Optimal reinsurance via BSDEs in a partially observable contagion model with jump clusters

– Joint work with M. Brachetta, G. Callegaro and C. Sgarra [1] –

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9th colloquium on Backward Stochastic Differential Equations and Mean Field System ,
Annecy, June 27- July 1, 2022

[1] Preprint available soon! []

Contents

Main ingredients

Our stochastic control problem setting:

- **Reinsurance:** insurance companies cannot hedge against every source of risk in the real world[2].
- **Pure jump setting**: the (compound) Poisson process is the essential building block for claims arrival (Cramér-Lundberg model, 1903).
- **Jump clustering:** in catastrophic situations the jumps in the claims arrival process can exhibit clustering feature. We combine Cox with shot-noise intensity and Hawkes processes (with exponential kernel) and we get a shot-noise self-exciting counting process
- **Partial information:** insurer has partial information about claims arrival intensity.

[2] Think of what happened during the last two years!

Related literature

Partial Information:

- Liang, Z., Bayraktar, E. (2014): Optimal reinsurance and investment with unobservable claim size and intensity. *Insurance Math. Econom.* 55.
- Brachetta, M., Ceci, C. (2020): A BSDE-based approach for the optimal reinsurance problem under partial information, *Insurance Math. Econom.* 95

Contagion model:

- Dassios A. , Zhao, H. (2011): A dynamic contagion process, *Adv. Appl. Prob.* 43.
- Cao Y., Landriault D., Li, B. (2020): Optimal reinsurance-investment strategy for a dynamic contagion claim model. *Insurance Math. Econom.* 93.

The Mathematical Model

On $(\Omega, \mathscr{F}, \mathbf{P}; \mathbb{F})$ with $T > 0$ the maturity of a reinsurance contract, introduce the cumulative claim process $C = \{C_t,\ t \in [0,T]\}$:

$$C_t = \sum_{j=1}^{N_t^{(1)}} \underbrace{Z_j^{(1)}}_{claims\ size}, \quad t \in [0,T]$$

where the claims arrival process $N^{(1)}$ is a point process with intensity:

$$\lambda_t = \beta + (\lambda_0 - \beta) e^{-\alpha t} + \underbrace{\sum_{j=1}^{N_t^{(1)}} e^{-\alpha(t - T_j^{(1)})} \ell(\underbrace{Z_j^{(1)}}_{Int-exc.jump})}_{CLUSTERING} + \underbrace{\sum_{j=1}^{N_t^{(2)}} e^{-\alpha(t - T_j^{(2)})} \underbrace{Z_j^{(2)}}_{Ext-exc.jump}}_{CLUSTERING}$$

Assumption

$N^{(2)}$, $\{Z_n^{(1)}\}_{n \ge 1}$ and $\{Z_n^{(2)}\}_{n \ge 1}$ independent.

- $\beta > 0$: constant reversion level, $\lambda_0 > 0$: initial value of λ at $t = 0$;
- $\alpha > 0$: rate of exponential decay;
- $N^{(2)}$: Poisson process with intensity $\rho > 0$;
- $\{T_n^{(1)}\}_{n\geq 1}$: jump times of $N^{(1)}$ (claims are reported!)
- $\{T_n^{(2)}\}_{n\geq 1}$: jump times of $N^{(2)}$, when exogenous/external factors make intensity jump;
- $\{Z_n^{(1)}\}_{n\geq 1}$: claims size, i.i.d.$\mathbb{R}^+$-valued rv with distribution function $F^{(1)} : (0, +\infty) \to [0, 1]$ s.t. $\mathbb{E}[Z^{(1)}] < \infty$;
- $\ell : [0, +\infty) \to [0, +\infty)$ is a measurable function (e.g. $g(z) = az$: self-exciting jumps proportional to claims sizes) s.t. $\mathbb{E}[\ell(Z^{(1)})] < \infty$;
- $\{Z_n^{(2)}\}_{n\geq 1}$: externally-excited jumps, i.i.d. $\mathbb{R}^+$-valued rv with distribution function $F^{(2)} : (0, +\infty) \to [0, 1]$, such that $\mathbb{E}[Z^{(2)}] < \infty$.

