Tamed Milstein-type scheme for the McKean-Vlasov SDEs

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Probabilistic set-up

- $(\Omega^0, \mathcal{F}^0, P^0)$ and $(\Omega^1, \mathcal{F}^1, P^1)$ are complete probability spaces equipped with the filtrations $\mathbb{F}^0 := (\mathcal{F}^0_t)_{t \geq 0}$ and $\mathbb{F}^1 := (\mathcal{F}^1_t)_{t \geq 0}$, satisfying usual conditions.

- Wiener processes $W^0$ and $W$ are defined on $(\Omega^0, \mathcal{F}^0, P^0)$ and $(\Omega^1, \mathcal{F}^1, P^1)$.

- Define a product space $(\Omega, \mathcal{F}, P)$, where $\Omega = \Omega^0 \times \Omega^1$, $(\mathcal{F}, P)$ is the completion of $(\mathcal{F}^0 \otimes \mathcal{F}^1, P^0 \otimes P^1)$ and $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is the complete and right-continuous augmentation of $(\mathcal{F}^0_t \otimes \mathcal{F}^1_t)_{t \geq 0}$.

- For $X : \Omega \to \mathbb{R}^d$, $L^1(X) : \Omega^0 \ni \omega^0 \mapsto L(X(\omega^0, \cdot))$ is a well-defined random variable from $(\Omega^0, \mathcal{F}^0, P^0)$ into $\mathcal{P}_2(\mathbb{R}^d)$ ($P^0$-a.s.) and is seen as a conditional law of $X$ given $\mathcal{F}^0$ (Lemma 2.4 in Carmona and Delarue (2018b)).

- If the $\mathbb{F}$-adapted unique solution $(X_t)_{0 \leq t \leq T}$ of (1) has continuous paths and has uniformly bounded second moment, then one can find a version of $L^1(X_t)$, for every $t \geq 0$, such that $(L^1(X_t))_{t \geq 0}$ has continuous paths and is $\mathbb{F}^0$-adapted (Lemma 2.5 in Carmona and Delarue (2018b)).

- $X_0$ of (1) is assumed to be defined on $(\Omega^1, \mathcal{F}^0_0, P^1)$, which means that only $W^0$ plays the role of the common noise. In light of Proposition 2.9 in Carmona and Delarue (2018b), $L^1(X_t)$ is a version of the conditional law of $X_t$ given $W^0$. For alternative choices of the initial data, we refer to Remark 2.10 in Carmona and Delarue (2018b).
Probabilistic set-up

- $\mathcal{P}_2(\mathbb{R}^d)$: space of probability measures $\mu$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ with $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$ and a $\mathcal{L}^2$-Wasserstein metric given by

  $$W_2(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_1)} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{1/2}$$

- $b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times m}$ and $\sigma^0 : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \times m}$ are measurable functions

Consider McKean–Vlasov SDE,

$$X_t = X_0 + \int_0^t b_s(X_s, \mathcal{L}^1(X_s)) \, ds + \int_0^t \sigma_s(X_s, \mathcal{L}^1(X_s)) \, dW_s$$

$$+ \int_0^t \sigma^0_s(X_s, \mathcal{L}^1(X_s)) \, dW^0_s. \quad (1)$$

almost surely for any $t \in [0, T]$. 
Well-posedness and Moment Bound

Assumption 1 (Moment of Initial Value)

\[ \mathbb{E}|X_0|^{p_0} < \infty \text{ for a fixed constant } p_0 > 2. \]

Assumption 2 (Coercivity)

There exists a constant \( L > 0 \) such that for all \( t \in [0, T] \), \( x \in \mathbb{R}^d \) and \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \),

\[
2xb_t(x, \mu) + (p_0 - 1)|\sigma_t(x, \mu)|^2 + (p_0 - 1)|\sigma^0_t(x, \mu)|^2 \\
\leq L\{(1 + |x|)^2 + \mathcal{W}_2^2(\mu, \delta_0)\}.
\]

Assumption 3 (Monotonicity)

There exists a constant \( L > 0 \) such that for all \( t \in [0, T] \), \( x, \bar{x} \in \mathbb{R}^d \) and \( \mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d) \),

\[
2(x - \bar{x})(b_t(x, \mu) - b_t(\bar{x}, \bar{\mu})) + |\sigma_t(x, \mu) - \sigma_t(\bar{x}, \bar{\mu})|^2 + |\sigma^0_t(x, \mu) - \sigma^0_t(\bar{x}, \bar{\mu})|^2 \\
\leq L\{|x - \bar{x}|^2 + \mathcal{W}_2^2(\mu, \bar{\mu})\}.
\]
Assumption 4 (Continuity)

For every $t \in [0, T]$ and $\mu \in P_2(\mathbb{R}^d)$, $b_t(x, \mu)$ is a continuous function of $x \in \mathbb{R}^d$.

