Tamed Milstein-type scheme for the McKean-Vlasov SDEs

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Probabilistic set-up

- $\left(\Omega^0, \mathcal{F}^0, P^0\right)$ and $\left(\Omega^1, \mathcal{F}^1, P^1\right)$ are complete probability spaces equipped with the filtrations $\mathbb{F}^0 := (\mathcal{F}^0_t)_{t\geq 0}$ and $\mathbb{F}^1 := (\mathcal{F}^1_t)_{t\geq 0}$, satisfying usual conditions.
- Wiener processes W^0 and W are defined on $\left(\Omega^0, \mathcal{F}^0, P^0\right)$ and $\left(\Omega^1, \mathcal{F}^1, P^1\right)$.
- Define a product space $(\Omega, \mathcal{F}, P)$, where $\Omega = \Omega^0 \times \Omega^1$, $(\mathcal{F}, P)$ is the completion of $\left(\mathcal{F}^0 \otimes \mathcal{F}^1, P^0 \otimes P^1\right)$ and $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is the complete and right-continuous augmentation of $\left(\mathcal{F}^0_t \otimes \mathcal{F}^1_t\right)_{t\geq 0}$.
- For $X : \Omega \to \mathbb{R}^d$, $\mathcal{L}^1(X) : \Omega^0 \ni \omega^0 \mapsto \mathcal{L}(X(\omega^0, \cdot))$ is a well-defined random variable from $\left(\Omega^0, \mathcal{F}^0, P^0\right)$ into $\mathcal{P}_2(\mathbb{R}^d)$ (P^0-a.s.) and is seen as a conditional law of X given $\mathcal{F}^0$ (Lemma 2.4 in Carmona and Delarue (2018b)).
- If the $\mathbb{F}$-adapted unique solution $(X_t)_{0\leq t\leq T}$ of (1) has continuous paths and has uniformly bounded second moment, then one can find a version of $\mathcal{L}^1(X_t)$, for every $t \geq 0$, such that $\left(\mathcal{L}^1(X_t)\right)_{t\geq 0}$ has continuous paths and is $\mathbb{F}^0$-adapted (Lemma 2.5 in Carmona and Delarue (2018b)).
- X_0 of (1) is assumed to be defined on $\left(\Omega^1, \mathcal{F}^1_0, P^1\right)$, which means that only W^0 plays the role of the common noise. In light of Proposition 2.9 in Carmona and Delarue (2018b), $\mathcal{L}^1(X_t)$ is a version of the conditional law of X_t given W^0. For alternative choices of the initial data, we refer to Remark 2.10 in Carmona and Delarue (2018b).

Probabilistic set-up

- $\mathcal{P}_2(\mathbb{R}^d)$: space of probability measures μ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$ and a $\mathcal{L}^2$-Wasserstein metric given by

$$\mathcal{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_1)} \Big(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 \pi(dx, dy) \Big)^{1/2}$$

- $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times m}$ and $\sigma^0 : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times m}$ are measurable functions

Consider McKean–Vlasov SDE,

$$\begin{aligned} X_t = X_0 + \int_0^t b_s\big(X_s, \mathcal{L}^1(X_s)\big) ds + \int_0^t \sigma_s\big(X_s, \mathcal{L}^1(X_s)\big) dW_s \\ + \int_0^t \sigma_s^0\big(X_s, \mathcal{L}^1(X_s)\big) dW_s^0. \end{aligned} \tag{1}$$

almost surely for any $t \in [0, T]$.

Well-posedness and Moment Bound

Assumption 1 (Moment of Initial Value)

$E|X_0|^{p_0} < \infty$ for a fixed constant $p_0 > 2$.

Assumption 2 (Coercivity)

There exists a constant $L > 0$ s. t. $\forall\ t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$2xb_t(x,\mu) + (p_0 - 1)\big|\sigma_t(x,\mu)\big|^2 + (p_0 - 1)\big|\sigma_t^0(x,\mu)\big|^2 \\ \leq L\big\{\big(1 + |x|\big)^2 + \mathcal{W}_2^2\big(\mu, \delta_0\big)\big\}.$$

Assumption 3 (Monotonicity)

There exists a constant $L > 0$ s. t. $\forall\ t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$,

$$2\big(x - \bar{x}\big)\big(b_t(x,\mu) - b_t(\bar{x},\bar{\mu})\big) + \big|\sigma_t(x,\mu) - \sigma_t(\bar{x},\bar{\mu})\big|^2 + \big|\sigma_t^0(x,\mu) - \sigma_t^0(\bar{x},\bar{\mu})\big|^2 \\ \leq L\big\{|x - \bar{x}|^2 + \mathcal{W}_2^2(\mu,\bar{\mu})\big\}.$$

Well-posedness and Moment Bound

Assumption 4 (Continuity)

For every $t \in [0, T]$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $b_t(x, \mu)$ is a continuous function of $x \in \mathbb{R}^d$.

