martingales.*

Remark. If X is an $\mathbb{F}$-step process and τ satisfies ($\mathscr{H}$), then $\nu^{\mathbb{G},X} = \nu^{\mathbb{F},X}$.

The $\mathbb{G}$-dual predictable projection of μ^Z under $(\mathscr{A})$-$(\mathscr{H})$

Theorem

Let X be a step process and let τ be a random time satisfying $(\mathscr{A})$-$(\mathscr{H})$. Then,

$$\begin{aligned}&\nu^{\mathbb{G},Z}(\omega,\mathrm{d}t,\mathrm{d}x_1,\mathrm{d}x_2)\\&=\nu^{\mathbb{F},X}(\omega,\mathrm{d}t,\mathrm{d}x_1)\delta_0(\mathrm{d}x_2)+\delta_1(\mathrm{d}x_2)\delta_0(\mathrm{d}x_1)\mathrm{d}\Lambda_t^{\mathbb{G}}(\omega).\end{aligned}$$

Remark 1. If μ^X is $\mathbb{F}$-quasi-left continuous and τ satisfies $(\mathscr{A})$-$(\mathscr{H})$, then μ^Z is $\mathbb{G}$-quasi-left continuous.

Remark 2. One can quite easily construct random times τ satisfying $(\mathscr{A})$-$(\mathscr{H})$, see e.g. Di Tella and Engelbert (2021).

Jacod's absolute continuity property

Let $\eta(\mathrm{d}u)$ denote the law of τ and $P_t(\mathrm{d}u)$ a regular version of its conditional distribution, i.e., for every $A \in \mathscr{B}(\mathbb{R})$, $\eta(A) := \mathbb{P}(\tau^{-1}(A))$ and $P_t(A) := \mathbb{P}(\tau \in A | \mathscr{F}_t)$.

Definition (Assumption (*JAC*))

A random time τ satisfies ***Jacod's absolute continuity condition*** *if*

$$\eta(\mathrm{d}u) \quad \textit{is a diffused probability measure,} \tag{1}$$

$$P_t(\mathrm{d}u) \ << \ \eta(\mathrm{d}u). \tag{2}$$

Remark 1. Condition (1) ensures property $(\mathscr{A})$.

Remark 2. Random times satisfying (*JAC*) can be constructed following the approach presented by Jeanblanc and Le Cam (2009).

The $\mathbb{G}$-dual predictable projection of μ^Z under (*JAC*)

Theorem

Let $(X,\mathbb{F})$ be an $\mathbb{F}$-quasi left continuous step process, and τ be a random time satisfying (JAC). Then

$$\begin{aligned}
&\nu^{\mathbb{G},Z}(\omega,\mathrm{d}t,\mathrm{d}x_1,\mathrm{d}x_2)\\
&=\Big(1_{[0,\tau]}(\omega,t)\Big(1+\frac{W'(\omega,t,x_1)}{A_{t-}(\omega)}\Big)\\
&\qquad+1_{(\tau,+\infty)}(\omega,t)(1+U(\omega,t,x_1))\Big)\nu^{\mathbb{F},X}(\omega,\mathrm{d}t,\mathrm{d}x_1)\delta_0(\mathrm{d}x_2)\\
&+1_{[0,\tau]}(\omega,t)\frac{1}{A_{t-}(\omega)}\mathrm{d}H_t^{p,\mathbb{F}}(\omega)\delta_0(\mathrm{d}x_1)\delta_1(\mathrm{d}x_2),
\end{aligned}$$

where W' is an $\mathbb{F}$-predictable function such that $A_-+W'\geq 0$ and U is a $\mathbb{G}$-predictable function such that $1+U\geq 0$ identically.

Remark. If μ^X is $\mathbb{F}$-quasi left continuous and τ satisfies (*JAC*), then μ^Z is $\mathbb{G}$-quasi left continuity.

Outline

Background

Optimization problem for a step process $(X, \mathbb{F})$, in presence of an additional exogenous risk source that cannot be inferred from $\mathbb{F}$ (as the death of the investor or the default of part of the market), whose occurrence time is modeled by τ.

Setting 1

X is $\mathbb{F}$-quasi-left continuous and τ satisfies $(\mathscr{A})$-$(\mathscr{H})$.

or

Setting 2

X is $\mathbb{F}$-quasi-left continuous and τ satisfies (JAC) (hence $(\mathscr{A})$).