Model Construction

The key idea is based on equivalent change probability measure on $(\Omega, \mathscr{F}; \mathbb{F})$. Under $\mathbf{Q}$:

- $N^{(1)}$ and $N^{(2)}$ are Poisson processes with intensity 1 and $\rho > 0$, respectively;
- $\{Z_n^{(1)}\}_{n\geq 1}$ and $\{Z_n^{(2)}\}_{n\geq 1}$ are i.i.d. positive r.v. with distribution functions $F^{(1)}$ and $F^{(2)}$, s.t. $\mathbb{E}^{\mathbf{Q}}[\ell(Z^{(1)})] < \infty$ and $E^{\mathbf{Q}}[Z^{(2)}] < \infty$.
- $N^{(1)}$, $N^{(2)}$, $\{Z_n^{(1)}\}_{n\geq 1}$ and $\{Z_n^{(2)}\}_{n\geq 1}$ are independent of each other.

Let us introduce the integer valued random measures $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$

$$m^{(i)}(\mathrm{d}t, \mathrm{d}z) = \sum_{n\geq 1} \delta_{(T_n^{(i)}, Z_n^{(i)})}(\mathrm{d}t, \mathrm{d}z) \mathbb{1}_{\{T_n^{(i)} < \infty\}}.$$

Under $\mathbf{Q}$: $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$, are independent Poisson measures with compensator measures given respectively by

$$\nu^{(1),\mathbf{Q}}(\mathrm{d}t, \mathrm{d}z) = F^{(1)}(\mathrm{d}z)\mathrm{d}t, \quad \nu^{(2),\mathbf{Q}}(\mathrm{d}t, \mathrm{d}z) = \rho F^{(2)}(\mathrm{d}z)\mathrm{d}t.$$

Proposition

We assume that there exists $\varepsilon > 0$ *s.t.* $\mathbb{E}^{\mathbf{Q}}\left[e^{\varepsilon \ell(Z^{(1)})}\right] < \infty, \quad \mathbb{E}^{\mathbf{Q}}\left[e^{\varepsilon Z^{(2)}}\right] < \infty$. *Let* λ_t *be given in Eq. (1) and*

$$L_t = \mathcal{E}\left(\int_0^t (\lambda_{s^-} - 1)(dN_s^{(1)} - \mathrm{d}s)\right).$$

(here $\mathcal{E}(.)$ *denotes the Doléans-Dade exponential). Then* $\{L_t\}_{t\in[0,T]}$ *is a* $(\mathbf{Q}, \mathbb{F})$*-martingale.*

Define $\mathbf{P}$ via

$$\left.\frac{d\mathbf{P}}{d\mathbf{Q}}\right|_{\mathcal{F}_T} = L_T.$$

By Girsanov Theorem we have the $(\mathbf{P}, \mathbb{F})$-predictable projections measures of the random measure $m^{(i)}(\mathrm{d}t, \mathrm{d}z)$, $i = 1, 2$ are given by:

$$\nu^{(1)}(\mathrm{d}t, \mathrm{d}z) = \lambda_{t^-} F^{(1)}(\mathrm{d}z)\mathrm{d}t, \quad \nu^{(2)}(\mathrm{d}t, \mathrm{d}z) = \rho F^{(2)}(\mathrm{d}z)\mathrm{d}t. \tag{1}$$

In particular, $N^{(1)}$ is a point process with $(\mathbf{P}, \mathbb{F})$-predictable intensity $\{\lambda_{s^-}\}_{s\in[0,T]}$.

Partial information: filtering

- The externally-exciting component in the intensity λ is not observable, the insurance company observes the cumulative claim process C, that is, $\{(T_n^{(1)}, Z_n^{(1)})\}_{n\geq 1}$:
$$\mathbb{H} := \mathbb{F}^C \subset \mathbb{F}.$$
- The insurance company has to estimate λ given $\mathbb{H}$.
- The filter process $\pi = \{\pi_t, t \geq 0\}$ provides the conditional distribution of λ_t given $\mathcal{H}_t$, for any time t: it is the $\mathbb{H}$-càdlàg process process taking values in the space of probability measures on $[0, +\infty)$ such that
$$\pi_t(f) = \mathbb{E}[f(\lambda_t)|\mathcal{H}_t].$$
for any function f s.t. $\mathbb{E}[\int_0^t |f(\lambda_s)|ds] < \infty$.
- $\{\pi_{t^-}(\lambda)\}_{t\geq 0}$, where $\pi_t(\lambda) = \mathbb{E}[\lambda_t|\mathcal{H}_t]$, $t \geq 0$, is the $(\mathbf{P}, \mathbb{H})$-predictable intensity of $N^{(1)}$.