Theorem 1 (Existence, Uniqueness and Moment Bound)

Let Assumptions 1, 2, 3 and 4 be satisfied. Then, there exists a unique strong solution of (1) and the following holds,

$$\sup_{0 \le t \le T} E|X_t|^{p_0} \le K,$$

where $K := K(L, E|X_0|^{p_0}, d, m, m_0) > 0$ is a constant. Moreover,

$$E \sup_{0 \le t \le T} |X_t|^q \le K,$$

for any $q < p_0$.

Propagation of Chaos

- *System of conditional non-interacting particles*:

$$X_t^i = X_0^i + \int_0^t b_s\big(X_s^i, \mathcal{L}^1(X_s^i)\big)\,ds + \int_0^t \sigma_s\big(X_s^i, \mathcal{L}^1(X_s^i)\big)\,dW_s^i + \int_0^t \sigma_s^0\big(X_s^i, \mathcal{L}^1(X_s^i)\big)\,dW_s^0. \text{ (a. s.)} \tag{2}$$

$$P^0\Big[\,\mathcal{L}^1(X_t^i) = \mathcal{L}^1(X_t^1) \text{ for all } t \in [0,T]\Big] = 1.$$

(Proposition 2.11 in Carmona and Delarue (2018b))

- *System of interacting particles*:

$$X_t^{i,N} = X_0^i + \int_0^t b_s\big(X_s^{i,N}, \mu_s^{X,N}\big)\,ds + \int_0^t \sigma_s\big(X_s^{i,N}, \mu_s^{X,N}\big)\,dW_s^i + \int_0^t \sigma_s^0\big(X_s^{i,N}, \mu_s^{X,N}\big)\,dW_s^0. \text{ (a. s.)} \tag{3}$$

$$\mu_t^{X,N}(\cdot) := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}(\cdot)$$

Propagation of chaos

Proposition 1 (Propagation of chaos)

Let Assumptions 1, 2, 3 and 4 be satisfied with $p_0 > 4$. Then,

$$\sup_{i\in\{1,\dots,N\}} \sup_{t\in[0,T]} E\left|X_t^i - X_t^{i,N}\right|^2 \leq K \begin{cases} N^{-1/2}, & \text{if } d < 4, \\ N^{-1/2}\ln(N), & \text{if } d = 4, \\ N^{-2/d}, & \text{if } d > 4, \end{cases}$$

where the constant $K > 0$ does not depend on N.

Tamed Milstein-type scheme

Assumption 5

For some $p_1 > 2$, there exists a constant $L > 0$ such that

$$2\big(x-\bar{x}\big)\big(b_t(x,\mu)-b_t(\bar{x},\bar{\mu})\big)+(p_1-1)\big|\sigma_t(x,\mu)-\sigma_t(\bar{x},\bar{\mu})\big|^2 + (p_1-1)\big|\sigma_t^0(x,\mu)-\sigma_t^0(\bar{x},\bar{\mu})\big|^2 \leq L\big\{|x-\bar{x}|^2+\mathcal{W}_2^2(\mu,\bar{\mu})\big\},$$

for all $t \in [0,T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

Assumption 6

There exist constants $L > 0$ and $\rho > 0$ such that

$$|b_t(x,\mu)-b_t(\bar{x},\bar{\mu})| \leq L\big\{(1+|x|+|\bar{x}|)^{\rho/2}|x-\bar{x}|+\mathcal{W}_2(\mu,\bar{\mu})\big\},$$

for all $t \in [0,T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

Tamed Milstein-type scheme

Remark 1

Due to Assumptions 5 and 6, there exists a constant $K := K(L) > 0$ *such that*

$$|\sigma_t(x,\mu) - \sigma_t(\bar{x},\bar{\mu})| + |\sigma_t^0(x,\mu) - \sigma_t^0(\bar{x},\bar{\mu})| \leq K\big\{(1+|x|+|\bar{x}|)^{\rho/4}|x-\bar{x}| + \mathcal{W}_2(\mu,\bar{\mu})\big\},$$

for all $t \in [0,T]$, $x, \bar{x} \in \mathbb{R}^d$ *and* $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.