Theorem 1 (Existence, Uniqueness and Moment Bound)

Let Assumptions 1, 2, 3 and 4 be satisfied. Then, there exists a unique strong solution of (1) and the following holds,

$$\sup_{0 \leq t \leq T} E|X_t|^{p_0} \leq K,$$

where $K := K(L, E|X_0|^{p_0}, d, m, m_0) > 0$ is a constant. Moreover,

$$E\sup_{0 \leq t \leq T} |X_t|^q \leq K,$$

for any $q < p_0$. 
Propagation of Chaos

- **System of conditional non-interacting particles:**

\[
X^i_t = X^i_0 + \int_0^t b_s(X^i_s, \mathcal{L}^1(X^i_s)) \, ds + \int_0^t \sigma_s(X^i_s, \mathcal{L}^1(X^i_s)) \, dW^i_s \\
+ \int_0^t \sigma^0_s(X^i_s, \mathcal{L}^1(X^i_s)) \, dW^0_s. \quad (a. s.) \tag{2}
\]

\[
P^0 \left[ \mathcal{L}^1(X^i_t) = \mathcal{L}^1(X^1_t) \text{ for all } t \in [0, T] \right] = 1.
\]

(Proposition 2.11 in Carmona and Delarue (2018b))

- **System of interacting particles:**

\[
X^i,N_t = X^i_0 + \int_0^t b_s(X^i,N_s, \mu^X,N_s) \, ds + \int_0^t \sigma_s(X^i,N_s, \mu^X,N_s) \, dW^i_s \\
+ \int_0^t \sigma^0_s(X^i,N_s, \mu^X,N_s) \, dW^0_s. \quad (a. s.) \tag{3}
\]

\[
\mu^X,N_t(\cdot) := \frac{1}{N} \sum_{i=1}^N \delta_{X^i,N}(\cdot)
\]
Proposition 1 (Propagation of chaos)

Let Assumptions 1, 2, 3 and 4 be satisfied with \( p_0 > 4 \). Then,

\[
\sup_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} \mathbb{E} \left| X_t^i - X_t^{i,N} \right|^2 \leq K \begin{cases} 
N^{-1/2}, & \text{if } d < 4, \\
N^{-1/2} \ln(N), & \text{if } d = 4, \\
N^{-2/d}, & \text{if } d > 4,
\end{cases}
\]

where the constant \( K > 0 \) does not depend on \( N \).
### Assumption 5

*For some $p_1 > 2$, there exists a constant $L > 0$ such that*

\[
2(x - \bar{x})(b_t(x, \mu) - b_t(\bar{x}, \bar{\mu})) + (p_1 - 1)|\sigma_t(x, \mu) - \sigma_t(\bar{x}, \bar{\mu})|^2 \\
+ (p_1 - 1)|\sigma_0^0(x, \mu) - \sigma_0^0(\bar{x}, \bar{\mu})|^2 \leq L\{|x - \bar{x}|^2 + \mathcal{W}_2^2(\mu, \bar{\mu})\},
\]

*for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.*

### Assumption 6

*There exist constants $L > 0$ and $\rho > 0$ such that*

\[
|b_t(x, \mu) - b_t(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/2}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},
\]

*for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.*
Remark 1

Due to Assumptions 5 and 6, there exists a constant $K := K(L) > 0$ such that

$$
|\sigma_t(x, \mu) - \sigma_t(\bar{x}, \bar{\mu})| + |\sigma^0_t(x, \mu) - \sigma^0_t(\bar{x}, \bar{\mu})| \leq K \{(1 + |x| + |\bar{x}|)^{\rho/4} |x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},
$$

for all $t \in [0, T]$, $x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$. 
There exists a constant $L > 0$ such that
\[
|b_t(x, \mu) - b_s(x, \mu)| + |\sigma_t(x, \mu) - \sigma_s(x, \mu)| + |\sigma_0^t(x, \mu) - \sigma_0^s(x, \mu)| \leq L|t - s|,
\]
for all $t, s \in [0, T], \, x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