Markov property

Here we work on $(\Omega, \mathscr{F}; \mathbb{F} = \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbf{P})$, with

$$d\lambda_t = \alpha(\beta - \lambda_t)dt + \int_0^{+\infty} \ell(z) m^{(1)}(dt, dz) + \int_0^{+\infty} z m^{(2)}(dt, dz).$$

Proposition

- *The process λ is a $(\mathbf{P}, \mathbb{F})$-Markov process with generator:*

$$\begin{aligned} \mathscr{L} f(\lambda) &= \alpha(\beta - \lambda)\frac{\partial f}{\partial \lambda} + \int_0^{+\infty} [f(\lambda + \ell(z)) - f(\lambda)]\lambda F^{(1)}(dz) \\ &\quad + \int_0^{+\infty} [f(\lambda + z) - f(\lambda)]\rho F^{(2)}(dz). \end{aligned}$$

  $\forall f \in \mathscr{D}(\mathscr{L}), f(\lambda_t) = f(\lambda_0) + \int_0^t \mathscr{L} f(\lambda_s) ds + m_t^f$ *with* m^f *a* $(\mathbf{P}, \mathbb{F})$*-mg.*

- *Under* $\mathbb{E}[(\ell(Z^{(1)})^k] < \infty, \quad \mathbb{E}[(Z^{(2)})^k] < \infty, \quad \forall k = 1, 2, \ldots,$ *then* $\mathbb{E}\left[\int_0^t \lambda_s^k ds\right] < \infty, \quad \forall k = 1, 2, \ldots, \forall t \geq 0$

- *The functions* $f_k(\lambda) := \lambda^k \in \mathscr{D}(\mathscr{L}), \; k = 1, 2, \ldots.$

The filter equation

We apply the innovation method (see e.g. Brémaud, Chap. IV (1981), Ceci-Colaneri (2012)).

Theorem (Kushner-Stratonovich equation)

For any $f \in \mathcal{D}(\mathcal{L})$, the filter is the unique strong solution to

$$\begin{aligned} \pi_t(f) = f(\lambda_0) + \int_0^t \pi_s(\mathcal{L} f) \mathrm{d}s \\ + \int_0^t \int_0^{+\infty} \left(\frac{\pi_{s^-}(f(\lambda + \ell(z))\lambda)}{\pi_{s^-}(\lambda)} - \pi_{s^-}(f) \right) \widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{2}$$

with $\widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z) := m^{(1)}(\mathrm{d}s, \mathrm{d}z) - \pi_{s^-}(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}s$ is the $(\mathbf{P}, \mathbb{H})-$compensated jump measure.

Remark (Recursive structure)

Between two consecutive jumps, for $t \in [T_n^{(1)}, T_{n+1}^{(1)})$ *Eq.* (2) *reads as*

$$d\pi_t(f) = \pi_t(\widetilde{\mathscr{L}} f)dt - [\pi_{t^-}(\lambda f) - \pi_{t^-}(\lambda)\pi_{t^-}(f)]dt,$$

where $\widetilde{\mathscr{L}} f(\lambda) = \alpha(\beta - \lambda)\frac{\partial f}{\partial \lambda} + \int_0^{+\infty}[f(\lambda + z) - f(\lambda)]\rho F^{(2)}(dz)$. *At a jump time* $T_n^{(1)}$*:*

$$\pi_{T_n^{(1)}}(f) = \frac{\pi_{T_n^{(1)-}}(\lambda f(\lambda + \ell(Z_n^{(1)})))}{\pi_{T_n^{(1)-}}(\lambda)}.$$

Remark

The dynamics of $\pi_t(\lambda)$ *is described by a system of countable equations:*

$$\begin{aligned} d\pi_t(\lambda) &= \alpha\left(\beta + \rho\frac{\mathbb{E}[Z^{(2)}]}{\alpha} - \pi_t(\lambda)\right)dt - (\pi_t(\lambda^2) - \pi_t(\lambda)^2)dt \\ &+ \int_0^{+\infty} \ell(z) m^{(1)}(ds, dz) + \frac{\pi_{t^-}(\lambda^2) - \pi_{t^-}(\lambda)^2}{\pi_{t^-}(\lambda)} dN_t^{(1)}. \end{aligned}$$