Tamed Milstein-type scheme

Assumption 7

There exists a constant $L > 0$ such that

$$\left|b_t(x,\mu) - b_s(x,\mu)\right| + \left|\sigma_t(x,\mu) - \sigma_s(x,\mu)\right| + \left|\sigma_t^0(x,\mu) - \sigma_s^0(x,\mu)\right| \leq L|t-s|,$$

for all $t, s \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

Assumption 8

There exists a constant $L > 0$ such that, for every $j \in \{1, \dots m\}$ and $j' \in \{1, \dots m^0\}$

$$\begin{aligned}
\left|\partial_x b_t(x,\mu) - \partial_x b_t(\bar{x},\bar{\mu})\right| &\leq L\Big\{\big(1+|x|+|\bar{x}|\big)^{\rho/2-1}\big|x-\bar{x}\big| + \mathcal{W}_2\big(\mu,\bar{\mu}\big)\Big\},\\
\left|\partial_x \sigma_t^{(j)}(x,\mu) - \partial_x \sigma_t^{(j)}(\bar{x},\bar{\mu})\right| &\leq L\Big\{\big(1+|x|+|\bar{x}|\big)^{\rho/4-1}\big|x-\bar{x}\big| + \mathcal{W}_2\big(\mu,\bar{\mu}\big)\Big\},\\
\left|\partial_x \sigma_t^{0,(j')}(x,\mu) - \partial_x \sigma_t^{0,(j')}(\bar{x},\bar{\mu})\right| &\leq L\Big\{\big(1+|x|+|\bar{x}|\big)^{\rho/4-1}\big|x-\bar{x}\big| + \mathcal{W}_2\big(\mu,\bar{\mu}\big)\Big\},
\end{aligned}$$

for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2\big(\mathbb{R}^d\big)$.

Tamed Milstein scheme

Assumption 9

There exists a constant $L > 0$ such that, for every $k \in \{1, \dots, d\}$, $j \in \{1, \dots m\}$ and $j' \in \{1, \dots m^0\}$,

$$\begin{aligned}
\left|\partial_\mu b_t^{(k)}(x,\mu,y) - \partial_\mu b_t^{(k)}(\bar{x},\bar{\mu},\bar{y})\right| &\leq L\big\{\big(1+|x|+|\bar{x}|\big)^{\rho/2}\big|x-\bar{x}\big| \\
&\quad + \mathcal{W}_2\big(\mu,\bar{\mu}\big) + |y-\bar{y}|\big\}, \\
\left|\partial_\mu \sigma_t^{(k,j)}(x,\mu,y) - \partial_\mu \sigma_t^{(k,j)}(\bar{x},\bar{\mu},\bar{y})\right| &\leq L\big\{\big(1+|x|+|\bar{x}|\big)^{\rho/4}\big|x-\bar{x}\big| \\
&\quad + \mathcal{W}_2\big(\mu,\bar{\mu}\big) + \big|y-\bar{y}\big|\big\}, \\
\left|\partial_\mu \sigma_t^{0,(k,j')}(x,\mu,y) - \partial_\mu \sigma_t^{0,(k,j')}(\bar{x},\bar{\mu},\bar{y})\right| &\leq L\big\{\big(1+|x|+|\bar{x}|\big)^{\rho/4}\big|x-\bar{x}\big|^2 \\
&\quad + \mathcal{W}_2\big(\mu,\bar{\mu}\big) + \big|y-\bar{y}\big|\big\},
\end{aligned}$$

for all $t \in [0, T]$, $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2\big(\mathbb{R}^d\big)$.

Tamed Milstein-type scheme

Partition $[0, T]$ into n sub-intervals of size $h := T/n$ and define $\kappa_n(t) := \lfloor nt \rfloor / n$ for any $t \in [0, T]$ and $n \in \mathbb{N}$. Further, for every $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$b_t^n(x, \mu) := \frac{b_t(x, \mu)}{1 + n^{-1}|x|^\rho}, \quad \sigma_t^n(x, \mu) := \frac{\sigma_t(x, \mu)}{1 + n^{-1}|x|^\rho}, \quad \sigma_t^{0,n}(x, \mu) := \frac{\sigma_t^0(x, \mu)}{1 + n^{-1}|x|^\rho},$$

Remark 2

Using above equation, one obtains,

$$|b_t^n(x, \mu)| \leq K \min\left\{ n^{1/2}\left(1 + |x| + \mathcal{W}_2(\mu, \delta_0)\right), |b_t(x, \mu)| \right\},$$

$$|\sigma_t^n(x, \mu)| \leq K \min\left\{ n^{1/4}\left(1 + |x| + \mathcal{W}_2(\mu, \delta_0)\right), |\sigma_t(x, \mu)| \right\},$$

$$|\sigma_t^{0,n}(x, \mu)| \leq K \min\left\{ n^{1/4}\left(1 + |x| + \mathcal{W}_2(\mu, \delta_0)\right), |\sigma_t^0(x, \mu)| \right\},$$

$$|\partial_x \sigma_t^{(u,v)}(x, \mu)||\sigma_t^n(x, \mu)| \leq K \min\left\{ n^{1/2}\left(1 + |x| + \mathcal{W}_2(\mu, \delta_0)\right), |\partial_x \sigma_t^{(u,v)}(x, \mu)||\sigma_t(x, \mu)| \right\},$$

