Numerical approximation of singular FBSDEs: application to carbon markets

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based on joint works with
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Outline

Carbon markets

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  - A central authority set a limit on pollutant emission during a given period. Allowances are allocated to participating installations (via auctioning).
  - The total amount of allowances is the aggregated *cap*.
  - At the end of the period, each participating installation has to surrender an allowance for each unit of emission or pay a penalty.
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- China, whose carbon emissions make up approximately one quarter of the global total, has launched a national emissions trading scheme in July 2021 (with various pilot schemes already running)

EUA price (*tradingeconomics.com*)

Euros per tCO_2 (compare with China ETS price: 8.4 euros/tCO_2 on 1 April 2022)

Main features

• Model based on FBSDEs see e.g. Carmona, Delarue, Espinosa & Touzi (2013), Carmona & Delarue (2013), Howison & Schwarz (2015), C.-Chotai-Crisan (2020)

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- Three main processes on one period $[0, T]$.
  1. The spot allowance price Y: we assume that the market is frictionless and arbitrage-free and that there is a probability such that $(e^{-rt}Y_t)_{0 \le t \le T}$ is a martingale, namely

$$\mathrm{d}Y_t = rY_t\mathrm{d}t + Z_t\mathrm{d}W_t$$

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2. Auxiliary process P:

$$\mathrm{d}P_t = b(P_t)\mathrm{d}t + \sigma(P_t)\mathrm{d}W_t$$

   Represent state variables that trigger the emission process (Electricity price or demand & fuel prices etc.) Fundamentals that are linked to goods emitting CO_2.

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3. Emission process E: cumulative process with impact from the allowance price

$$\mathrm{d}E_t = \mu(P_t, Y_t)\mathrm{d}t$$

   $\hookrightarrow$ μ is decreasing in Y to take into account feedback of the allowance price

Associated singular FBSDE

- System of Equations: $0 \leq t \leq T$

$$
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\mathrm{d}P_t &= b(P_t)\,\mathrm{d}t + \sigma(P_t)\,\mathrm{d}W_t, && \text{(forward)}\\
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- The terminal condition for the Allowance price Y. There is a cap Λ on the total emission set by the regulator
  1. If non-compliance i.e. $E_T > \Lambda$ then the penalty ρ is paid so $Y_T = \rho$
  2. If compliance i.e. $E_T < \Lambda$ then the Allowance is worth nothing (Emission regulation stops at the end of the period) so $Y_T = 0$

  $\hookrightarrow$ $Y_T = \phi(E_T) := \rho \mathbf{1}_{\{E_T > \Lambda\}}$ and $Y_t = e^{-r(T-t)}\mathbb{E}[Y_T | \mathcal{F}_t]$

Results for one-period model

- Carmona and Delarue (2013), there exists a unique solution to:

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with terminal condition: $\phi(E_T) = \rho \mathbf{1}_{\{E_T > \Lambda\}} \le Y_T \le \rho \mathbf{1}_{\{E_T \ge \Lambda\}} =: \phi_+(E_T)$.
There exists a decoupling field s.t. $Y_t = v(t, P_t, E_t)$ for $t < T$.

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- The decoupling field v is the "entropy" solution to

$$\partial_t v + \mu(p, v)\partial_e v + \mathcal{L}_p v = rv, \quad \text{and } v(T, e, p) = \phi(e)$$

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- Multi-period model (finite or infinite number of period): Dan Crisan's talk!

Outline

A numerical Toy model

- One-period Toy model ($r = 0$), dimension $d + 1$, $\sigma > 0$:

$$\mathrm{d}P_t = \sigma \mathrm{d}W_t,\ \mathrm{d}E_t = \left(\frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} P_t^\ell - Y_t \right) \mathrm{d}t,\ \mathrm{d}Y_t = Z_t \cdot \mathrm{d}W_t,$$

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- Reduced to one dimension via $v(t,p,e) = u(t, e + (T-t)\frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} p^{\ell})$ with

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$\hookrightarrow$ Particle method associated to scalar conservation law can be used (Bossy, Jourdain, Tallay...) to get a proxy for the true solution: $e \mapsto v(0,0,e)$.