Under Setting 1 or 2,

$$\nu^{\mathbb{F},X}(\mathrm{d}t, \mathrm{d}x_1) = \phi_t^{\mathbb{F},X}(\mathrm{d}x_1)\mathrm{d}C_t^{\mathbb{F},X},$$

where $\phi^{\mathbb{F},X}$ is a transition probability from $(\Omega \times [0,T], \mathscr{P}(\mathbb{F}))$ into $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, and $C^{\mathbb{F},X} \in \mathscr{A}^+_{\mathrm{loc}}(\mathbb{F})$ is a continuous process.

Under Setting 1

$$\phi_t^{\mathbb{G},X}(\mathrm{d}x_1) = \phi_t^{\mathbb{F},X}(\mathrm{d}x_1),$$
$$\mathrm{d}C_t^{\mathbb{G},X} = \mathrm{d}C_t^{\mathbb{F},X}.$$

Under Setting 2

$$\phi_t^{\mathbb{G},X}(\mathrm{d}x_1) = \frac{\kappa(t,x_1)}{\int_{\mathbb{R}^d}\kappa(t,x_1)\,\phi_t^{\mathbb{F},X}(\mathrm{d}x_1)}\phi_t^{\mathbb{F},X}(\mathrm{d}x_1),$$
$$\mathrm{d}C_t^{\mathbb{G},X} = \int_{\mathbb{R}^d}\kappa(t,x_1)\,\phi_t^{\mathbb{F},X}(\mathrm{d}x_1)\,\mathrm{d}C_t^{\mathbb{F},X},$$

where

$$\kappa(\omega,t,x_1) := 1_{[0,\tau]}(\omega,t)\left(1+\frac{W'(\omega,t,x_1)}{A_{t-}(\omega)}\right)+1_{(\tau,+\infty)}(\omega,t)\big(1+U(\omega,t,x_1)\big).$$

Formulation of the problem

Consider the problem of an agent whose available information is $\mathbb{F}$ (that is, she pursues $\mathbb{F}$-predictable strategies) but, for some reasons, has only access to the market up to the occurrence of the exogenous shock event at time τ.

The Data

$(U, \mathscr{U})$ *is a measurable space.*

$r, l : \Omega \times [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}$ *are* $\mathscr{P}(\mathbb{F}) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{U}$*-measurable and there exist constants* $M_r > 1$, $M_l > 0$ *such that,* $\mathbb{P}$*-a.s.,*

$$0 \le r_t(x_1,u) \le M_r, \quad |l_t(x_1,u)| \le M_l, \qquad t \in [0,T], x_1 \in \mathbb{R}^d, u \in U.$$

$g : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *is* $\mathscr{G}_{T\wedge\tau} \otimes \mathscr{B}(\mathbb{R}^d)$*-measurable, and there exists a constant* β *such that* $\beta > \sup|r-1|^2$*, and*

$$\mathbb{E}[e^{\beta C_T}] < +\infty, \quad \mathbb{E}[|g(X_{T\wedge\tau})|^2 e^{\beta C_T}] < +\infty, \tag{3}$$

with $C_t := C_t^{\mathbb{G},X} + \Lambda_t^{\mathbb{G}}$.

The optimal control problem

For simplicity we consider only the case where Assumption 1 is satisfied, so that, $\mathrm{d}C^{\mathbb{F},X} = \mathrm{d}C^{\mathbb{G},X}$ and $\phi^{\mathbb{F},X}(\mathrm{d}x_1) = \phi^{\mathbb{G},X}(\mathrm{d}x_1)$.

Let $\mathscr{C} = \{U\text{-valued } \mathbb{G}\text{-predictable processes } u(\cdot)\}$ and set

$$\hat{\mathscr{C}} := \{u \in \mathscr{C} : 1_{[T\wedge\tau, T]} u = 0\} \subseteq \mathscr{C}.$$

Remark. $\mathbb{F}$- and $\mathbb{G}$-predictable processes coincide on $[0,\tau]$, so the set $\hat{\mathscr{C}}$ consists of strategies which are morally $\mathbb{F}$-predictable.

