Numerical approximation of singular FBSDEs: application to carbon markets

J-F Chassagneux (Université Paris Cité)
based on joint works with
M. Yang (Université Paris Cité)

9th Colloquium on BSDEs and MF system
Annecy, 2022, June 27 - July 1
Introduction

Emission Trading Scheme
FBSDEs approach
One-period model

Approximation schemes

Classical FBSDE schemes
Splitting scheme

Numerical results

Numerical schemes
Examples
Outline

Introduction
- Emission Trading Scheme
- FBSDEs approach
- One-period model

Approximation schemes
- Classical FBSDE schemes
- Splitting scheme

Numerical results
- Numerical schemes
- Examples
Carbon markets

- Carbon dioxide (CO₂) emission have a negative impact on the environment.
- Carbon markets are implemented to ‘price’ this and hopefully carbon emission reduction could be achieved.
Carbon markets

- Carbon dioxide (CO₂) emission have a negative impact on the environment.
- Carbon markets are implemented to ‘price’ this and hopefully carbon emission reduction could be achieved.
- Since 2005, the EU has had its own emissions trading system (ETS): an example of *cap-and-trade scheme*
  - A central authority set a limit on pollutant emission during a given period. Allowances are allocated to participating installations (via auctioning).
  - The total amount of allowances is the aggregated *cap*.
  - At the end of the period, each participating installation has to surrender an allowance for each unit of emission or pay a penalty.
  - During the period, participants can trade the allowances.
Carbon markets

- Carbon dioxide (CO₂) emission have a negative impact on the environment.
- Carbon markets are implemented to ‘price’ this and hopefully carbon emission reduction could be achieved.
- Since 2005, the EU has had its own emissions trading system (ETS): an example of *cap-and-trade scheme*
  - A central authority set a limit on pollutant emission during a given period. Allowances are allocated to participating installations (via auctioning).
  - The total amount of allowances is the aggregated *cap*.
  - At the end of the period, each participating installation has to surrender an allowance for each unit of emission or pay a penalty.
  - During the period, participants can trade the allowances.
- China, whose carbon emissions make up approximately one quarter of the global total, has launched a national emissions trading scheme in July 2021 (with various pilot schemes already running).
EUA price (tradingeconomics.com)

EU Carbon Permits (EUR) 78.23  +1.75 (+2.29%)

Euros per $tCO_2$ (compare with China ETS price: 8.4 euros/$tCO_2$ on 1 April 2022)
Main features

- Model based on FBSDEs see e.g. Carmona, Delarue, Espinosa & Touzi (2013), Carmona & Delarue (2013), Howison & Schwarz (2015), C.-Chotai-Crisan (2020)
Main features

- Model based on FBSDEs see e.g. Carmona, Delarue, Espinosa & Touzi (2013), Carmona & Delarue (2013), Howison & Schwarz (2015), C.-Chotai-Crisan (2020)
- Three main processes on one period $[0, T]$. 

1. The spot allowance price $Y_t$: we assume that the market is frictionless and arbitrage-free and that there is a probability such that $(e^{-rt}Y_t)_0 \leq t \leq T$ is a martingale, namely
   \[
   dY_t = rY_t dt + Z_t dW_t
   \]
   $r$ is the interest rate, $Z_t$ is a square integrable process.

2. Auxiliary process $P_t$:
   \[
   dP_t = b(P_t) dt + \sigma(P_t) dW_t
   \]
   Represent state variables that trigger the emission process (Electricity price or demand & fuel prices etc.) Fundamentals that are linked to goods emitting CO₂.