The Unnormalized Filter

By the Kallianpur-Striebel formula we get that

$$\pi_t(f) = \frac{\mathbb{E}^{\mathbf{Q}}[L_t f(\lambda_t)|\mathscr{H}_t]}{\mathbb{E}^{\mathbf{Q}}[L_t|\mathscr{H}_t]} = \frac{\sigma_t(f)}{\sigma_t(1)}$$

The process $\sigma_t(f) = \mathbb{E}^{\mathbf{Q}}[L_t f(\lambda_t)|\mathscr{H}_t]$ denotes the unnormalized filter and is a finite measure-valued $\mathbb{F}$-càdlàg process.

Proposition (Zakai equation)

For any $f \in \mathscr{D}(\mathscr{L})$, the unnormalized filter is the unique strong solution to the Zakai equation

$$\sigma_t(f)=f(\lambda_0)+\int_0^t \sigma_s(\mathscr{L}f)\mathrm{d}s+\int_0^t\int_0^{+\infty}\Big(\sigma_{s^-}(\lambda f(\lambda+\ell(z)))-\sigma_{s^-}(f)\Big)(m^{(1)}(\mathrm{d}s,\mathrm{d}z)-F^{(1)}(\mathrm{d}z)\mathrm{d}s).$$

As the KS-equation the Zakai equation has a recursive structure. Between two consecutive jump times, for $t \in [T_n, T_{n+1})$

$$d\sigma_t(f) = [\sigma_t(\widetilde{\mathscr{L}} f) - \sigma_t((\lambda - 1) f)]dt$$

and at a jump time $T_n^{(1)}$

$$\sigma_{T_n^{(1)}}(f) = \sigma_{T_{n^-}^{(1)}}(\lambda f(\lambda + \ell(Z_n)).$$

By the linear structure of the Zakai between consecutive jumps we get a computable expression of the filter.

Proposition

The following representation holds, for any $f \in \mathscr{D}(\mathscr{L})$ and $\forall\, n = 1, 2, \dots$

$$\pi_t(f) = \frac{\mathbb{E}[f(\widetilde{\lambda}_t^n) e^{-\int_s^t (\widetilde{\lambda}_u^n - 1) du}]|_{s = T_{n-1}^{(1)}}}{\mathbb{E}[e^{-\int_s^t (\widetilde{\lambda}_u^n - 1) du}]|_{s = T_{n-1}}}, \quad t \in (T_{n-1}^{(1)}, T_n^{(1)})$$

where $\widetilde{\lambda}^n$ follows the dynamics of a Cox with shot noise's intensity

$$d\widetilde{\lambda}_t^n = \alpha(\beta - \widetilde{\lambda}_t^n) dt + \int_0^{+\infty} z m^{(1)}(dt, dz), \quad \mathscr{L}(\widetilde{\lambda}^n_{T_{n-1}^{(1)}}) = \pi_{T_{n-1}^{(1)}}.$$

Reinsurance Contract

The insurer selects a reinsurance strategy $\{u_t\}_{t\in[0,T]}$, $u \in [0, I]$, so that the aggregate losses covered by the insurer are

$$C_t^u = \sum_{j=1}^{N_t^{(1)}} \Phi(Z_j^{(1)}, u_{T_j^{(1)}}) = \int_0^t \int_0^{+\infty} \Phi(z, u_s) m^{(1)}(\mathrm{d}s, \mathrm{d}z), \quad t \in [0, T],$$

(the remaining $C_t - C_t^u$ will be undertaken by the reinsurer).
The retention function $\Phi(z, u)$ satisfies:

- continuous in $u \in [0, I]$ and increasing in both z, u
- $\Phi(z, u) \leq z \; \forall u \in [0, I]$,
- $\Phi(z, 0) = 0$ ($u = 0$: full reinsurance), $\Phi(z, I) = z$ ($u = I$: null reinsurance).

Example

a) Proportional reinsurance: the insurer transfers a percentage $(1 - u)$ of any future loss to the reinsurer, so $I = 1$ and $\Phi(z, u) = uz$, $u \in [0, 1]$.
b) Excess-of-loss: the reinsurer covers all the losses exceeding a threshold u, hence $I = +\infty$ and $\Phi(z, u) = u \wedge z$, $u \in [0, +\infty)$.