Toward a probabilistic scheme

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- We test "classical" FBSDE methods ($d = 1$):
  - Bender-Zhang Method (decoupling via Picard iteration+regression)
  - Delarue-Menozzi Method (probabilistic layer method, decoupling via predictor method)
  - Deep FBSDE solver (E-Han-Jentzen) learning method+DNN

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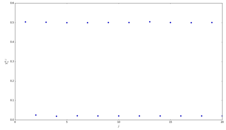
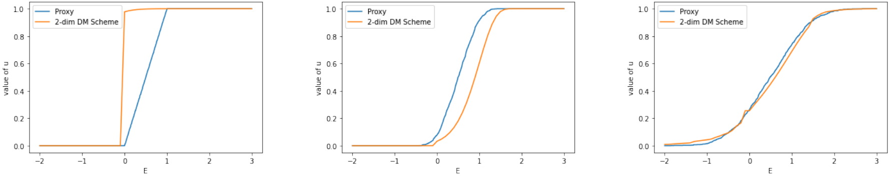
• Results for Bender-Zhang Method to compute Y_0:



Iterations do not converge (regularisation would help but difficult to tune)

Other methods

- Results for Delarue-Menozzi scheme:



(a) $\sigma = 0.01$ (b) $\sigma = 0.3$ (c) $\sigma = 1.0$

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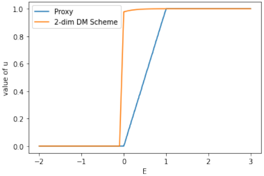
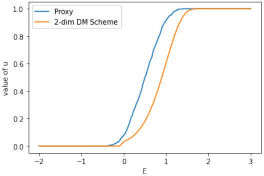
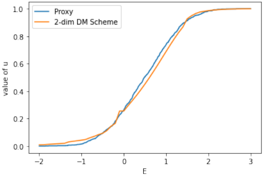


(g) $\sigma = 0.01$

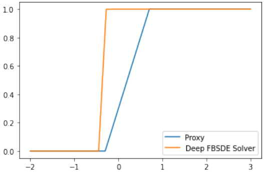


(h) $\sigma = 0.3$



(i) $\sigma = 1.0$

- Results for the deep FBSDE solver (learning error is small):



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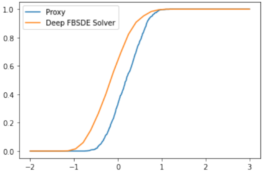

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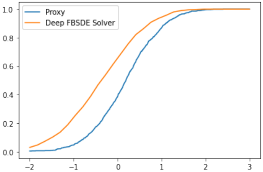

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- Results:
  1. we prove the convergence of the splitting scheme with rate $\frac{1}{2}$, in the setting of existence and uniqueness for singular FBSDEs.
  2. we test the splitting scheme using various approximations of the transport operator and the diffusion part (regression).

Theoretical splitting

- Recall the pde "satisfied" by the decoupling field ($r = 0$):

$$\partial_t v + \mu(p, v)\partial_e v + \mathcal{L}_p v = 0, \quad \text{and } v(T, e, p) = \phi(e)$$

and we know $v \in \mathcal{K} := \{(p, e) \mapsto \psi(p, e) : |\partial_p \psi| \leq L, \partial_e \psi \geq 0\}$

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  - Transport step: $\mathcal{T}_h(\psi) = \tilde{v}(0, \cdot)$ with $\tilde{v}$ solution to

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  - Diffusion step: $\mathcal{D}_h(\psi) = \bar{v}(0, \cdot)$ with $\bar{v}(t, p, e) = \mathbb{E}\big[\psi(P_h^{t,p}, e)\big]$

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- We obtain, under minimal Lipschitz assumption + structure conditions

$$\int_{\mathbb{R}} |v(0, p, e) - u_0^\pi(p, e)| \mathrm{d}e \le C(1 + |p|^2) N^{-\frac{1}{2}},$$

Convergence 'proof'

Classically, we study stability of the scheme and truncation error.

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  - We want to control $\int |V_0| \mathrm{d}e$: study $t \mapsto \int |V_t \partial_e \tilde{E}_t| \mathrm{d}e$.
    $\hookrightarrow$ we get terms like

$$\int \partial_y \mu(\tilde{v}(t,p,\tilde{E}_t), p) \partial_e \tilde{v}() \partial_e \tilde{E} \mathrm{d}e = \int \partial_e M(\tilde{v}(t,p,\tilde{E}_t), p)) \mathrm{d}e$$

that we can control...