There exists a constant $L > 0$ such that, for every $j \in \{1, \ldots, m\}$ and $j' \in \{1, \ldots, m^0\}$
\[
|\partial_x b_t(x, \mu) - \partial_x b_t(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/2-1}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},
\]
\[
|\partial_x \sigma_t^{(j)}(x, \mu) - \partial_x \sigma_t^{(j)}(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/4-1}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},
\]
\[
|\partial_x \sigma_t^{0,(j')}^t(x, \mu) - \partial_x \sigma_t^{0,(j')}^t(\bar{x}, \bar{\mu})| \leq L\{(1 + |x| + |\bar{x}|)^{\rho/4-1}|x - \bar{x}| + \mathcal{W}_2(\mu, \bar{\mu})\},
\]
for all $t \in [0, T], \, x, \bar{x} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$.
Tamed Milstein scheme

Assumption 9

There exists a constant $L > 0$ such that, for every $k \in \{1, \ldots, d\}$, $j \in \{1, \ldots, m\}$ and $j' \in \{1, \ldots, m^0\}$,

$$
\begin{align*}
|\partial_\mu b_t^{(k)}(x, \mu, y) - \partial_\mu b_t^{(k)}(\bar{x}, \bar{\mu}, \bar{y})| &\leq L \{ (1 + |x| + |\bar{x}|)^{\rho/2} |x - \bar{x}| \\
&+ \mathcal{W}_2(\mu, \bar{\mu}) + |y - \bar{y}| \}, \\
|\partial_\mu \sigma_t^{(k,j)}(x, \mu, y) - \partial_\mu \sigma_t^{(k,j)}(\bar{x}, \bar{\mu}, \bar{y})| &\leq L \{ (1 + |x| + |\bar{x}|)^{\rho/4} |x - \bar{x}| \\
&+ \mathcal{W}_2(\mu, \bar{\mu}) + |y - \bar{y}| \}, \\
|\partial_\mu \sigma_t^{0,(k,j')} (x, \mu, y) - \partial_\mu \sigma_t^{0,(k,j')} (\bar{x}, \bar{\mu}, \bar{y})| &\leq L \{ (1 + |x| + |\bar{x}|)^{\rho/4} |x - \bar{x}|^2 \\
&+ \mathcal{W}_2(\mu, \bar{\mu}) + |y - \bar{y}| \},
\end{align*}
$$

for all $t \in [0, T]$, $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ and $\mu, \bar{\mu} \in \mathcal{P}_2(\mathbb{R}^d)$. 
Tamed Milstein-type scheme

Partition $[0, T]$ into $n$ sub-intervals of size $h := T / n$ and define $\kappa_n(t) := \lfloor nt \rfloor / n$ for any $t \in [0, T]$ and $n \in \mathbb{N}$. Further, for every $t \in [0, T]$, $x \in \mathbb{R}^d$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$b^n_t(x, \mu) := \frac{b_t(x, \mu)}{1 + n^{-1}|x|^\rho}, \quad \sigma^n_t(x, \mu) := \frac{\sigma_t(x, \mu)}{1 + n^{-1}|x|^\rho}, \quad \sigma^{0,n}_t(x, \mu) := \frac{\sigma^0_t(x, \mu)}{1 + n^{-1}|x|^\rho},$$

**Remark 2**

*Using above equation, one obtains,*

$$|b^n_t(x, \mu)| \leq K \min \left\{ n^{1/2} \left( 1 + |x| + \mathcal{W}_2(\mu, \delta_0) \right), |b_t(x, \mu)| \right\},$$

$$|\sigma^n_t(x, \mu)| \leq K \min \left\{ n^{1/4} \left( 1 + |x| + \mathcal{W}_2(\mu, \delta_0) \right), |\sigma_t(x, \mu)| \right\},$$

$$|\sigma^{0,n}_t(x, \mu)| \leq K \min \left\{ n^{1/4} \left( 1 + |x| + \mathcal{W}_2(\mu, \delta_0) \right), |\sigma^0_t(x, \mu)| \right\},$$