To every admissible control process $u \in \hat{\mathscr{C}}$ we will associate the cost functional

$$J(u) = \mathbb{E}_u \left[\int_0^{T\wedge\tau} l_t(X_t, u_t) \mathrm{d}C_t^{\mathbb{F},X} + g(X_{T\wedge\tau}) \right],$$

where $\mathbb{P}_u$ is a suitable probability measure, absolutely continuous with respect to $\mathbb{P}$. The control problem will be

$$\inf_{u \in \hat{\mathscr{C}}} J(u) \qquad (P)$$

The probability measure $\mathbb{P}_u$

For any $u \in \hat{\mathscr{C}}$ we introduce the Doléans-Dade exponential

$$L_t^u = \exp\Big(\int_0^t \int_{\mathbb{R}^{d+1}} (1 - R_s(x_1, x_2, u_s))\nu^{\mathbb{G},Z}(\mathrm{d}s, \mathrm{d}x_1, \mathrm{d}x_2)\Big) \prod_{n \geq 1 : T_n \leq t} R_{T_n}(X_{T_n}, H_{T_n}, u_{T_n}),$$

with $R_t(x_1, x_2, u) := r_t(x_1, u)\, 1_{\{x_2 = 0\}} + 1_{\{x_2 \neq 0\}}$.

Lemma

Assume that

$$\mathbb{E}[\mathrm{e}^{(3+M_r^4)C_T}] < +\infty. \tag{4}$$

Then, for every $u \in \hat{\mathscr{C}}$, L^u is a square integrable $\mathbb{G}$-martingale.

We can then define $\mathbb{P}_u(d\omega) := L_T^u(\omega)\mathbb{P}(d\omega)$.

$$\nu^{\mathbb{G},Z,u}(\omega, \mathrm{d}t, \mathrm{d}x_1, \mathrm{d}x_2) = R_t(\omega, X_t(\omega), H_t(\omega), u_t(\omega))\nu^{\mathbb{G},Z}(\omega, \mathrm{d}t, \mathrm{d}x_1, \mathrm{d}x_2)$$

is the $\mathbb{G}$-compensator of μ^Z under $\mathbb{P}_u$.

The associated BSDE

We consider the following BSDE: $\mathbb{P}$-a.s., for all $t \in [0,T]$,

$$Y_t + \int_t^T \int_{\mathbb{R}^{d+1}} \Theta_s(x_1,x_2)\,(\mu^Z - \nu^{\mathbb{G},Z})(\mathrm{d}s,\mathrm{d}x_1,\mathrm{d}x_2)$$
$$= g(X_{T\wedge\tau}) + \int_t^T f(s,X_s,\Theta_s(\cdot))\,1_{[0,T\wedge\tau]}(s)\,\mathrm{d}C_s^{\mathbb{F},X} \tag{5}$$

with

$$f(\omega,t,y_1,\theta(\cdot))$$
$$:= \inf_{u\in U}\left\{l_t(\omega,y_1,u) + \int_{\mathbb{R}^d} \theta(x_1,0)\,(r_t(\omega,x_1,u)-1)\,\phi_t^{\mathbb{F},X}(\omega,\mathrm{d}x_1)\right\}.$$

Assumption 1

$\forall \Theta \in L^1(\mu^Z)$, $\exists \underline{\hat{u}}^\Theta \in \hat{\mathscr{C}}$ *s.t., for a.a.* (ω,t) *wrt* $dC_t^{\mathbb{F},X}(\omega)\mathbb{P}(d\omega)$,

$$f(\omega,t,X_{t-}(\omega),\Theta_t(\omega,\cdot)) = l_t(\omega,X_{t-}(\omega),\underline{\hat{u}}^\Theta(\omega,t))$$
$$+\int_{\mathbb{R}^d} \Theta_t(\omega,x_1,0)\,(r_t(\omega,x_1,\underline{\hat{u}}^\Theta(\omega,t))-1)\,\phi_t^{\mathbb{F},X}(\omega,\mathrm{d}x_1).$$

$L^{2,\beta}_{\mathrm{Prog}}(\Omega \times [0,T],\mathbb{G})$ is the space of real-valued $\mathbb{G}$-progr. meas. Y s.t.