3. Emission process $E_t$: cumulative process with impact from the allowance price
   \[
   dE_t = \mu(P_t, Y_t) dt
   \]
   $\mu$ is decreasing in $Y_t$ to take into account feedback of the allowance price.
Main features

- Model based on FBSDEs see e.g. Carmona, Delarue, Espinosa & Touzi (2013), Carmona & Delarue (2013), Howison & Schwarz (2015), C.-Chotai-Crisan (2020)
- Three main processes on one period \([0, T]\).
  1. The spot allowance price \(Y\): we assume that the market is frictionless and arbitrage-free and that there is a probability such that \((e^{-rt}Y_t)_{0 \leq t \leq T}\) is a martingale, namely
     \[
     dY_t = rY_t dt + Z_t dW_t
     \]
     \(r\) is the interest rate, \(Z\) is a square integrable process.
Main features

- Model based on FBSDEs see e.g. Carmona, Delarue, Espinosa & Touzi (2013), Carmona & Delarue (2013), Howison & Schwarz (2015), C.-Chotai-Crisan (2020)
- Three main processes on one period $[0, T]$.
  1. The spot allowance price $Y$: we assume that the market is frictionless and arbitrage-free and that there is a probability such that $(e^{-rt}Y_t)_{0\leq t\leq T}$ is a martingale, namely
     \[ dY_t = rY_t\,dt + Z_t\,dW_t \]
     $r$ is the interest rate, $Z$ is a square integrable process.
  2. Auxiliary process $P$:
     \[ dP_t = b(P_t)\,dt + \sigma(P_t)\,dW_t \]
     Represent state variables that trigger the emission process (Electricity price or demand & fuel prices etc.) Fundamentals that are linked to goods emitting CO$_2$. 

J-F Chassagneux
Main features

- Model based on FBSDEs see e.g. Carmona, Delarue, Espinosa & Touzi (2013), Carmona & Delarue (2013), Howison & Schwarz (2015), C.-Chotai-Crisan (2020)
- Three main processes on one period $[0, T]$.

1. The spot allowance price $Y$: we assume that the market is frictionless and arbitrage-free and that there is a probability such that $(e^{-rt}Y_t)_{0 \leq t \leq T}$ is a martingale, namely

$$dY_t = rY_t dt + Z_t dW_t$$

$r$ is the interest rate, $Z$ is a square integrable process.

2. Auxiliary process $P$:

$$dP_t = b(P_t)dt + \sigma(P_t)dW_t$$

Represent state variables that trigger the emission process (Electricity price or demand & fuel prices etc.) Fundamentals that are linked to goods emitting CO$_2$.

3. Emission process $E$: cumulative process with impact from the allowance price

$$dE_t = \mu(P_t, Y_t)dt$$

$\mu$ is decreasing in $Y$ to take into account feedback of the allowance price
System of Equations: $0 \leq t \leq T$

\[
\begin{align*}
\frac{dP_t}{dt} &= b(P_t) dt + \sigma(P_t) dW_t, \quad \text{(forward)} \\
\frac{dE_t}{dt} &= \mu(P_t, Y_t) dt, \quad \text{(forward coupled)} \\
\frac{dY_t}{dt} &= rY_t dt + Z_t dW_t, \quad \text{(backward)}
\end{align*}
\]

$E_0, P_0$ is known but $Y_0$ is unknown!
Associated singular FBSDE

▶ System of Equations: $0 \leq t \leq T$

\[
\begin{align*}
    dP_t &= b(P_t) \, dt + \sigma(P_t) \, dW_t, \quad \text{(forward)} \\
    dE_t &= \mu(P_t, Y_t) \, dt, \quad \text{(forward coupled)} \\
    dY_t &= rY_t \, dt + Z_t \, dW_t, \quad \text{(backward)}
\end{align*}
\]

$E_0, P_0$ is known but $Y_0$ is unknown!