Tamed Milstein-type scheme

Remark 2 continued ...

$$
\begin{aligned}
&|\partial_x \sigma_t^{(u,v)}(x,\mu)||\sigma_t^{0,n}(x,\mu)| \\
&\quad \leq K \min\left\{ n^{1/2}\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right), |\partial_x \sigma_t^{(u,v)}(x,\mu)||\sigma_t^{0}(x,\mu)| \right\}, \\
&|\partial_\mu \sigma_t^{(u,v)}(x,\mu,y)||\sigma_t^{n}(x,\mu)| \\
&\quad \leq K \min\left\{ n^{1/4}\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right), |\partial_\mu \sigma_t^{(u,v)}(x,\mu,y)||\sigma_t(x,\mu)| \right\}, \\
&|\partial_\mu \sigma_t^{(u,v)}(x,\mu,y)||\sigma_t^{0,n}(x,\mu)| \\
&\quad \leq K \min\left\{ n^{1/4}\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right), |\partial_\mu \sigma_t^{(u,v)}(x,\mu,y)||\sigma_t^{0}(x,\mu)| \right\}, \\
&|\partial_x \sigma_t^{0,(u,v)}(x,\mu)||\sigma_t^{n}(x,\mu)| \\
&\quad \leq K \min\left\{ n^{1/2}\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right), |\partial_x \sigma_t^{0,(u,v)}(x,\mu)||\sigma_t(x,\mu)| \right\},
\end{aligned}
$$

Tamed Milstein-type scheme

Remark 2 continued . . .

$$
\begin{aligned}
&|\partial_x \sigma_t^{0,(u,v)}(x,\mu)||\sigma_t^{0,n}(x,\mu)| \\
&\quad \leq K \min\left\{ n^{1/2}\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right), |\partial_x \sigma_t^{0,(u,v)}(x,\mu)||\sigma_t^{0}(x,\mu)| \right\}, \\
&|\partial_\mu \sigma_t^{0,(u,v)}(x,\mu,y)||\sigma_t^{n}(x,\mu)| \\
&\quad \leq K \min\left\{ n^{1/4}\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right), |\partial_\mu \sigma_t^{0,(u,v)}(x,\mu,y)||\sigma_t(x,\mu)| \right\}, \\
&|\partial_\mu \sigma_t^{0,(u,v)}(x,\mu,y)||\sigma_t^{0,n}(x,\mu)| \\
&\quad \leq K \min\left\{ n^{1/4}\left(1+|x|+\mathcal{W}_2(\mu,\delta_0)\right), |\partial_\mu \sigma_t^{0,(u,v)}(x,\mu,y)||\sigma_t^{0}(x,\mu)| \right\},
\end{aligned}
$$

for all $t \in [0,T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and for some constant $K > 0$ independent of n.

Tamed Milstein-type scheme

We propose the following tamed Milstein-type scheme,

$$X_t^{i,N,n} = X_0^i + \int_0^t b^n_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)ds + \int_0^t \tilde{\sigma}^n_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)dW_s^i + \int_0^t \tilde{\sigma}^{0,n}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)dW_s^0 \tag{4}$$

almost surely for any $t \in [0, T]$, $i \in \{1, \ldots, N\}$ and $n, N \in \mathbb{N}$. The coefficients $\tilde{\sigma}^n$ and $\tilde{\sigma}^{0,n}$ are defined below. For any $s \in [0, T]$, $i \in \{1, \ldots, N\}$ and $n, N \in \mathbb{N}$,

$$\tilde{\sigma}^n_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) := \sigma^n_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) + \Gamma^{n,\sigma}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) \tag{5}$$

where $\Gamma^{n,\sigma}$ is further expressed as a sum of four matrices, i.e.,

$$\Gamma^{n,\sigma}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) := \Lambda^{n,\sigma\sigma}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) + \Lambda^{n,\sigma\sigma^0}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) + \bar{\Lambda}^{n,\sigma\sigma}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) + \bar{\Lambda}^{n,\sigma\sigma^0}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$