Outline

Backward scheme

- For the E-direction:
  - J be a positive integer and $\mathfrak{E} = (e_j)_{1 \le j \le J}$ a discrete grid of $\mathbb{R}$.
  - $\mathcal{T}_h^{\mathfrak{E}}$ an approximation of the transport operator on $\mathfrak{E}$:

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- Scheme:
  1. For $n = N$, $\Gamma^j_N = \phi(\widehat{P}^{\pi}_{t_N}, e_j)$ for $1 \le j \le J$.
  2. Then, compute for $n < N$

$$\tilde{\Gamma}^j_n = \mathbb{E}\Big[\Gamma^j_{n+1} | \widehat{P}^{\pi}_{t_n}\Big] \;\text{ for all } 1 \le j \le J,$$
$$\Gamma_n = \mathcal{T}_h^{\mathfrak{E}}(\widehat{P}^{\pi}_{t_n}, \tilde{\Gamma}_n)\,.$$

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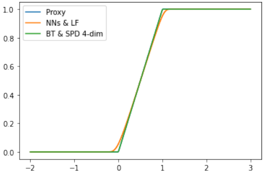
- We test also a multiplicative model:

$$\mathrm{d}P^\ell_t = \mu P^\ell_t \mathrm{d}t + \sigma P^\ell_t \mathrm{d}W^\ell_t,\ P^\ell_0 = 1,\ \text{and } \mathrm{d}E_t = \tilde{\mu}(Y_t, P_t)\mathrm{d}t$$

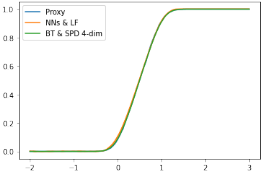
with $\tilde{\mu}(y,p) = \left(\prod_{\ell=1}^d p^\ell\right)^{\frac{1}{\sqrt{d}}} e^{-\theta y}$, for some $\theta > 0$ and $\phi(p,e) = \mathbf{1}_{\{e \geq 0\}}$.

$\hookrightarrow$ *it can be reduced to a 2-dimensional model*!

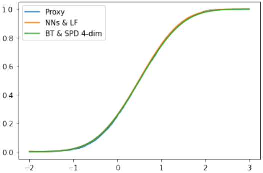
Some numerics on the Toy model



(m) $\sigma = 0.01$



(n) $\sigma = 0.3$

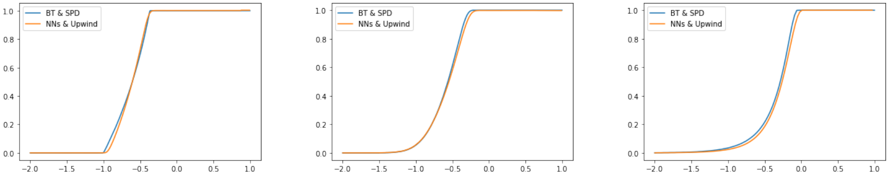


(o) $\sigma = 1.0$

Figure: Linear Toy Model: Comparison of the three methods:
- Neural Nets & Lax-Friedrichs (NN&LF) with $d = 10$
- an alternative scheme (BT&SPD) with $d = 4$
- The Proxy solution given by particle method.

Lax-Friedrichs scheme implemented with discretization of space $J = 1500, 1000, 500$, for $\sigma = 0.01, 0.3, 1$ respectively and number of time step $K = 30$. The number of time step for the splitting is $N = 64$. For $BT\&SPD$, the number of particles is $M = 3500$ and the number of time steps $N = 20$.

On the multiplicative model



(a) $\sigma = 0.01$ (b) $\sigma = 0.3$ (c) $\sigma = 1.0$

Figure: A multiplicative model in dimension $d = 10$. Comparison of two methods:
- Neural nets & Upwind scheme
- the alternative scheme on equivalent 4-dimensional model (BT&SPD).

The Upwind scheme used discretization of space $J = 100, 400, 500$ respectively for $\sigma = 1, 0.3, 0.01$ and number of time step $K = 20$. The number of time step for the splitting is $N = 32$. For $BT\&SPD$, the number of particles is $M = 3500$, and the number of time steps $N = 20$.