▶ The terminal condition for the Allowance price $Y$. There is a cap $\Lambda$ on the total emission set by the regulator

1. If non-compliance i.e. $E_T > \Lambda$ then the penalty $\rho$ is paid so $Y_T = \rho$
2. If compliance i.e. $E_T < \Lambda$ then the Allowance is worth nothing (Emission regulation stops at the end of the period) so $Y_T = 0$

$\iff Y_T = \phi(E_T) := \rho 1_{\{E_T > \Lambda\}}$ and $Y_t = e^{-r(T-t)} \mathbb{E}[Y_T | \mathcal{F}_t]$
Results for one-period model

Carmona and Delarue (2013), there exists a unique solution to:

\[ dP_t = b(P_t) \, dt + \sigma(P_t) \, dW_t, \]
\[ dE_t = \mu(P_t, Y_t) \, dt, \]
\[ dY_t = rY_t \, dt + Z_t \, dW_t, \]

with terminal condition: \( \phi(E_T) = \rho 1_{\{E_T > \Lambda\}} \leq Y_T \leq \rho 1_{\{E_T \geq \Lambda\}} =: \phi^+(E_T) \).

There exists a decoupling field s.t. \( Y_t = v(t, P_t, E_t) \) for \( t < T \).
Results for one-period model

- Carmona and Delarue (2013), there exists a unique solution to:

\[
\begin{align*}
\frac{d}{dt}P_t &= b(P_t) dt + \sigma(P_t) dW_t, \\
\frac{d}{dt}E_t &= \mu(P_t, Y_t) dt, \\
\frac{d}{dt}Y_t &= rY_t dt + Z_t dW_t,
\end{align*}
\]

with terminal condition: \( \phi(E_T) = \rho 1_{\{E_T > \Lambda\}} \leq Y_T \leq \rho 1_{\{E_T \geq \Lambda\}} =: \phi_+(E_T) \).

There exists a decoupling field s.t. \( Y_t = \nu(t, P_t, E_t) \) for \( t < T \).

- The decoupling field \( \nu \) is the “entropy” solution to

\[
\partial_t \nu + \mu(p, \nu) \partial_e \nu + \mathcal{L}_p \nu = rv, \quad \text{and} \quad \nu(T, e, p) = \phi(e)
\]
Results for one-period model

- Carmona and Delarue (2013), there exists a unique solution to:

\[
\begin{align*}
    dP_t &= b(P_t) \, dt + \sigma(P_t) \, dW_t, \\
    dE_t &= \mu(P_t, Y_t) \, dt, \\
    dY_t &= rY_t \, dt + Z_t \, dW_t,
\end{align*}
\]

with terminal condition: \( \phi(E_T) = \rho \mathbf{1}_{\{E_T > \Lambda\}} \leq Y_T \leq \rho \mathbf{1}_{\{E_T \geq \Lambda\}} =: \phi_+(E_T) \).

There exists a decoupling field s.t. \( Y_t = \nu(t, P_t, E_t) \) for \( t < T \).

- The decoupling field \( \nu \) is the “entropy” solution to

\[
\partial_t \nu + \mu(p, \nu) \partial_e \nu + \mathcal{L}_p \nu = r \nu, \quad \text{and} \quad \nu(T, e, p) = \phi(e)
\]

\( \leftrightarrow \) \( \nu \) is Lipschitz in \( p \) and non decreasing in \( e \).
Results for one-period model

Carmona and Delarue (2013), there exists a unique solution to:

\[\begin{align*}
    dP_t &= b(P_t) \, dt + \sigma(P_t) \, dW_t, \\
    dE_t &= \mu(P_t, Y_t) \, dt, \\
    dY_t &= rY_t \, dt + Z_t \, dW_t,
\end{align*}\]

with terminal condition: \(\phi(E_T) = \rho \mathbb{1}_{\{E_T > \Lambda\}} \leq Y_T \leq \rho \mathbb{1}_{\{E_T \geq \Lambda\}} =: \phi_+(E_T)\).

There exists a decoupling field s.t. \(Y_t = \nu(t, P_t, E_t)\) for \(t < T\).