$$|\partial_x \sigma_t^{(u,v)}(x, \mu)||\sigma^n_t(x, \mu)|$$

$$\leq K \min \left\{ n^{1/2} \left( 1 + |x| + \mathcal{W}_2(\mu, \delta_0) \right), |\partial_x \sigma_t^{(u,v)}(x, \mu)||\sigma_t(x, \mu)| \right\},$$
\[
\begin{align*}
|\partial_x \sigma_t^{(u,v)}(x, \mu)||\sigma_t^0(x, \mu)| & \\
& \leq K \min \left\{ n^{1/2} (1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_x \sigma_t^{(u,v)}(x, \mu)||\sigma_t^0(x, \mu)| \right\}, \\
|\partial_\mu \sigma_t^{(u,v)}(x, \mu, y)||\sigma_t^n(x, \mu)| & \\
& \leq K \min \left\{ n^{1/4} (1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_\mu \sigma_t^{(u,v)}(x, \mu, y)||\sigma_t(x, \mu)| \right\}, \\
|\partial_\mu \sigma_t^{(u,v)}(x, \mu, y)||\sigma_t^{0,n}(x, \mu)| & \\
& \leq K \min \left\{ n^{1/4} (1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_\mu \sigma_t^{(u,v)}(x, \mu, y)||\sigma_t^0(x, \mu)| \right\}, \\
|\partial_x \sigma_t^{0,(u,v)}(x, \mu)||\sigma_t^n(x, \mu)| & \\
& \leq K \min \left\{ n^{1/2} (1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_x \sigma_t^{0,(u,v)}(x, \mu)||\sigma_t(x, \mu)| \right\},
\end{align*}
\]
Remark 2 continued . . .

\[
|\partial_x \sigma_t^0, (u, v)(x, \mu)| |\sigma_t^n(x, \mu)| \\
\leq K \min \left\{ n^{1/2} (1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_x \sigma_t^0, (u, v)(x, \mu)| |\sigma_t^0(x, \mu)| \right\}, \\
|\partial_\mu \sigma_t^0, (u, v)(x, \mu, y)| |\sigma_t^n(x, \mu)| \\
\leq K \min \left\{ n^{1/4} (1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_\mu \sigma_t^0, (u, v)(x, \mu, y)| |\sigma_t^0(x, \mu)| \right\}, \\
|\partial_\mu \sigma_t^0, (u, v)(x, \mu, y)| |\sigma_t^n(x, \mu)| \\
\leq K \min \left\{ n^{1/4} (1 + |x| + \mathcal{W}_2(\mu, \delta_0)), |\partial_\mu \sigma_t^0, (u, v)(x, \mu, y)| |\sigma_t^0(x, \mu)| \right\},
\]

for all \( t \in [0, T] \), \( x \in \mathbb{R}^d \) and \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) and for some constant \( K > 0 \) independent of \( n \).
Tamed Milstein-type scheme

We propose the following tamed Milstein-type scheme,

\[
X_t^{i,N,n} = X_0^i + \int_0^t b_{\kappa_n}(s) (X_s^{i,N,n}, \mu_{\kappa_n}(s)) \, ds + \int_0^t \tilde{\sigma}^n_{\kappa_n}(s) (s, X_s^{i,N,n}, \mu_{\kappa_n}(s)) \, dW_s^i + \int_0^t \tilde{\sigma}^0_{\kappa_n}(s) (s, X_s^{i,N,n}, \mu_{\kappa_n}(s)) \, dW_s^0
\]

almost surely for any \( t \in [0, T] \), \( i \in \{1, \ldots, N\} \) and \( n, N \in \mathbb{N} \). The coefficients \( \tilde{\sigma}^n \) and \( \tilde{\sigma}^0,n \) are defined below. For any \( s \in [0, T] \), \( i \in \{1, \ldots, N\} \) and \( n, N \in \mathbb{N} \),

\[
\tilde{\sigma}^n_{\kappa_n}(s) (s, X_s^{i,N,n}, \mu_{\kappa_n}(s)) := \sigma^n_{\kappa_n}(s) (X_s^{i,N,n}, \mu_{\kappa_n}(s)) + \Gamma^{n,\sigma}_{\kappa_n}(s) (s, X_s^{i,N,n}, \mu_{\kappa_n}(s))
\]

where \( \Gamma^{n,\sigma} \) is further expressed as a sum of four matrices, i.e.,

\[
\Gamma^{n,\sigma}_{\kappa_n}(s) (s, X_s^{i,N,n}, \mu_{\kappa_n}(s)) := \Lambda^{n,\sigma\sigma}_{\kappa_n}(s) (s, X_s^{i,N,n}, X_s^{i,N,n}, \mu_{\kappa_n}(s)) + \Lambda^{n,\sigma\sigma^0}_{\kappa_n}(s) (s, X_s^{i,N,n}, X_s^{i,N,n}, \mu_{\kappa_n}(s)) + \bar{\Lambda}^{n,\sigma\sigma}_{\kappa_n}(s) (s, X_s^{i,N,n}, X_s^{i,N,n}, \mu_{\kappa_n}(s)) + \bar{\Lambda}^{n,\sigma\sigma^0}_{\kappa_n}(s) (s, X_s^{i,N,n}, X_s^{i,N,n}, \mu_{\kappa_n}(s))
\]
Tamed Milstein-type scheme