The surplus and the reinsurance premium

Under $\{u_t\}_{t\in[0,T]}$, the surplus process R^u of the primary insurer follows:

$$dR_t^u = \left(c_t - q_t^u\right) dt - dC_t^u, \quad R_0^u = R_0 \in \mathbb{R}^+$$

with $\mathbb{H}-$predictable processes

- c_t: insurance premium rate;
- q_t^u: the reinsurance premium rate, $q_t^u(\omega) = q(t,\omega,u)$ satisfying
  - $q(t,\omega,u)$ continuous and decreasing in u, with $\frac{\partial q(t,\omega,u)}{\partial u}$ continuous in u,
  - $q(t,\omega,I) = 0 \quad \forall (t,\omega) \in [0,T] \times \Omega$ (null protection is not expensive)
  - $q(t,\omega,0) > c_t \quad \forall (t,\omega) \in [0,T] \times \Omega$ (no risk-free profit).

Assumption

$$\mathbb{E}\left[\int_0^T q_t^0 \mathrm{d}t\right] < \infty,$$

This implies $\mathbb{E}\left[\int_0^T q_t^u \mathrm{d}t\right] < \infty, \forall u \in \mathscr{U}$ and $\mathbb{E}\left[\int_0^T c_t \mathrm{d}t\right] < \infty.$

The wealth and the problem to solve

The insurance company invests its surplus in a risk-free asset with interest rate $r > 0$, so that the wealth is $X_0^u = R_0 \in \mathbb{R}^+$

$$dX_t^u = dR_t^u + rX_t^u\,dt = \left(c_t - q_t^u\right)dt - \int_0^{+\infty} \Phi(z, u_t)\, m^{(1)}(dt, dz) + rX_t^u\,dt$$

($m^{(1)}(dt, dz)$ has $(\mathbf{P}, \mathbb{H})$-compensator measure $\pi_{t^-}(\lambda) F^{(1)}(\mathrm{d}z)\mathrm{d}t$)) and it aims at solving (with $\eta > 0$ the insurer's risk aversion)

$$\sup_{u \in \mathcal{U}} \mathbb{E}\left[1 - e^{-\eta X_T^u}\right] = 1 - \inf_{u \in \mathcal{U}} \mathbb{E}\left[e^{-\eta X_T^u}\right]$$

Definition (Admissible strategies)

$\mathcal{U}$: all the $[0, I]$-valued, $\mathbb{H}$-predictable processes s.t. $\mathbb{E}\left[e^{-\eta X_T^u}\right] < +\infty$.

Proposition

Assume $\forall a > 0$: $\mathbb{E}\left[e^{a\ell(Z^{(1)})}\right] < \infty$, $\quad \mathbb{E}\left[e^{aZ^{(2)}}\right] < \infty$, $\quad \mathbb{E}\left[e^{a\int_0^T q_t^0\,dt}\right] < \infty$. *Then any process* $[0, I]$*-valued,* $\mathbb{H}$*-predictable process is an admissible control.*

The Value Process

We define, for $\mathscr{U}(t,u) = \left\{\bar{u} \in \mathscr{U} : \bar{u}_s = u_s \text{ a.s., } s \le t \le T\right\}$, the Snell envelope

$$W_t^u = \operatorname*{ess\,inf}_{\bar{u}\in\mathscr{U}(t,u)} \mathbb{E}\left[e^{-\eta X_T^{\bar{u}}} \mid \mathscr{H}_t\right],$$

so that if $\widehat{X}_t^u := e^{-rt} X_t^u$ is the discounted wealth, then

$$W_t^u = e^{-\eta \widehat{X}_t^u e^{rT}} V_t,$$

where V is the value process: $V_t = \operatorname{ess\,inf}_{\bar{u}\in\mathscr{U}_t} \mathbb{E}\left[e^{-\eta e^{rT}(\widehat{X}_T^{\bar{u}} - \widehat{X}_t^{\bar{u}})} \mid \mathscr{H}_t\right] = v(t,\pi_t)$
Moreover,

$$V_t = e^{\eta \widehat{X}_t^{I} e^{rT}} W_t^{I}.$$

Idea: develop a BSDEs characterization for $\{W_t^I\}_{t\ge 0}$ ($u = I$: null reinsurance) to get a complete description of V.
Why BSDEs? Well suited to solve stochastic control problems under partial information (infinite-dimensional filter).