The decoupling field \(\nu\) is the “entropy” solution to

\[\begin{align*}
    \partial_t \nu + \mu(p, \nu) \partial_e \nu + \mathcal{L}_p \nu &= rv, \quad \text{and} \quad \nu(T, e, p) = \phi(e) \\
    \partial_e \nu \text{ is Lipschitz in } p \text{ and non decreasing in } e, \\
    \partial_e \nu \text{ explodes at } T \text{ near } \Lambda, \text{ we only know } |\partial_e \nu(t, p, e)| \leq \frac{C}{T-t}
\end{align*}\]
Results for one-period model

- Carmona and Delarue (2013), there exists a unique solution to:

\[
\begin{align*}
\frac{dP_t}{dt} &= b(P_t) \, dt + \sigma(P_t) \, dW_t, \\
\frac{dE_t}{dt} &= \mu(P_t, Y_t) \, dt, \\
\frac{dY_t}{dt} &= rY_t \, dt + Z_t \, dW_t,
\end{align*}
\]

with terminal condition: \( \phi(E_T) = \rho 1_{\{E_T > \Lambda\}} \leq Y_T \leq \rho 1_{\{E_T \geq \Lambda\}} =: \phi^+(E_T) \).

There exists a decoupling field s.t. \( Y_t = \nu(t, P_t, E_t) \) for \( t < T \).

- The decoupling field \( \nu \) is the “entropy” solution to

\[
\partial_t \nu + \mu(p, \nu) \partial_e \nu + \mathcal{L}_p \nu = rv, \quad \text{and} \quad \nu(T, e, p) = \phi(e)
\]

\( \nu \) is Lipschitz in \( p \) and non-decreasing in \( e \).

\( \partial_e \nu \) explodes at \( T \) near \( \Lambda \), we only know \( |\partial_e \nu(t, p, e)| \leq \frac{C}{T-t} \).

Set \( \mu(p, \nu) = -\nu \) and \( \mathcal{L}_p = 0, r = 0 \). One obtains a ‘backward’ inviscid Burgers equation...
Results for one-period model

- Carmona and Delarue (2013), there exists a unique solution to:

\[
\begin{align*}
    dP_t &= b(P_t) \, dt + \sigma(P_t) \, dW_t, \\
    dE_t &= \mu(P_t, Y_t) \, dt, \\
    dY_t &= rY_t \, dt + Z_t \, dW_t,
\end{align*}
\]

with terminal condition: \( \phi(E_T) = \rho_1 \{ E_T > \Lambda \} \leq Y_T \leq \rho_1 \{ E_T \geq \Lambda \} =: \phi_+(E_T) \).

There exists a decoupling field s.t. \( Y_t = \nu(t, P_t, E_t) \) for \( t < T \).

- The decoupling field \( \nu \) is the “entropy” solution to

\[
\partial_t \nu + \mu(p, \nu) \partial_e \nu + \mathcal{L}_p \nu = rv, \quad \text{and} \quad \nu(T, e, p) = \phi(e)
\]

\( \nu \) is Lipschitz in \( p \) and non decreasing in \( e \).

\( \partial_e \nu \) explodes at \( T \) near \( \Lambda \), we only know \( |\partial_e \nu(t, p, e)| \leq \frac{C}{T-t} \).

Set \( \mu(p, \nu) = -\nu \) and \( \mathcal{L}_p = 0, r = 0 \). One obtains a ‘backward’ inviscid Burgers equation...

- Multi-period model (finite or infinite number of period): Dan Crisan’s talk!