$\Lambda^{n,\sigma}, \Lambda^{n,\sigma^0}, \bar{\Lambda}^{n,\sigma}$ and $\bar{\Lambda}^{n,\sigma^0}$ are $d \times m$-matrices given below,

$\Lambda_n^{\kappa_n(s)}(u, v) \left( s, X_{\kappa_n(s)}, \mu_{\kappa_n(s)} \right)$

\[= \partial_x \sigma_n^{\kappa_n(s)}(X_{\kappa_n(s)}, \mu_{\kappa_n(s)}) \int_{\kappa_n(s)}^{s} \sigma_n^{\kappa_n(r)}(X_{\kappa_n(r)}, \mu_{\kappa_n(r)}) \, dW_r,\]

$\Lambda_n^{\kappa_n(s)}(u, v) \left( s, X_{\kappa_n(s)}, \mu_{\kappa_n(s)} \right)$

\[= \partial_x \sigma_n^{\kappa_n(s)}(X_{\kappa_n(s)}, \mu_{\kappa_n(s)}) \int_{\kappa_n(s)}^{s} \sigma_n^{0,\kappa_n(r)}(X_{\kappa_n(r)}, \mu_{\kappa_n(r)}) \, dW_r,\]

$\bar{\Lambda}_n^{\kappa_n(s)}(u, v) \left( s, X_{\kappa_n(s)}, \mu_{\kappa_n(s)} \right)$

\[= \frac{1}{N} \sum_{j=1}^{N} \partial_{\mu} \sigma_n^{\kappa_n(s)}(X_{\kappa_n(s)}, \mu_{\kappa_n(s)}, X_{\kappa_n(s)}) \int_{\kappa_n(s)}^{s} \sigma_n^{\kappa_n(r)}(X_{\kappa_n(r)}, \mu_{\kappa_n(r)}) \, dW_r,\]

$\bar{\Lambda}_n^{\kappa_n(s)}(u, v) \left( s, X_{\kappa_n(s)}, \mu_{\kappa_n(s)} \right)$

\[= \frac{1}{N} \sum_{j=1}^{N} \partial_{\mu} \sigma_n^{\kappa_n(s)}(X_{\kappa_n(s)}, \mu_{\kappa_n(s)}, X_{\kappa_n(s)}) \int_{\kappa_n(s)}^{s} \sigma_n^{0,\kappa_n(r)}(X_{\kappa_n(r)}, \mu_{\kappa_n(r)}) \, dW_r.\]
Tamed Milstein-type scheme

For any $s \in [0, T]$, $i \in \{1, \ldots, N\}$ and $n, N \in \mathbb{N}$,

$$
\tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}) := \sigma_{\kappa_n(s)}^{0,n}(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}) + \Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)})
$$

(6)

where $\Gamma_{\kappa_n(s)}^{n,\sigma^0}$ is further expressed as a sum of four matrices, i.e.,

$$
\Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}) := \Lambda_{\kappa_n(s)}^{n,\sigma^0}(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}) + \Lambda_{\kappa_n(s)}^{n,\sigma^0}\sigma^0(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)})
$$

$$
+ \Lambda_{\kappa_n(s)}^{n,\sigma^0}(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}) + \Lambda_{\kappa_n(s)}^{n,\sigma^0}(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)})
$$

$\Lambda_{\kappa_n(s)}^{n,\sigma^0}$, $\Lambda_{\kappa_n(s)}^{n,\sigma^0}\sigma^0$, $\Lambda_{\kappa_n(s)}^{n,\sigma^0}\sigma^0$ and $\Lambda_{\kappa_n(s)}^{n,\sigma^0}\sigma^0$ are $d \times m^0$-matrices whose $(u, \nu)$-th elements are given in this order by

$$
\Lambda_{\kappa_n(s)}^{n,\sigma^0}(u, \nu)(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)})
$$

$$
:= \partial_x \sigma_{\kappa_n(s)}^{0, (u, \nu)}(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}) \int_{\kappa_n(s)}^{s} \sigma_{\kappa_n(r)}^{n}(X^{i,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}) dW^i_r,
$$

$$
\Lambda_{\kappa_n(s)}^{n,\sigma^0}(u, \nu)(s, X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)})
$$

