What if we (sort of) knew what the future brings?

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based on joint papers with
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Problem formulation: Frontrunning with price impact

Bachelier stock price dynamics, zero interest rate:

$$S_t = s_0 + \mu t + \sigma W_t, \quad t \geq 0$$

Information flow: peek Δ time units ahead $\rightsquigarrow$ "frontrunning"

$$\mathscr{G}_t^{\Delta} = \mathscr{F}_{t+\Delta}^{S}, \quad t \geq 0.$$

Egregious arbitrage opportunities

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Egregious arbitrage opportunities curtailed by price impact because execution price is

$$S_t^{\phi} = S_t + \frac{\Lambda}{2}\phi_t$$

where $\phi_t = \frac{d}{dt}\Phi_t$ is the frontrunner's turnover rate $\rightsquigarrow$ temporary price impact à la Almgren-Chriss

Problem formulation: Frontrunning with price impact

Profits and losses from trading according to $\Phi_t = \Phi_0 + \int_0^t \phi_s ds$:

$$
\begin{aligned}
V_T^{\Phi_0,\phi} =& - \int_0^T S_t^\phi d\Phi_t + \Phi_T S_T - \Phi_0 S_0 \\
=& \Phi_0(S_T - S_0) + \int_0^T \phi_t (S_T - S_t) dt - \frac{\Lambda}{2} \int_0^T \phi_t^2 dt,
\end{aligned}
$$

$\rightsquigarrow$ mark to market stock positions at time 0 and T

Natural class of **admissible strategies**:

$$
\mathscr{A}^\Delta = \left\{ \phi = (\phi_t)_{t \in [0,T]}\ \mathscr{G}^\Delta\text{-optional with } \int_0^T \phi_t^2 dt < \infty \text{ a.s.} \right\}
$$

Exponential **utility maximization**:

$$
\text{Maximize } \mathbb{E}\left[u(V_T^{\Phi_0,\phi}) \right] = \mathbb{E}\left[-\exp\left(-\alpha V_T^{\Phi_0,\phi} \right) \right] \text{ over } \phi \in \mathscr{A}^\Delta
$$

Related work

- enlargement of filtrations and small investor: Karatzas/Pikovsky '96, Amendinger/Imkeller/Schweizer '98, Amendinger/Becherer/Schweizer '03, Imkeller '03, ..., Jeanblanc '05–'21
- insider trading with equilibrium prices: Kyle '85, Back '92, ... Cetin '18, Back/Cocquemas/Ekren/Lioui '21
- insider trading with temporary price impact: Ankircher/BlanchetScalliet/Eyraud-Loisel '16, Barger/Donelly '20
- control with infinite-dimensional state space: Ji/Wang/Yang '13, Fabbri/Gozzi/Swiech '17, Saporito/Zhang '19
- short-term inside information: B./Körber '21+ $\rightsquigarrow$ 2nd part of this talk; B./Besslich '20

Main result: Optimal Policy

Maximize $\mathbb{E}\left[u(V_T^{\Phi_0,\phi})\right] = \mathbb{E}\left[-\exp\left(-\alpha V_T^{\Phi_0,\phi}\right)\right]$ over $\phi \in \mathscr{A}^\Delta$

Theorem

The **optimal turnover rate** *at time $t \in [0, T]$ is*

$$\hat{\phi}_t = \frac{d}{dt}\hat{\Phi}_t = \frac{1}{\Lambda}\left(\bar{S}_t^\Delta - S_t\right) + \frac{\Upsilon^\Delta(T-t)}{\Delta}\left(\frac{\mu}{\alpha\sigma^2} - \hat{\Phi}_t\right),$$