© J-F Chassagneux
Outline

Introduction
  Emission Trading Scheme
  FBSDEs approach
  One-period model

Approximation schemes
  Classical FBSDE schemes
  Splitting scheme

Numerical results
  Numerical schemes
  Examples
A numerical Toy model

- One-period Toy model \((r = 0)\), dimension \(d + 1\), \(\sigma > 0\):

\[
\begin{align*}
\text{d}P_t &= \sigma \text{d}W_t, \\
\text{d}E_t &= \left(\frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} P_\ell^t - Y_t\right) \text{d}t, \\
\text{d}Y_t &= Z_t \cdot \text{d}W_t,
\end{align*}
\]

and "\(Y_T = 1_{[1, \infty)}(E_T)\)".
A numerical Toy model

- One-period Toy model ($r = 0$), dimension $d + 1$, $\sigma > 0$:

\[
\begin{align*}
\mathrm{d}P_t &= \sigma \mathrm{d}W_t, \\
\mathrm{d}E_t &= \left( \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} P_{t}^{\ell} - Y_{t} \right) \mathrm{d}t, \\
\mathrm{d}Y_t &= Z_t \cdot \mathrm{d}W_t,
\end{align*}
\]

and "\(Y_T = 1_{[1,\infty)}(E_T)\)".

- The quasi-linear pde associated is:

\[
\partial_t v + \left( \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} P_{\ell} - v \right) \partial_e v + \frac{\sigma^2}{2} \sum_{\ell=1}^{d} \partial_{\ell\ell}^2 v = 0
\]
A numerical Toy model

- One-period Toy model \((r = 0)\), dimension \(d + 1\), \(\sigma > 0\):

\[
\begin{align*}
    dP_t &= \sigma dW_t, \\
    dE_t &= \left( \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} P_{t}^{\ell} - Y_t \right) dt, \\
    dY_t &= Z_t \cdot dW_t,
\end{align*}
\]

and “\(Y_T = 1_{[1,\infty)}(E_T)\)”.

- The quasi-linear pde associated is:

\[
\begin{align*}
    \partial_t v + \left( \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} p^{\ell} - v \right) \partial_e v + \frac{\sigma^2}{2} \sum_{\ell=1}^{d} \partial_{p\ell p\ell} v &= 0
\end{align*}
\]

- Reduced to one dimension via \(v(t, p, e) = u(t, e + (T - t) \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} p^{\ell})\) with

\[
\begin{align*}
    \partial_t u - u \partial_\xi u + \frac{\sigma^2(T - t)^2}{2} \partial_{\xi\xi} u &= 0 \text{ and } u(T, \xi) = 1_{\{\xi \geq 1\}}
\end{align*}
\]
A numerical Toy model

- One-period Toy model ($r = 0$), dimension $d + 1$, $\sigma > 0$:
  \[
  \begin{aligned}
  dP_t &= \sigma dW_t, \\
  dE_t &= \left( \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} P_t^\ell - Y_t \right) dt, \\
  dY_t &= Z_t \cdot dW_t,
  \end{aligned}
  \]
  and "$Y_T = 1_{[1, \infty)}(E_T)$".

- The quasi-linear pde associated is:
  \[
  \partial_t v + \left( \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} p^\ell - v \right) \partial_e v + \frac{\sigma^2}{2} \sum_{\ell=1}^{d} \partial_{p^\ell p^\ell}^2 v = 0
  \]

- Reduced to one dimension via $v(t, p, e) = u(t, e + (T - t) \frac{1}{\sqrt{d}} \sum_{\ell=1}^{d} p^\ell)$ with
  \[
  \partial_t u - u \partial_\xi u + \frac{\sigma^2(T - t)^2}{2} \partial_{\xi\xi}^2 u = 0 \text{ and } u(T, \xi) = 1_{\{\xi \geq 1\}}
  \]

  → Particle method associated to scalar conservation law can be used (Bossy, Jourdain, Tallay...) to get a proxy for the true solution: $e \mapsto v(0, 0, e)$. 
Toward a probabilistic scheme

- When $d = 1$ (dimension of $P$), PDE methods can be used see e.g. Howison-Schwarz (2012), C.-Chotai-Crisan (2020).
Toward a probabilistic scheme

- When $d = 1$ (dimension of $P$), PDE methods can be used see e.g. Howison-Schwarz (2012), C.-Chotai-Crisan (2020).