$$
:= \partial_x \sigma_{\kappa_n(s)}^{0, (u, \nu)}(X^{i,N,n}_{\kappa_n(s)}, \mu^{X,N,n}_{\kappa_n(s)}) \int_{\kappa_n(s)}^{s} \sigma_{\kappa_n(r)}^{0, n}(X^{i,N,n}_{\kappa_n(r)}, \mu^{X,N,n}_{\kappa_n(r)}) dW^0_r,
$$
\[\bar{\Lambda}_{n,\sigma^0,\sigma^0}(u,v)(s, X_{\kappa_n}(s), \mu_{\kappa_n}(s)) = \frac{1}{N} \sum_{j=1}^{N} \partial_{\mu} \sigma_{\kappa_n}(s)(X_{\kappa_n}(s), \mu_{\kappa_n}(s), X_{\kappa_n}(s)) \int_{\kappa_n(s)}^{s} \sigma_{\kappa_n}(s)(X_{\kappa_n}(r), \mu_{\kappa_n}(r)) dW_r^j,\]

\[\tilde{\Lambda}_{n,\sigma^0,\sigma^0}(u,v)(s, X_{\kappa_n}(s), \mu_{\kappa_n}(s)) = \frac{1}{N} \sum_{j=1}^{N} \partial_{\mu} \sigma_{\kappa_n}(s)(X_{\kappa_n}(s), \mu_{\kappa_n}(s), X_{\kappa_n}(s)) \int_{\kappa_n(s)}^{s} \sigma_{0,\kappa_n}(s)(X_{\kappa_n}(r), X_{\kappa_n}(r)) dW_r^0,\]
Tamed Milstein-type scheme

**Lemma 2**

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for each \(i \in \{1, \ldots, N\}\),

\[
E\left| \Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, X_{\kappa_n(s)}^{N,n}) \right|^{p_0} \leq KE \left( 1 + \left| X_{\kappa_n(s)}^{i,N,n} \right|^2 \right)^{p_0/2} + KE \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}, \delta_0),
\]

\[
E\left| \Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, X_{\kappa_n(s)}^{N,n}) \right|^{p_0} \leq KE \left( 1 + \left| X_{\kappa_n(s)}^{i,N,n} \right|^2 \right)^{p_0/2} + KE \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}, \delta_0),
\]

for all \(s \in [0, T]\) and \(n, N \in \mathbb{N}\) where \(K > 0\) does not depend on \(n\) and \(N\).

**Corollary 1**

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for each \(i \in \{1, \ldots, N\}\),

\[
E\left| \tilde{\sigma}_{\kappa_n(s)}^{n}(s, X_{\kappa_n(s)}^{i,N,n}, X_{\kappa_n(s)}^{N,n}) \right|^{p_0} \leq Kn^{p_0/4} \left\{ E \left( 1 + \left| X_{\kappa_n(s)}^{i,N,n} \right|^2 \right)^{p_0/2} + E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}, \delta_0) \right\},
\]

\[
E\left| \tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, X_{\kappa_n(s)}^{N,n}) \right|^{p_0} \leq Kn^{p_0/4} \left\{ E \left( 1 + \left| X_{\kappa_n(s)}^{i,N,n} \right|^2 \right)^{p_0/2} + E \mathcal{W}_2^{p_0}(\mu_{\kappa_n(s)}, \delta_0) \right\},
\]

for all \(s \in [0, T]\) and \(n, N \in \mathbb{N}\) where \(K > 0\) does not depend on \(n\) and \(N\).
**Tamed Milstein-type scheme**

**Lemma 3**

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for every $i \in \{1, \ldots, N\}$,

$$E|X_{s}^{i,N,n} - X_{\kappa_{n}(s)}^{i,N,n}|_{p_{0}} \leq Kn^{-p_{0}/4}\left\{ E\left(1 + |X_{\kappa_{n}(s)}^{i,N,n}|^{2}\right)^{p_{0}/2} + KE\mathcal{W}_{p_{0}}^{2}\left(\mu_{\kappa_{n}(s)}^{X,N,n}, \delta_{0}\right) \right\},$$

for any $t \in [0, T]$ and $n, N \in \mathbb{N}$ where $K > 0$ is independent of $N$ and $n$.