$$\mathbb{E}\Big[\int_0^T e^{\beta C_t}|Y_t|^2 dC_t\Big] < \infty.$$

$L^{2,\beta}(\mu^Z,\mathbb{G})$ is the space of $\mathscr{P}(\mathbb{G})\otimes\mathscr{B}(\mathbb{R}^{d+1})$-meas. Θ s.t.

$$\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d} e^{\beta C_t}|\Theta_t(x_1,0)|^2\phi_t^{\mathbb{G},X}(dx_1)dC_t^{\mathbb{G},X}\Big] + \mathbb{E}\Big[\int_0^T e^{\beta C_t}|\Theta_t(0,1)|^2 d\Lambda_t^{\mathbb{G}}\Big] < \infty.$$

Theorem

Let Assumption 1 hold true. Set

$$L := \operatorname*{ess\,sup}_{\omega}\big(\sup\{|r_t(x_1,u)-1| : t\in[0,T], x_1\in\mathbb{R}^d, u\in U\}\big),$$

and let (3) *hold true with* $\beta > L^2$.

Then BSDE (5) *admits a unique solution*
$(Y,\Theta(\cdot)) \in L^{2,\beta}_{Prog}(\Omega\times[0,T],\mathbb{G})\times L^{2,\beta}(\mu^Z,\mathbb{G})$.
In particular, $Y = Y_{\cdot\wedge\tau}$, $\mathbb{P}(d\omega)$*-a.e., and* $\Theta = \Theta 1_{[0,T\wedge\tau]}$
$\phi_t(\omega,dx_1,dx_2)\,dC_t(\omega)\,\mathbb{P}(d\omega)$*-a.e.*

Solution to the optimal control problem (P)

Theorem

Let Assumption 1 hold true, condition (3) *hold true with* $\beta > L^2$, *and condition* (4) *be satisfied. Let* $(Y, \Theta(\cdot))$ *be the unique solution to BSDE* (5), *with corresponding admissible control* $\underline{\hat{u}}^{\Theta} \in \hat{\mathscr{C}}$.
Then $\underline{\hat{u}}^{\Theta}$ *is optimal and* Y_0 *is the optimal cost , i.e.*

$$Y_0 = J(\underline{\hat{u}}^{\Theta}) = \inf_{u \in \hat{\mathscr{C}}} J(u).$$

Proof. Let $u \in \hat{\mathscr{C}}$. Thank to the previous result, (adding and subtracting $\mathbb{E}_u\big[\int_0^{T\wedge\tau} l_s(X_s, \hat{u})\,\mathrm{d}C_s^{X,\mathbb{F}}\big]$) we obtain

$$Y_0 = J(u) + \mathbb{E}_u\Big[\int_0^{T\wedge\tau} \big[f(s, X_s, \Theta_s(\cdot)) - l(s, X_s, \hat{u}) - \int_{\mathbb{R}^d} \Theta_s(x_1, 0)\,(\bar{r}_s(x_1, u) - 1)\,\phi_s^{\mathbb{F},X}(\mathrm{d}x_1)\big]\,\mathrm{d}C_s^{\mathbb{F},X}\Big].$$

The conclusion comes from the definition of f and Assumption 3. $\square$

An auxiliary control problem

The functional cost in the optimal control problem (P) can be equivalently rewritten as

$$J(u) = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t)\, 1_{[0,T\wedge\tau(\omega)]}(t) \mathrm{d}C_t^{\mathbb{F},X} + g(X_{T\wedge\tau}) \right], \quad u \in \hat{\mathscr{C}}.$$

Clearly $l 1_{[0,T\wedge\tau]}$ is $\mathscr{P}(\mathbb{G}) \otimes \mathscr{B}(\mathbb{R}^d) \otimes \mathscr{U}$-measurable and $g(X_{T\wedge\tau})$ is $\mathscr{G}_T$-measurable.