where $\bar{S}^\Delta$ denotes the **stock price average**

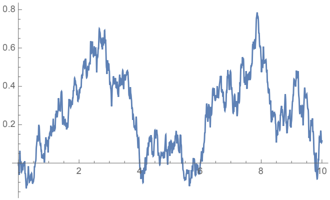
$$\bar{S}_t^\Delta := \left(1 - \Upsilon^\Delta(T-t)\right) S_{(t+\Delta)\wedge T} + \Upsilon^\Delta(T-t)\frac{1}{\Delta}\int_0^\Delta S_{t+s}\,ds$$

with

$$\Upsilon^\Delta(\tau) = \frac{\Delta\sqrt{\rho}\tanh(\sqrt{\rho}(\tau-\Delta)^+)}{1+\Delta\sqrt{\rho}\tanh(\sqrt{\rho}(\tau-\Delta)^+)}, \quad \tau \geq 0,$$

where ρ is the risk/liquidity ratio $\rho = \frac{\alpha\sigma^2}{\Lambda}$.

Discussion: Optimal policy illustration



- $S_0 = 0$, $\mu = .1$, $\sigma = .3$, $\alpha = .03$, $T = 10\Delta$, $\Lambda = .01$
- stock price S

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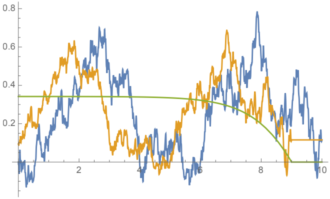

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- weight Υ^{Δ}

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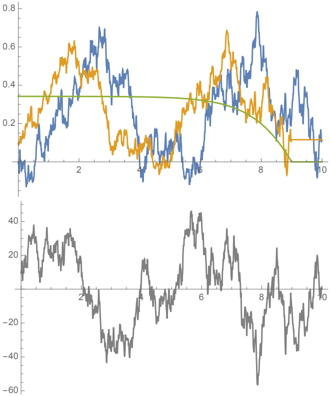

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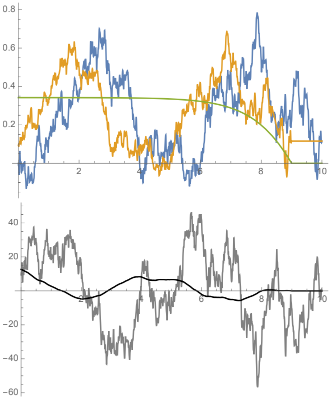

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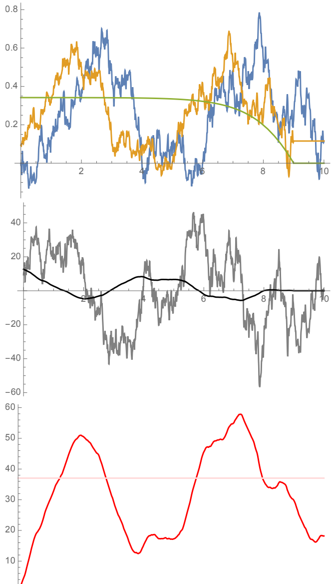

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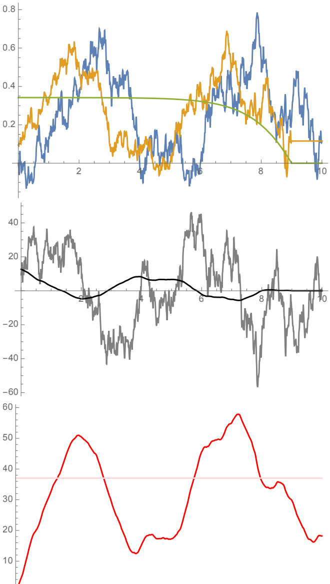

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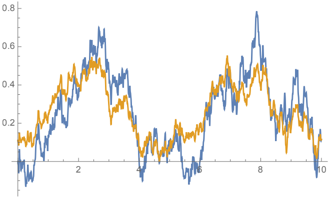
- execution prices:



Main result (ctd): Value of frontrunning

Theorem

The certainty equivalent of extra information is

$$C(\Delta) = -\frac{1}{\alpha}\log\frac{\max_{\phi\in\mathscr{A}^{\Delta}}\mathbb{E}\left[-\exp\left(-\alpha V_T^{\Phi_0,\phi}\right)\right]}{\max_{\phi\in\mathscr{A}^{0}}\mathbb{E}\left[-\exp\left(-\alpha V_T^{\Phi_0,\phi}\right)\right]}$$
$$= \frac{1}{2\alpha}\int_0^T \frac{(s\wedge\Delta)\rho}{1+(s\wedge\Delta)\sqrt{\rho}\tanh\left(\sqrt{\rho}(T-s)\right)}ds.$$

- independent of risk premium μ
- governed by risk/liquidity ratio $\rho = \alpha\sigma^2/\Lambda$ and ability to peek ahead Δ

Proof proceeds via duality

Theorem (cf. Guasoni, Rasonyi '15)

Denoting by $\mathscr{Q}$ the set of all probability measures $\mathbb{Q} \sim \mathbb{P}$ with finite entropy $\mathbb{E}_{\mathbb{Q}}\left[\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] < \infty$ relative to $\mathbb{P}$, we have

$$\max_{\phi \in \mathscr{A}} \left\{ -\frac{1}{\alpha} \log \mathbb{E}\left[\exp\left(-\alpha V_T^{\Phi_0,\phi}\right)\right]\right\}$$
$$= \inf_{\mathbb{Q}\in\mathscr{Q}} \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\alpha}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) + \Phi_0(S_T - S_0) + \frac{1}{2\Lambda}\int_0^T \left|\mathbb{E}_{\mathbb{Q}}[S_T|\mathscr{G}_t^\Delta] - S_t\right|^2 dt\right]$$

There is a minimizer $\hat{\mathbb{Q}}$ and it yields via

$$\hat{\phi}_t = \frac{\mathbb{E}_{\hat{\mathbb{Q}}}\left[S_T|\mathscr{G}_t^\Delta\right] - S_t}{\Lambda}, \quad t \in [0, T],$$

the unique optimal turnover rates for the primal problem.

Note: This holds for **any** $\mathscr{G}$ and **any** càdlàg S with finite exponential moments: $\sup_{t\in[0,T]} \mathbb{E}\exp(\varepsilon S_t^2) < \infty$ for some $\varepsilon > 0$.

Step 1: Rewrite the dual target value

Without loss of generality: $S = W$ (rescale parameters).

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Lemma

The dual infimum coincides with the one taken over $\mathbb{Q}$ with

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt\right)$$

for some bounded and adapted θ changing values only at finitely many deterministic times. For such θ, the induced $\mathbb{Q}$-Brownian motion $W_s^{\mathbb{Q}} = W_s + \int_0^s \theta_r dr$, $s \geq r$ generates the same filtration as W and so we get, for any $t \in [0, T]$, the Itô-representations

$$\theta_t = a_t + \int_0^t l_{t,s} dW_s^{\mathbb{Q}}$$

for a suitable constant a_t and square-integrable predictable $l_{t,\cdot}$.

Step 1: Rewrite the dual target value (ctd)

Lemma

With the previous notation the dual target value reads

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{\alpha}\log\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)+\Phi_0(S_T-S_0)+\frac{1}{2\Lambda}\int_0^T\left|\mathbb{E}_{\mathbb{Q}}(S_T|\mathscr{G}_t^{\Delta})-S_t\right|^2 dt\right]$$
$$=\frac{1}{2\alpha}\int_0^T a_t^2 dt-\Phi_0\int_0^T a_t dt+\frac{1}{2\Lambda}\int_0^T\left(\int_t^T a_u du\right)^2 dt$$
$$+\int_0^T\mathbb{E}_{\mathbb{Q}}\left[\frac{1}{2\alpha}\int_s^T l_{t,s}^2 dt+\frac{1}{2\Lambda}\int_s^T\left(\int_t^T l_{u,s}du\right)^2 dt\right.$$
$$\left.+\frac{s\wedge\Delta}{2\Lambda}\left(1-\int_s^T l_{u,s}du\right)^2\right]ds.$$

$\rightsquigarrow$ **family of deterministic (!) variational problems:**
Minimize separately over $a_\cdot$ and over $l_{\cdot,s}$ for each $s\in[0,T]$!