Tamed Milstein-type scheme

$\Lambda^{n,\sigma\sigma}$, $\Lambda^{n,\sigma\sigma^0}$, $\bar{\Lambda}^{n,\sigma\sigma}$ and $\bar{\Lambda}^{n,\sigma\sigma^0}$ are $d \times m$-matrices given below,

$$\Lambda^{n,\sigma\sigma,(u,v)}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$:= \partial_x \sigma^{(u,v)}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) \int_{\kappa_n(s)}^{s} \sigma^{n}_{\kappa_n(r)}\big(X^{i,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}\big) dW^i_r,$$

$$\Lambda^{n,\sigma\sigma^0,(u,v)}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$:= \partial_x \sigma^{(u,v)}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) \int_{\kappa_n(s)}^{s} \sigma^{0,n}_{\kappa_n(r)}\big(X^{i,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}\big) dW^0_r,$$

$$\bar{\Lambda}^{n,\sigma\sigma,(u,v)}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$:= \frac{1}{N}\sum_{j=1}^{N} \partial_\mu \sigma^{(u,v)}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}, X^{j,N,n}_{\kappa_n(s)}\big) \int_{\kappa_n(s)}^{s} \sigma^{n}_{\kappa_n(s)}\big(X^{j,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}\big) dW^j_r,$$

$$\bar{\Lambda}^{n,\sigma\sigma^0,(u,v)}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$:= \frac{1}{N}\sum_{j=1}^{N} \partial_\mu \sigma^{(u,v)}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}, X^{j,N,n}_{\kappa_n(s)}\big) \int_{\kappa_n(s)}^{s} \sigma^{0,n}_{\kappa_n(s)}\big(X^{j,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}\big) dW^0_r.$$

Tamed Milstein-type scheme

For any $s \in [0, T]$, $i \in \{1, \ldots, N\}$ and $n, N \in \mathbb{N}$,

$$\tilde{\sigma}^{0,n}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) := \sigma^{0,n}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) + \Gamma^{n,\sigma^0}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) \quad (6)$$

where Γ^{n,σ^0} is further expressed as a sum of four matrices, i.e.,

$$\Gamma^{n,\sigma^0}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) := \Lambda^{n,\sigma^0\sigma}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) + \Lambda^{n,\sigma^0\sigma^0}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$+ \bar{\Lambda}^{n,\sigma^0\sigma}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) + \bar{\Lambda}^{n,\sigma^0\sigma^0}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$

$\Lambda^{n,\sigma^0\sigma}$, $\Lambda^{n,\sigma^0\sigma^0}$, $\bar{\Lambda}^{n,\sigma^0\sigma}$ and $\bar{\Lambda}^{n,\sigma^0\sigma^0}$ are $d \times m^0$-matrices whose (u, v)-th elements are given in this order by

$$\Lambda^{n,\sigma^0\sigma,(u,v)}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$:= \partial_x \sigma^{0,(u,v)}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) \int_{\kappa_n(s)}^{s} \sigma^{n}_{\kappa_n(r)}\big(X^{i,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}\big) dW^i_r,$$

$$\Lambda^{n,\sigma^0\sigma^0,(u,v)}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$:= \partial_x \sigma^{0,(u,v)}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) \int_{\kappa_n(s)}^{s} \sigma^{0,n}_{\kappa_n(r)}\big(X^{i,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}\big) dW^0_r,$$

Tamed Milstein-type scheme

$$\bar{\Lambda}^{n,\sigma^0\sigma,(u,v)}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$:= \frac{1}{N}\sum_{j=1}^{N} \partial_\mu \sigma^{0,(u,v)}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}, X^{j,N,n}_{\kappa_n(s)}\big) \int_{\kappa_n(s)}^{s} \sigma^{n}_{\kappa_n(s)}\big(X^{j,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}\big) dW^j_r,$$

$$\bar{\Lambda}^{n,\sigma^0\sigma^0,(u,v)}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)$$
$$:= \frac{1}{N}\sum_{j=1}^{N} \partial_\mu \sigma^{0,(u,v)}_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}, X^{j,N,n}_{\kappa_n(s)}\big) \int_{\kappa_n(s)}^{s} \sigma^{0,n}_{\kappa_n(s)}\big(X^{j,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}\big) dW^0_r,$$

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Lemma 2

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for each $i \in \{1, \ldots, N\}$,

$$E\left|\Gamma^{n,\sigma}_{\kappa_n(s)}\left(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right|^{p_0} \leq KE\left(1 + \left|X^{i,N,n}_{\kappa_n(s)}\right|^2\right)^{p_0/2} + KE\mathcal{W}_2^{p_0}\left(\mu^{X,N,n}_{\kappa_n(s)}, \delta_0\right),$$
$$E\left|\Gamma^{n,\sigma^0}_{\kappa_n(s)}\left(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right|^{p_0} \leq KE\left(1 + \left|X^{i,N,n}_{\kappa_n(s)}\right|^2\right)^{p_0/2} + KE\mathcal{W}_2^{p_0}\left(\mu^{X,N,n}_{\kappa_n(s)}, \delta_0\right),$$

for all $s \in [0, T]$ and $n, N \in \mathbb{N}$ where $K > 0$ does not depend on n and N.