- However in applications $d$ is larger... ($d \geq 3$ e.g. for electricity generation sector producer: demand and two fuel prices).
Toward a probabilistic scheme

- When $d = 1$ (dimension of $P$), PDE methods can be used see e.g. Howison-Schwarz (2012), C.-Chotai-Crisan (2020).

- However in applications $d$ is larger... ($d \geq 3$ e.g. for electricity generation sector producer: demand and two fuel prices).

- We test “classical” FBSDE methods ($d = 1$):
  - Bender-Zhang Method (decoupling via Picard iteration+regression)
  - Delarue-Menozzi Method (probabilistic layer method, decoupling via predictor method)
  - Deep FBSDE solver (E-Han-Jentzen) learning method+DNN
Toward a probabilistic scheme

- When \( d = 1 \) (dimension of \( P \)), PDE methods can be used see e.g. Howison-Schwarz (2012), C.-Chotai-Crisan (2020).
- However in applications \( d \) is larger... \( (d \geq 3 \text{ e.g. for electricity generation sector producer: demand and two fuel prices}) \).
- We test “classical” FBSDE methods \((d = 1)\):
  - Bender-Zhang Method (decoupling via Picard iteration+regression)
  - Delarue-Menozzi Method (probabilistic layer method, decoupling via predictor method)
  - Deep FBSDE solver (E-Han-Jentzen) learning method+DNN

- Results for Bender-Zhang Method to compute \( Y_0 \):

  Iterations do not converge (regularisation would help but difficult to tune)
Other methods

- Results for Delarue-Menozzi scheme:

  (a) $\sigma = 0.01$

  (b) $\sigma = 0.3$

  (c) $\sigma = 1.0$
Other methods

- Results for Delarue-Menozzi scheme:
  - (g) $\sigma = 0.01$
  - (h) $\sigma = 0.3$
  - (i) $\sigma = 1.0$

- Results for the deep FBSDE solver (learning error is small):
  - (j) $\sigma = 0.01$
  - (k) $\sigma = 0.3$
  - (l) $\sigma = 1.0$
A splitting scheme

- The numerical methods above fail to capture the correct weak solution.
- This comes from the degeneracy in $e$ and the irregularity of the final condition. Many PDE methods would work, however the dimension of $P$ is too 'big' in applications.
The numerical methods above fail to capture the correct weak solution.

This comes from the degeneracy in $e$ and the irregularity of the final condition. Many PDE methods would work, however the dimension of $P$ is too 'big' in applications.

We use a splitting scheme to treat both problem: on a time grid $\pi = (t_n)_{0 \leq n \leq N}$ we iterate a transport operator (fixing $p$) and a diffusion operator (fixing $e$).
A splitting scheme

- The numerical methods above fail to capture the correct weak solution.
- This comes from the degeneracy in $e$ and the irregularity of the final condition. Many PDE methods would work, however the dimension of $P$ is too ’big’ in applications.
- We use a splitting scheme to treat both problem: on a time grid $\pi = (t_n)_{0 \leq n \leq N}$ we iterate a transport operator (fixing $p$) and a diffusion operator (fixing $e$).
- The transport part is implemented using methods designed for discontinuous solution.
A splitting scheme

- The numerical methods above fail to capture the correct weak solution.
- This comes from the degeneracy in $e$ and the irregularity of the final condition. Many PDE methods would work, however the dimension of $P$ is too 'big' in applications.
- We use a splitting scheme to treat both problem: on a time grid $\pi = (t_n)_{0 \leq n \leq N}$ we iterate a transport operator (fixing $p$) and a diffusion operator (fixing $e$).
- The transport part is implemented using methods designed for discontinuous solution.
- Results:
  1. we prove the convergence of the splitting scheme with rate $\frac{1}{2}$, in the setting of existence and uniqueness for singular FBSDEs.
  2. we test the splitting scheme using various approximations of the transport operator and the diffusion part (regression).
Theoretical splitting