Step 2: Solve deterministic variational problems

Lemma

The minimum of the functional

$$\frac{1}{2\alpha}\int_0^T a_t^2 dt - \Phi_0 \int_0^T a_t dt + \frac{1}{2\Lambda}\int_0^T \left(\int_t^T a_u du\right)^2 dt$$

over $a \in L^2([0,T], dt)$ *is attained for* $\hat{a}\Phi_0$ *where*

$$\hat{a}_t = \frac{\alpha \cosh(\sqrt{\rho}(T-t))}{\cosh(\sqrt{\rho}T)}, \quad t \in [0,T].$$

The resulting minimum value is $-\hat{A}_T \Phi_0^2$ *where*

$$\hat{A}_T = \Lambda\sqrt{\rho}\tanh(\sqrt{\rho}T)/2.$$

Step 2: Solve deterministic variational problems

Lemma

For any $s \in [0, T]$, the minimum of the functional

$$\frac{1}{2\alpha}\int_s^T l_t^2 dt + \frac{1}{2\Lambda}\int_s^T \left(\int_t^T l_u du\right)^2 dt + \frac{s \wedge \Delta}{2\Lambda}\left(1 - \int_s^T l_u du\right)^2$$

over $l \in L^2([s, T], dt)$ is attained at

$$\hat{l}_{t,s} = \frac{\rho(s \wedge \Delta)\cosh(\sqrt{\rho}(T-t))}{\cosh(\sqrt{\rho}(T-s)) + \sqrt{\rho}(s \wedge \Delta)\sinh(\sqrt{\rho}(T-s))}, \quad t \in [s, T].$$

The corresponding minimum value is

$$\hat{L}_s = \frac{1}{2\Lambda}\frac{s \wedge \Delta}{1 + (s \wedge \Delta)\sqrt{\rho}\tanh(\sqrt{\rho}(T-s))}.$$

Step 3: Deal with Gaussian stochastic Volterra equation

Candidate for dual solution: $\hat{\mathbb{Q}} \sim \mathbb{P}$ with density represented by

$$\hat{\theta}_s = \hat{a}_s \Phi_0 + \int_0^s \hat{l}_{s,r} \, d\hat{W}_r^{\hat{\mathbb{Q}}}, \quad s \in [0, T].$$

For the associated Brownian motion $\hat{W} = W^{\hat{\mathbb{Q}}} = W + \int_0^{\cdot} \hat{\theta}_r dr$ this implies the **Gaussian Volterra-type integral equation**

$$\hat{W}_t = W_t + \int_0^t \hat{a}_s \Phi_0 ds + \int_0^t \int_0^s \hat{l}_{s,r} \, d\hat{W}_r \, ds, \quad t \in [0, T].$$

Step 3: Deal with Gaussian stochastic Volterra equation

Theorem (Hitsuda '68, Hida, Hitsuda '93)

The Gaussian Volterra-type integral equation has the unique solution

$$\begin{aligned}\hat{W}_t &= W_t + \Phi_0 \int_0^t \hat{a}_s ds - \int_0^t \int_0^s \hat{k}_{s,r} \left(dW_r + \Phi_0 a_r dr \right) ds \\ &= W_t - \int_0^t \int_0^s \hat{k}_{s,r} dW_r ds + \Phi_0 \left(\int_0^t \hat{a}_s ds - \int_0^t \int_0^s \hat{k}_{s,r} \hat{a}_r dr ds \right)\end{aligned}$$

where $\hat{k}$ *is the associated resolvent kernel characterized by*

$$\hat{k}_{t,s} + \hat{l}_{t,s} = \int_s^t \hat{l}_{t,u} \hat{k}_{u,s} du, \quad 0 \le s \le t \le T,$$

and this $\hat{W}$ *is the* $\hat{\mathbb{Q}}$*-Brownian motion induced by* W.