**Lemma 4 (Moment Bounds)**

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then,

$$\sup_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} E\left(1 + |X_{t}^{i,N,n}|^{2}\right)^{p_{0}/2} \leq K,$$

for any $n, N \in \mathbb{N}$ where $K > 0$ is independent of $n$ & $N$. Moreover, for any $q \leq p_{0}$,

$$\sup_{i \in \{1, \ldots, N\}} E \sup_{t \in [0, T]} \left(1 + |X_{t}^{i,N,n}|^{2}\right)^{q/2} \leq K.$$
Tamed Milstein-type scheme

Proof:

By Itô’s formula and Cauchy-Schwarz inequality,

\[ E \left( 1 + |X_t^{i,N,n}|^2 \right)^{p_0/2} \leq E \left( 1 + |X_0^{i,N,n}|^2 \right)^{p_0/2} \]

\[ + \frac{p_0}{2} E \int_0^t \left( 1 + |X_s^{i,N,n}|^2 \right)^{p_0/2 - 1} \left\{ 2X_{\kappa(s)}^{i,N,n} b_{\kappa(s)}^{n}(X_s^{i,N,n}, \mu_{\kappa(s)}^{X,N,n}) \right\} ds \]

\[ + (p_0 - 1) \left| \tilde{\sigma}_{\kappa(s)}^{n}(s, X_{\kappa(s)}^{i,N,n}, \mu_{\kappa(s)}^{X,N,n}) \right|^2 + (p_0 - 1) \left| \tilde{\sigma}_{\kappa(s)}^{0,n}(s, X_{\kappa(s)}^{i,N,n}, \mu_{\kappa(s)}^{X,N,n}) \right|^2 \] ds

\[ + p_0 E \int_0^t \left( 1 + |X_s^{i,N,n}|^2 \right)^{p_0/2 - 1} \left( X_s^{i,N,n} - X_{\kappa(s)}^{i,N,n} \right) b_{\kappa(s)}^{n}(X_s^{i,N,n}, \mu_{\kappa(s)}^{X,N,n}) ds \]

for any \( t \in [0, T] \), \( i \in \{1, \ldots, N\} \) and \( n, N \in \mathbb{N} \). Observe that \( \tilde{\sigma}^n \) and \( \tilde{\sigma}^{0,n} \) are sum of two matrices, see equations (5) and (6). Thus, by \( |A + B|^2 = |A|^2 + |B|^2 + 2 \sum_{u=1}^d \sum_{v=1}^m A^{(u,v)} B^{(u,v)} \) for matrices \( A \) and \( B \) along with Corollary 1, Lemmas [3, 2] and Gronwall’s inequality, the proof is completed (4 pages). □
Tamed Milstein-type scheme

Lemma 5

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for every $i \in \{1, \ldots, N\}$,

$$E\left| \Gamma_{\kappa_n(s)}^{n,\sigma}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \right|^p \leq Kn^{-p/2},$$

$$E\left| \Gamma_{\kappa_n(s)}^{n,\sigma^0}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \right|^p \leq Kn^{-p/2},$$

for all $p \leq p_0/\left(\rho/2 + 1\right)$, $s \in [0, T]$, and $n, N \in \mathbb{N}$ where $K > 0$ does not depend on $n$ and $N$.

Corollary 2

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then, for every $i \in \{1, \ldots, N\}$,

$$E\left| \tilde{\sigma}_{\kappa_n(s)}^{n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \right|^p \leq K,$$

$$E\left| \tilde{\sigma}_{\kappa_n(s)}^{0,n}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X,N,n}) \right|^p \leq K,$$

for any $p \leq p_0/\left(\rho/2 + 1\right)$, $s \in [0, T]$, $n, N \in \mathbb{N}$ where $K > 0$ does not depend on $n$ and $N$. 
Lemma 6

Let Assumptions 1, 2, 6, 8 and 9 be satisfied. Then,

$$E|X_s^{i,N,n} - X_{\kappa_n(s)}^{i,N,n}|^p \leq Kn^{-p/2},$$

for any $p \leq p_0/(\rho/2 + 1)$, $s \in [0, T]$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on $n$ and $N$.

Lemma 7

Let Assumptions 1, 2, 5, 6, 7, 8 and 9 be satisfied. Then,

$$E|\sigma_s(X_s^{i,N,n}, \mu_s X_s^{i,N,n}) - \tilde{\sigma}_{\kappa_n(s)}(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X_{\kappa_n(s)}})|^p \leq Kn^{-p},$$
$$E|\sigma_s^0(X_s^{i,N,n}, \mu_s X_s^{i,N,n}) - \tilde{\sigma}_{\kappa_n(s)}^0(s, X_{\kappa_n(s)}^{i,N,n}, \mu_{\kappa_n(s)}^{X_{\kappa_n(s)}})|^p \leq Kn^{-p},$$

for any $p \leq p_0/(2\rho + 4)$, $s \in [0, T]$, $i \in \{1, \ldots, N\}$ and $n, N \in \mathbb{N}$ where the constant $K > 0$ does not depend on $n$ and $N$. 
Lemma 8