The BSDE approach: road-map

Proposition (Bellman Optimality Principle)

- $\{W_t^u\}_{t\in[0,T]}$ *is a* $(\mathbf{P},\mathbb{H})$*-sub-martingale for any* $u\in\mathcal{U}$*;*
- $\{W_t^{u^*}\}_{t\in[0,T]}$ *is a* $(\mathbf{P},\mathbb{H})$*-martingale if and only if* $u^*\in\mathcal{U}$ *is an optimal control.*

- We prove that (W^I,Θ^{W^I}) is a solution to BSDE (3) under the assumption that there exists an optimal control.
- Verification Theorem states that any solution to BSDE (3) coincides with (W^I,Θ^{W^I}).
- Existence and uniqueness for the BSDE (3).

The BSDE

- $\mathcal{S}^2$: càdlàg $\mathbb{H}$-adapted pr. Y s.t. $\mathbb{E}[(\sup_{t\in[0,T]}|Y_t|)^2] < +\infty$.
- $\widehat{\mathcal{L}}^2$: $[0,+\infty)$-indexed $\mathbb{H}$-predictable random fields Θ s.t. $\mathbb{E}\left[\int_0^T \int_0^{+\infty} \Theta_t^2(z)\pi_{t^-}(\lambda)F^{(1)}(dz)\,dt\right] < +\infty$.

Theorem

Let $u^* \in \mathcal{U}$ be an optimal control. Then, $(W^I, \Theta^{W^I}) \in \mathcal{S}^2 \times \widehat{\mathcal{L}}^2$ solves:

$$W_t^I = \xi - \int_t^T \int_0^{+\infty} \Theta_s^{W^I}(z)\, \widetilde{m}^{(1)}(ds, dz) - \int_t^T \operatorname*{ess\,sup}_{u\in\mathcal{U}} \widetilde{f}(s, W_s^I, \Theta_s^{W^I}(z), u_s)\, ds, \quad (3)$$

with terminal condition $\xi = e^{-\eta X_T^I}$, where

$$\widetilde{f}(t,W_t^I,\Theta_t^{W^I}(\cdot),u_t) = -W_{t-}^I \eta e^{R(T-t)} q_t^u - \int_0^{+\infty} [W_{t-}^I + \Theta_t^{W^I}(z)]\left(e^{-\eta e^{R(T-t)}(z-\Phi(z,u_t))} - 1\right)\pi_{t-}(\lambda)F^{(1)}(dz).$$

Moreover, u^* is such that $\widetilde{f}(t,W_t^I,\Theta_t^{W^I}(\cdot),u_t^*) = \operatorname{ess\,sup}_{u\in\mathcal{U}} \widetilde{f}(t,W_t^I,\Theta_t^{W^I}(\cdot),u_t)$, $\forall\, t \in [0, T]$.

Theorem (Verification Theorem)

Let $(Y,\Theta^Y) \in \mathscr{L}^2 \times \widehat{\mathscr{L}}^2$ *be a solution to the BSDE* (3) *and let* $u^* \in \mathscr{U}$ *be the maximizer of* $\widetilde{f}(t, Y_t, \Theta_t^Y(\cdot), u_t)$. *Then* $Y = W^I$ *and*

$$V_t = e^{\eta \bar{X}_t^I e^{rT}} Y_t \qquad \forall t \in [0,T],$$

and u^* *is an optimal control.*

Theorem (Existence and uniqueness result)

There exists a unique solution to the BSDE (3).

Lemma

$\forall a > 0$

$$\mathbb{E}\left[e^{aN_T^{(1)}}\right] < +\infty \quad \mathbb{E}\left[e^{a\int_0^T \lambda_s ds}\right] < +\infty, \quad \mathbb{E}\left[e^{a\int_0^T \pi_s(\lambda) ds}\right] < +\infty, \quad \mathbb{E}[e^{aC_T}] < +\infty.$$

Existence and uniqueness result.