Remark: Result used previously in math finance by Cheridito '01.

Solution to the dual problem

Lemma

For our $\hat{l}$, the resolvent kernel is

$$\hat{k}_{t,s} = -\exp\left(\int_s^t \hat{l}_{u,u} du\right) \hat{l}_{t,s}, \quad 0 \le s \le t \le T,$$

and the dual infimum is attained by $\hat{\mathbb{Q}} \sim \mathbb{P}$ with density

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \exp\left(-\int_0^T \hat{\theta}_t dW_t - \frac{1}{2}\int_0^T \hat{\theta}_t^2 dt\right)$$

for $\hat{\theta}$, $\hat{W}$ as constructed above; this $\hat{W}$ coincides with the $\hat{\mathbb{Q}}$-Brownian motion induced by the $\mathbb{P}$-Brownian motion W via Girsanov's theorem. The value of the dual problem is

$$-\frac{\Lambda \Phi_0^2 \sqrt{\rho}}{2\coth\left(\sqrt{\rho}T\right)} + \int_0^T \frac{1}{2\Lambda} \frac{(s \wedge \Delta)}{1 + (s \wedge \Delta)\sqrt{\rho}\tanh\left(\sqrt{\rho}(T-s)\right)} ds.$$

From dual to primal solution: Optimal open loop control

Recall from general duality: Optimal primal turnover policy is

$$\hat{\phi}_t = \frac{\mathbb{E}_{\hat{\mathbb{Q}}}\left[S_T | \mathscr{G}_t^{\Delta}\right] - S_t}{\Lambda}, \quad t \in [0, T],$$

when $\hat{\mathbb{Q}}$ is the dual solution. Using the relation between $S = W$ and $\hat{W}$, we find

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left[S_T \middle| \mathscr{G}_t^{\Delta}\right] = \int_0^{(t+\Delta)\wedge T} \left(1 - \int_s^T \hat{l}_{u,s} du\right) d\hat{W}_s - \int_0^T \hat{a}_u du \Phi_0$$

which readily yields an **"open loop"** description of the optimal policy.

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This is clearly explicit, **but** not really expressive . . .

From open loop to closed loop: Getting feedback

Proving that dynamic programming holds in the primal optimization problem, allows us to back out the feedback form of the optimal policy from its open loop description.

A rather tedious computation finally leads to the initially described optimal feedback policy:

$$\hat{\phi}_t = \frac{1}{\Lambda}\left(\bar{S}_t^{\Delta} - S_t\right) + \frac{\Upsilon^{\Delta}(T-t)}{\Delta}\left(\frac{\mu}{\alpha\sigma^2} - \hat{\Phi}_t\right)$$

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Bachelier stock price dynamics, zero interest rate:

$$S_t = s_0 + \mu t + \sigma(\gamma W_t + \sqrt{1-\gamma^2} B_t), \quad t \geq 0$$

for two independent Brownian motions W, B, $\gamma \in [0,1]$.
Information flow: at time t noisy signal on stock price evolution up to time $\tau(t) \geq t$ with $\tau : [0,T] \to [0,T]$ continuous, increasing:
$\rightsquigarrow$ "frontrunning"

$$\mathscr{G}_t^\tau = \sigma(S_s, s \leq t) \vee \sigma(W_s, s \leq \tau(t)), \quad t \geq 0.$$

S is *not* a G^τ-martingale $\rightsquigarrow$ arbitrage opportunities curtailed by price impact because execution price is