Let Assumptions 1, 2, 5, 6, 7, 8 and 9 be satisfied. Then,

\[ E\left| b_s(X_s^{i,N,n}, \mu_s X_s^{N,n}) - b_n^{\kappa_n(s)}(X_{\kappa_n(s)}, \mu_{\kappa_n(s)}) \right|^p \leq Kn^{-p/2}, \]

for any \( p \leq p_0/(2\rho + 4) \), \( s \in [0, T] \), \( i \in \{1, \ldots, N\} \) and \( n, N \in \mathbb{N} \) where constant \( K > 0 \) does not depend on \( n \) and \( N \).

Lemma 9

Let Assumptions 1, 2, 5, 6, 7, 8 and 9 be satisfied. Then,

\[ E\left| X_s^{i,N} - X_s^{i,N,n} \right|^{p-2} (X_s^{i,N} - X_s^{i,N,n}) (b_s(X_s^{i,N,n}, \mu_s X_s^{N,n}) - b_n^{\kappa_n(s)}(X_{\kappa_n(s)}, \mu_{\kappa_n(s)})) \]
\[ \leq Kn^{-p} + K \sup_{i \in \{1, \ldots, N\}} \sup_{r \in [0,s]} E\left| X_r^{i,N} - X_r^{i,N,n} \right|^p, \]

for any \( p \leq p_0/(2\rho + 4) \), \( s \in [0, T] \), \( i \in \{1, \ldots, N\} \) and \( n, N \in \mathbb{N} \) where constant \( K > 0 \) does not depend on \( n \) and \( N \).
Tamed Milstein-type scheme

Theorem 10 (Rate of convergence)

Let Assumptions 1, 2, 5, 6, 7, 8 and 9 be satisfied. Then, the explicit Milstein-type scheme (4) converges to the true solution of the interacting particle system (3) associated with McKean–Vlasov SDE (1) in strong sense with the $L^p$ rate of convergence equal to 1 i.e.,

$$\sup_{i \in \{1,\ldots,N\}} E \sup_{t \in [0,T]} |X_t^{i,N} - X_t^{i,N,n}|^p \leq Kn^{-p},$$

for any $p < \min\{p_1, p_0/(2\rho + 4)\}$, where the constant $K > 0$ does not depend on $n, N \in \mathbb{N}$. 
Numerical Example

Mean-field stochastic double well dynamics:

\[ X_t = X_0 + \int_0^t (X_s - X_s^3 + E X_s) ds + \int_0^t (1 - X_s^2) dW_s \]

\[ X_t = X_0 + \int_0^t (X_s - X_s^3 + E^1 X_s) ds + \int_0^t (1 - X_s^2) dW_s + \int_0^t (1 - X_s^2) dW_s^0 \]

Tamed Milstein scheme:

\[ X_{i,N,n}^{i,N,n} = x_{lh}^{i,N,n} + \left( \frac{X_{lh}^{i,N,n} - (X_{lh}^{i,N,n})^3}{1 + n^{-1}|X_{lh}^{i,N,n}|^4} \right) h + \frac{1 - (X_{lh}^{i,N,n})^2}{1 + n^{-1}|X_{lh}^{i,N,n}|^4} \Delta W_{lh}^i + \frac{(X_{lh}^{i,N,n})^3 - X_{lh}^{i,N,n}}{1 + h|X_{lh}^{i,N,n}|^4} \left( ((\Delta W_{lh}^i)^2 - h) \right) \]

\[ X_{(i+1)h}^{i,N,n} = x_{lh}^{i,N,n} + \left( \frac{X_{lh}^{i,N,n} - (X_{lh}^{i,N,n})^3}{1 + n^{-1}|X_{lh}^{i,N,n}|^4} \right) h + \frac{1 - (X_{lh}^{i,N,n})^2}{1 + n^{-1}|X_{lh}^{i,N,n}|^4} \Delta W_{lh}^i + \frac{(X_{lh}^{i,N,n})^3 - X_{lh}^{i,N,n}}{1 + h|X_{lh}^{i,N,n}|^4} \left( ((\Delta W_{lh}^i)^2 + (\Delta W_{lh}^0)^2 - 2h) \right) \]
Mean-field stochastic double well dynamics

Figure: Double-well

Figure: Double-well with common noise