Let us now consider the *enlarged* optimal control problem obtained by taking the infimum over all the $\mathbb{G}$-predictable processes:

$$\inf_{u\in\mathscr{C}} J(u) = \mathbb{E}_u \left[\int_0^T l_t(X_t, u_t)\, 1_{[0,T\wedge\tau(\omega)]}(t) \mathrm{d}C_t^{\mathbb{F},X} + g(X_{T\wedge\tau}) \right]. \quad (P')$$

One can solve optimal control problem (P') by considering the following BSDE: $\mathbb{P}$-a.s., for all $t \in [0,T]$,

$$
\begin{aligned}
&\tilde{Y}_t + \int_t^T \int_{\mathbb{R}^{d+1}} \tilde{\Theta}_s(x_1,x_2)\,(\mu^Z - \nu^{\mathbb{G},Z})(\mathrm{d}s,\mathrm{d}x_1,\mathrm{d}x_2) \\
&= g(X_{T\wedge\tau}) + \int_t^T \tilde{f}(s,X_s,\tilde{\Theta}_s(\cdot))\,\mathrm{d}C_t^{\mathbb{F},X},
\end{aligned} \tag{6}
$$

where

$$
\begin{aligned}
\tilde{f}(\omega,t,y_1,\theta(\cdot)) := \inf_{u\in U}\Big\{ & l_t(\omega,y_1,u)1_{[0,T\wedge\tau(\omega)]}(t) \\
& + \int_{\mathbb{R}^d} \theta(x_1,0)\,(r_t(\omega,x_1,u)-1)\,\phi_t^{\mathbb{F},X}(\omega,\mathrm{d}x_1)\Big\}
\end{aligned}
$$

for every $\omega \in \Omega$, $t \in [0,T]$, $y_1 \in \mathbb{R}^d$, and $\theta \in \mathscr{L}^1(\mathbb{R}^{d+1},\mathscr{B}(\mathbb{R}^{d+1}),\phi_t(\omega,\mathrm{d}x_1,\mathrm{d}x_2))$.

Assumption 2

$\forall \Theta \in L^1(\mu^Z)$, $\exists$ $\mathbb{G}$-*predictable process* $\underline{u}^{\Theta} : \Omega \times [0,T] \to U$, *s.t., for a.a.* (ω, t) *wrt* $dC_t^{\mathbb{F},X}(\omega)\mathbb{P}(d\omega)$,

$$\tilde{f}(\omega, t, X_{t-}(\omega), \Theta_t(\omega, \cdot)) = l_t(\omega, X_{t-}(\omega), \underline{u}^{\Theta}(\omega, t))1_{[0, T\wedge\tau(\omega)]}(t)$$
$$+ \int_{\mathbb{R}^d} \Theta_t(\omega, x_1, 0)\,(r_t(\omega, x_1, \underline{u}^{\Theta}(\omega, t)) - 1)\,\phi_t^{\mathbb{F},X}(\omega, \mathrm{d}x_1).$$

Theorem

Let Assumption 2 hold true. Let condition (3) *hold true with* $\beta > L^2$, *and that condition* (4) *is satisfied. Let* $(\tilde{Y}, \tilde{\Theta}(\cdot))$ *denote the unique solution to BSDE* (6), *with corresponding admissible control* $\underline{u}^{\tilde{\Theta}} \in \mathscr{C}$. *Then*

$$\tilde{Y}_0 = J(\underline{u}^{\tilde{\Theta}}) = \inf_{u \in \mathscr{C}} J(u).$$

Let us now go back to our original optimal control problem (P): we aim at finding an admissible process $\underline{\hat{u}} \in \hat{\mathscr{C}}$ such that

$$J(\underline{\hat{u}}) = \inf_{u \in \hat{\mathscr{C}}} J(u).$$

Such an optimal control process exists and is provided by $\underline{u}^{\tilde{\Theta}} 1_{[0,T\wedge\tau]}$. Moreover, the value functions of (P) and (P') coincide:

Theorem

Let Assumption 2 holds true. Assume that (3) *holds true with* $\beta > L^2$, *and that condition* (4) *holds true. Let* $(\tilde{Y}, \tilde{\Theta}(\cdot))$ *denote the unique solution to BSDE* (6), *with corresponding admissible control* $\underline{u}^{\tilde{\Theta}} \in \mathscr{C}$. *Then* $\underline{\hat{u}} := \underline{u}^{\tilde{\Theta}} 1_{[0,T\wedge\tau]} \in \hat{\mathscr{C}}$ *is an optimal control process for* (P) *and*

$$\tilde{Y}_0 = J(\underline{u}^{\tilde{\Theta}}) = J(\underline{\hat{u}}) = Y_0.$$

Thank you for your attention!