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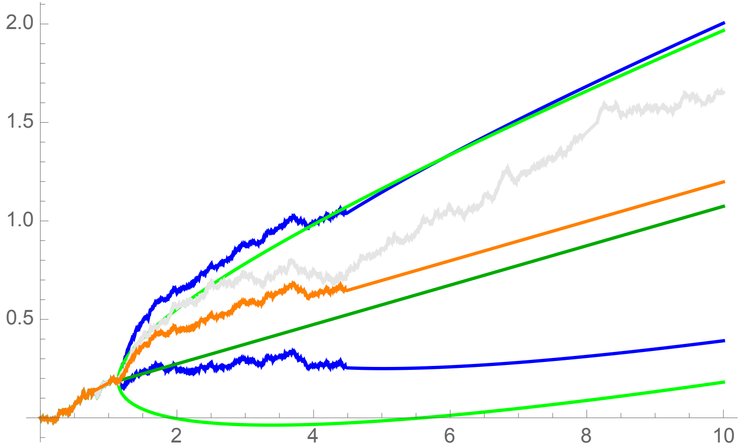
Noisy information in the course of time



As of time $t \in [0, T]$, the standard deviation of S from $\bar{S}^{\tau}$ for $s \geq 0$ into the future is

$$\sigma\sqrt{s(1-\gamma^2)} \text{ if } s \leq \tau(t) - t \text{ and after that } \sigma\sqrt{s - \gamma^2(\tau(t) - t)}$$

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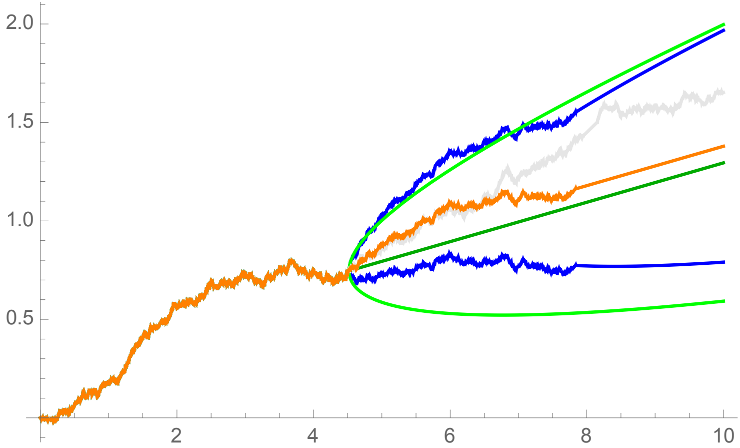

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Main result: Optimal Policy

The optimal policy takes a similar form as in the special case with perfect information corresponding to $\gamma = 1$; the technique of proof still works, but computations and formulae are more involved:

- at any time $t \in [0, T]$, one considers the best guess of the future price evolution

$$\hat{S}^{\tau}_{t+s} = S_t + \mu s + \sigma\gamma(W_{(t+s)\wedge\tau(t)} - W_t), \quad s \geq 0.$$

- the guess $\hat{S}^{\tau}$ is averaged over $s \in [0, \tau[t] - t]$ with some explicitly given kernel $\Upsilon(t, s)$ to yield an indicator $\bar{S}^{\tau}_t$ to assess the earnings potential and hence optimal inside trades: $(\bar{S}^{\tau}_t - S_t)/\Lambda$
- $\Upsilon(t, \tau(t))$ also yields urgency for trading towards Merton ratio $\mu/(\alpha\sigma^2)$
- value of extra information as measured by certainty equivalent can be computed explicitly as well

Conclusion and Outlook

- insider model with dynamic information advantages: frontrunning
- price impact keeps utility maximization viable: super-linear transaction costs
- explicit solution with financial-economic meaning: inside trades combined with standard optimal investment
- no dynamic programming, but dual approach
- convex duality result holds beyond Bachelier setting and frontrunning
- noisy inside information process
- Open: How to allow for more flexible signal quality deterioration (instead of abruptly no info beyond $\tau(t)$)? Stochastically changing signal quality?...

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