Corollary 1

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for each $i \in \{1, \ldots, N\}$,

$$E\left|\tilde{\sigma}^{n}_{\kappa_n(s)}\left(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right|^{p_0} \leq Kn^{\frac{p_0}{4}}\left\{E\left(1 + \left|X^{i,N,n}_{\kappa_n(s)}\right|^2\right)^{p_0/2} + E\mathcal{W}_2^{p_0}\left(\mu^{X,N,n}_{\kappa_n(s)}, \delta_0\right)\right\},$$
$$E\left|\tilde{\sigma}^{0,n}_{\kappa_n(s)}\left(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right|^{p_0} \leq Kn^{\frac{p_0}{4}}\left\{E\left(1 + \left|X^{i,N,n}_{\kappa_n(s)}\right|^2\right)^{p_0/2} + E\mathcal{W}_2^{p_0}\left(\mu^{X,N,n}_{\kappa_n(s)}, \delta_0\right)\right\},$$

for all $s \in [0, T]$ and $n, N \in \mathbb{N}$ where $K > 0$ does not depend on n and N.

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Lemma 3

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for every $i \in \{1, \ldots, N\}$,

$$E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^{p_0} \leq Kn^{-p_0/4}\Big\{E\big(1 + |X_{\kappa_n(s)}^{i,N,n}|^2\big)^{p_0/2} + KE\mathcal{W}_2^{p_0}\big(\mu_{\kappa_n(s)}^{X,N,n}, \delta_0\big)\Big\},$$

for any $t \in [0, T]$ and $n, N \in \mathbb{N}$ where $K > 0$ is independent of N and n.

Lemma 4 (Moment Bounds)

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then,

$$\sup_{i\in\{1,\ldots,N\}} \sup_{t\in[0,T]} E\big(1 + |X_t^{i,N,n}|^2\big)^{p_0/2} \leq K,$$

for any $n, N \in \mathbb{N}$ where $K > 0$ is independent of n & N. Moreover, for any $q < p_0$,

$$\sup_{i\in\{1,\ldots,N\}} E \sup_{t\in[0,T]} \big(1 + |X_t^{i,N,n}|^2\big)^{q/2} \leq K.$$

Tamed Milstein-type scheme

Proof:

By Itô's formula and Cauchy-Schwarz inequality,

$$
\begin{aligned}
E\big(1+|X_t^{i,N,n}|^2\big)^{p_0/2} &\leq E\big(1+|X_0^{i,N,n}|^2\big)^{p_0/2} \\
&+ \frac{p_0}{2} E\int_0^t \big(1+|X_s^{i,N,n}|^2\big)^{p_0/2-1} \Big\{ 2X^{i,N,n}_{\kappa_n(s)} b^n_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) \\
&+ (p_0-1)\big|\tilde{\sigma}^n_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)\big|^2 + (p_0-1)\big|\tilde{\sigma}^{0,n}_{\kappa_n(s)}\big(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big)\big|^2 \Big\} ds \\
&+ p_0 E\int_0^t \big(1+|X_s^{i,N,n}|^2\big)^{p_0/2-1} \big(X^{i,N,n}_s - X^{i,N,n}_{\kappa_n(s)}\big) b^n_{\kappa_n(s)}\big(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\big) ds
\end{aligned}
$$

for any $t \in [0,T]$, $i \in \{1,\ldots,N\}$ and $n, N \in \mathbb{N}$. Observe that $\tilde{\sigma}^n$ and $\tilde{\sigma}^{0,n}$ are sum of two matrices, see equations (5) and (6). Thus, by $|A+B|^2 = |A|^2 + |B|^2 + 2\sum_{u=1}^d \sum_{v=1}^m A^{(u,v)} B^{(u,v)}$ for matrices A and B along with Corollary 1, Lemmas [3, 2] and Gronwall's inequality, the proof is completed (4 pages). □

Tamed Milstein-type scheme

Lemma 5

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for every $i \in \{1, \ldots, N\}$,

$$E\left|\Gamma^{n,\sigma}_{\kappa_n(s)}\left(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right|^p \leq Kn^{-p/2},$$
$$E\left|\Gamma^{n,\sigma^0}_{\kappa_n(s)}\left(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right|^p \leq Kn^{-p/2},$$

for all $p \leq p_0/(\rho/2+1)$, $s \in [0, T]$, and $n, N \in \mathbb{N}$ where $K > 0$ does not depend on n and N.