- $\xi = e^{-\eta X_T^I} \leq e^{\eta e^{rT} C_T}$, hence ξ has finite moments of any order;
- The generator f satisfies a stochastic Lipschitz condition:

$$\left| f(t,y,\theta(\cdot)) - f(t,y',\theta'(\cdot)) \right|^2 \leq \gamma_t |y-y'|^2 + \bar{\gamma}_t \int_0^{+\infty} |\theta(z) - \theta'(z)|^2 \pi_{t^-}(\lambda) F^{(1)}(\mathrm{d}z),$$

with

$$\gamma_t = 3\eta^2 e^{2r(T-t)} (q_t^0)^2 + 3\pi_{t^-}^2(\lambda), \quad \bar{\gamma}_t = 3\pi_{t^-}(\lambda).$$

- We prove that, $\forall \beta > 0$

$$\mathbb{E}\left[e^{\beta \int_0^T \max\{\sqrt{\gamma_t}, \bar{\gamma}_t\} dt} e^{-2\eta X_T^I} \right] < \infty,$$

$$\mathbb{E}\left[\int_0^T e^{\beta \int_0^t \max\{\sqrt{\gamma_s}, \bar{\gamma}_s\} ds} \frac{|f(t,0,0,0)|^2}{\alpha_t^2} dt \right] < \infty.$$

- We can apply Theorem 3.5 in Papapantoleon, Possamaï, Saplaouras EJP (2018).

Proportional reinsurance and expected value principle (EVP)

- Proportional reinsurance: $\Phi(z,u) = zu$, $u \in [0,1]$.
- EVP: the expected revenue covers the expected losses plus a profit

$$\underbrace{q_t^u}_{\text{Reinsurance premium}} = (1 + \underbrace{\theta_R}_{\textbf{Safety loading}})\mathbb{E}[Z^{(1)}]\pi_{t^-}(\lambda)(1-u_t).$$

The optimal control u^* is obtained “explicitly” and

$$u_t^*(\omega) = \begin{cases} 0 & \text{if } \theta_R < \theta_t^F(\omega) \\ 1 & \text{if } \theta_R > \theta_t^N(\omega) \\ \bar{u}(t,\omega,W_{t^-}^I(\omega),\Theta_t^{W^I}(\cdot)(\omega)) & \text{otherwise,} \end{cases} \tag{4}$$

where the stochastic thresholds are:

$$\theta_t^F = \frac{1}{\mathbb{E}[Z^{(1)}]}\int_0^\infty \frac{W_{t^-}^I + \Theta_t^{W^I}(z)}{W_{t^-}^I} z e^{-\eta e^{r(T-t)z}} F^{(1)}(dz) - 1, \quad \theta_t^N = \frac{1}{\mathbb{E}[Z^{(1)}]}\int_0^\infty \frac{W_{t^-}^I + \Theta_t^{W^I}(z)}{W_{t^-}^I} z F^{(1)}(dz) - 1.$$

THANKS FOR YOUR KIND ATTENTION!

Bibliography

- Brachetta M. and Ceci C. Optimal proportional reinsurance and investment for stochastic factor model, *Insurance: Mathematics and Economics*, 2019.
- Brachetta M.and Ceci C. Optimal Excess-of-Loss Reinsurance for Stochastic Factor Risk Models, *RISKS*, 2019.
- Brémaud, P. (1981) Point Processes and Queues. Martingale dynamics. Springer-Verlag.
- C. Ceci and A. Gerardi. A model for high frequency data under partial information: a filtering approach, *International Journal of Theoretical and Applied Finance*, 2006.
- C. Ceci and K. Colaneri. Nonlinear filtering for jump diffusion observations, *Advances in Applied Probability*, 2012.
- C. Ceci and K. Colaneri. The Zakai equation of nonlinear filtering for jump-diffusion observation: existence and uniqueness, *Applied Mathematics and Optimization*, 2014.
- Confortola, F., Fuhrman, M.. Backward stochastic differential equations and optimal control of marked point processes, *SIAM J. Control Optim.*, 2013.
- Confortola, F., Fuhrman, M., and Jacod, J. . Backward stochastic differential equation driven by a marked point process: An elementary approach with an application to optimal control, *The Annals of Applied Probability*, 2016.
- Papapantoleon, A., Possamai, D. and Saplaouras, A.. Existence and uniqueness results for BSDE with jumps: the whole nine yards, *Electron. J. Probab.*, 2018.