Corollary 2

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for every $i \in \{1, \ldots, N\}$,

$$E\left|\tilde{\sigma}^{n}_{\kappa_n(s)}\left(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right|^p \leq K,$$
$$E\left|\tilde{\sigma}^{0,n}_{\kappa_n(s)}\left(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right|^p \leq K,$$

for any $p \leq p_0/(\rho/2+1)$, $s \in [0, T]$, $n, N \in \mathbb{N}$ where $K > 0$ does not depend on n and N.

Tamed Milstein-type scheme

Lemma 6

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then,

$$E\left|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}\right|^p \leq Kn^{-p/2},$$

for any $p \leq p_0/(\rho/2+1)$, $s \in [0,T]$ *and* $n, N \in \mathbb{N}$ *where the constant* $K > 0$ *does not depend on* n *and* N.

Lemma 7

Let Assumptions 1, 2, 5, 6, 7, 8 and 9 be satisfied. Then,

$$E\left|\sigma_s\left(X_s^{i,N,n}, \mu_s^{X,N,n}\right) - \tilde{\sigma}_{\kappa_n(s)}^{n}\left(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}\right)\right|^p \leq Kn^{-p},$$
$$E\left|\sigma_s^0\left(X_s^{i,N,n}, \mu_s^{X,N,n}\right) - \tilde{\sigma}_{\kappa_n(s)}^{0,n}\left(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}\right)\right|^p \leq Kn^{-p},$$

for any $p \leq p_0/(2\rho+4)$, $s \in [0,T]$, $i \in \{1,\ldots,N\}$ *and* $n, N \in \mathbb{N}$ *where the constant* $K > 0$ *does not depend on* n *and* N.

Tamed Milstein-type scheme

Lemma 8

Let Assumptions 1, 2, 5, 6, 7, 8 and 9 be satisfied. Then,

$$E\left|b_s(X_s^{i,N,n}, \mu_s^{X,N,n}) - b^n_{\kappa_n(s)}(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)})\right|^p \leq Kn^{-p/2},$$

for any $p \leq p_0/(2\rho+4)$, $s \in [0,T]$, $i \in \{1,\ldots,N\}$ and $n, N \in \mathbb{N}$ where constant $K > 0$ does not depend on n and N.

Lemma 9

Let Assumptions 1, 2, 5, 6, 7, 8 and 9 be satisfied. Then,

$$E\left|X_s^{i,N} - X_s^{i,N,n}\right|^{p-2}\left(X_s^{i,N} - X_s^{i,N,n}\right)\left(b_s\left(X_s^{i,N,n}, \mu_s^{X,N,n}\right) - b^n_{\kappa_n(s)}\left(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}\right)\right)$$
$$\leq Kn^{-p} + K \sup_{i\in\{1,\cdots,N\}} \sup_{r\in[0,s]} E\left|X_r^{i,N} - X_r^{i,N,n}\right|^p,$$

for any $p \leq p_0/(2\rho+4)$, $s \in [0,T]$, $i \in \{1,\ldots,N\}$ and $n, N \in \mathbb{N}$ where constant $K > 0$ does not depend on n and N.

Tamed Milstein-type scheme

Theorem 10 (Rate of convergence)

Let Assumptions 1, 2, 5, 6, 7, 8 and 9 be satisfied. Then, the explicit Milstein-type scheme (4) *converges to the true solution of the interacting particle system* (3) *associated with McKean–Vlasov SDE* (1) *in strong sense with the L^p rate of convergence equal to* 1 *i.e.,*

$$\sup_{i\in\{1,\ldots,N\}} E \sup_{t\in[0,T]} |X_t^{i,N} - X_t^{i,N,n}|^p \leq Kn^{-p},$$

for any $p < \min\{p_1, p_0/(2\rho+4)\}$, where the constant $K > 0$ does not depend on $n, N \in \mathbb{N}$.

Numerical Example

Mean-field stochastic double well dynamics:

$$X_t = X_0 + \int_0^t (X_s - X_s^3 + EX_s)ds + \int_0^t (1 - X_s^2)dW_s$$

$$X_t = X_0 + \int_0^t (X_s - X_s^3 + E^1X_s)ds + \int_0^t (1 - X_s^2)dW_s + \int_0^t (1 - X_s^2)dW_s^0$$

Tamed Milstein scheme:

$$\begin{aligned} X_{(l+1)h}^{i,N,n} =& X_{lh}^{i,N,n} + \Big(\frac{X_{lh}^{i,N,n} - (X_{lh}^{i,N,n})^3}{1 + n^{-1}|X_{lh}^{i,N,n}|^4} + \frac{1}{N}\sum_{i=1}^{N} X_{lh}^{i,N,n}\Big)h \\ &+ \frac{1 - (X_{lh}^{i,N,n})^2}{1 + n^{-1}|X_{lh}^{i,N,n}|^4}\Delta W_{lh}^i + \frac{(X_{lh}^{i,N,n})^3 - X_{lh}^{i,N,n}}{1 + h|X_{lh}^{i,N,n}|^4}((\Delta W_{lh}^i)^2 - h) \end{aligned}$$

$$\begin{aligned} X_{(l+1)h}^{i,N,n} =& X_{lh}^{i,N,n} + \Big(\frac{X_{lh}^{i,N,n} - (X_{lh}^{i,N,n})^3}{1 + n^{-1}|X_{lh}^{i,N,n}|^4} + \frac{1}{N}\sum_{i=1}^{N} X_{lh}^{i,N,n}\Big)h \\ &+ \frac{1 - (X_{lh}^{i,N,n})^2}{1 + n^{-1}|X_{lh}^{i,N,n}|^4}\Delta W_{lh}^i + \frac{(X_{lh}^{i,N,n})^3 - X_{lh}^{i,N,n}}{1 + h|X_{lh}^{i,N,n}|^4}((\Delta W_{lh}^i)^2 + (\Delta W_{lh}^0)^2 - 2h) \end{aligned}$$

Mean-field stochastic double well dynamics

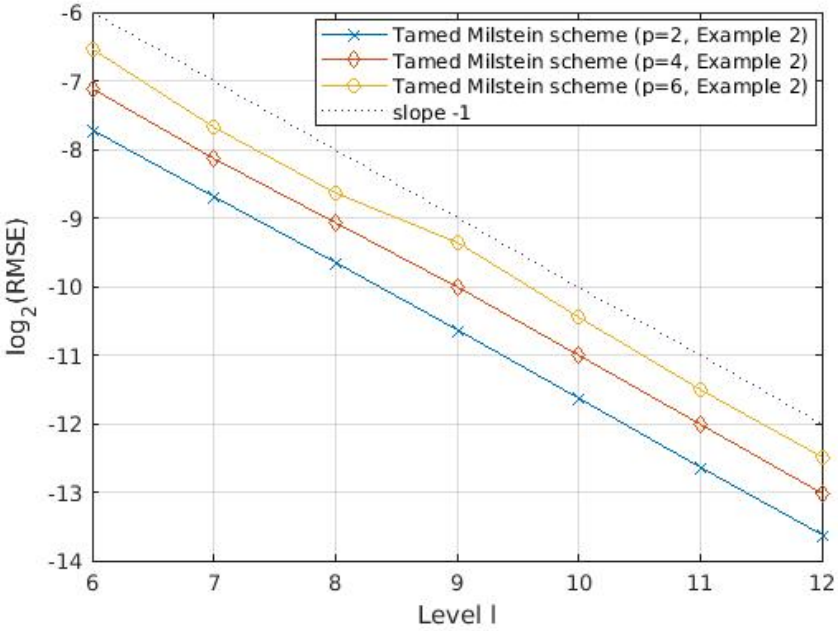

Figure: Double-well

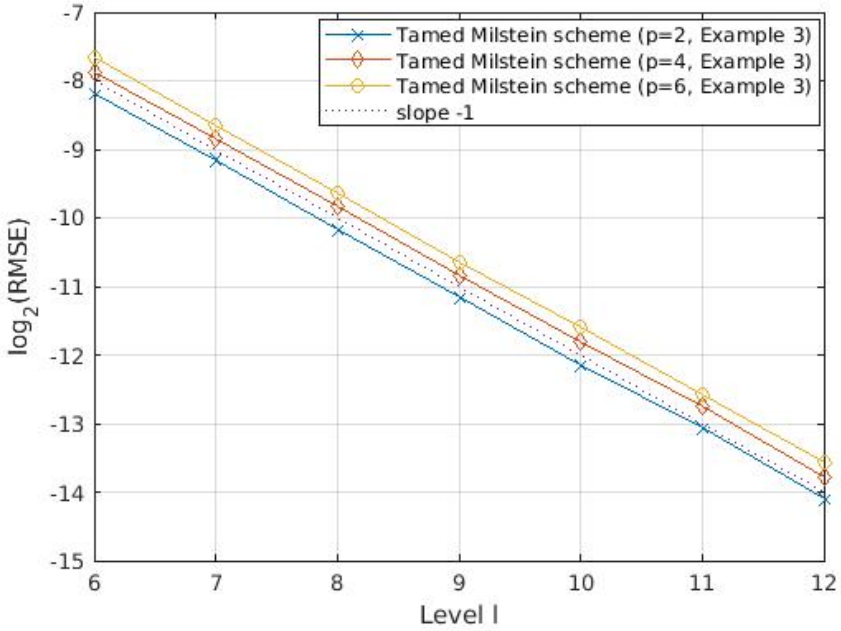

Figure: Double-well with common noise

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