Recall the pde “satisfied” by the decoupling field $(r = 0)$:

$$ \partial_t v + \mu(p, v) \partial_e v + L_p v = 0, \quad \text{and} \quad v(T, e, p) = \phi(e) $$

and we know $v \in \mathcal{K} := \{(p, e) \mapsto \psi(p, e) : |\partial_p \psi| \leq L, \partial_e \psi \geq 0\}$
Theoretical splitting

- Recall the pde “satisfied” by the decoupling field \((r = 0)\):

\[
\partial_t \nu + \mu(p, \nu) \partial_e \nu + \mathcal{L}_p \nu = 0, \quad \text{and} \quad \nu(T, e, p) = \phi(e)
\]

and we know \(\nu \in \mathcal{K} := \{(p, e) \mapsto \psi(p, e) : |\partial_p \psi| \leq L, \partial_e \psi \geq 0\}\)

- We define operators from \((0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto \text{op}_h \in \mathcal{K}\).
Theoretical splitting

Recall the pde “satisfied” by the decoupling field \( r = 0 \):
\[
\partial_t v + \mu(p, v) \partial_e v + \mathcal{L}_p v = 0, \quad \text{and} \quad v(T, e, p) = \phi(e)
\]
and we know \( v \in \mathcal{K} := \{(p, e) \mapsto \psi(p, e) : |\partial_p \psi| \leq L, \partial_e \psi \geq 0\} \)

We define operators from \((0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto \text{op}_h \in \mathcal{K}\).
- Transport step: \( T_h(\psi) = \tilde{v}(0, \cdot) \) with \( \tilde{v} \) solution to
\[
\partial_t w + \mu(w, p) \partial_e w = 0 \quad \forall p \in \mathbb{R}^d
\]
\[ \rightarrow \] better to consider the scalar conservation law to avoid issues at \( T \)
Theoretical splitting

- Recall the pde “satisfied” by the decoupling field \((r = 0)\):
  \[
  \partial_t \nu + \mu(p, \nu) \partial_e \nu + L_p \nu = 0, \quad \text{and } \nu(T, e, p) = \phi(e)
  \]
  and we know \(\nu \in K := \{(p, e) \mapsto \psi(p, e) : |\partial_p \psi| \leq L, \partial_e \psi \geq 0\}\)

- We define operators from \((0, \infty) \times K \ni (h, \psi) \mapsto \text{op}_h \in K\).
  - Transport step: \(T_h(\psi) = \tilde{\nu}(0, \cdot)\) with \(\tilde{\nu}\) solution to
    \[
    \partial_t \nu + \mu(w, p) \partial_e \nu = 0 \quad \forall p \in \mathbb{R}^d
    \]
    \(\leftrightarrow\) better to consider the scalar conservation law to avoid issues at \(T\)
  - Diffusion step: \(D_h(\psi) = \bar{\nu}(0, \cdot)\) with \(\bar{\nu}(t, p, e) = \mathbb{E}[\psi(P_{h,P}^t, e)]\)
Theoretical splitting

Recall the pde “satisfied” by the decoupling field \((r = 0)\):

\[
\partial_t v + \mu(p, v)\partial_e v + \mathcal{L}_p v = 0, \quad \text{and} \quad v(T, e, p) = \phi(e)
\]

and we know \(v \in \mathcal{K} := \{(p, e) \mapsto \psi(p, e) : |\partial_p \psi| \leq L, \partial_e \psi \geq 0\}\)

We define operators from \((0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto \text{op}_h \in \mathcal{K}\).

- Transport step: \(T_h(\psi) = \tilde{v}(0, \cdot)\) with \(\tilde{v}\) solution to

\[
\partial_t w + \mu(w, p)\partial_e w = 0 \quad \forall p \in \mathbb{R}^d
\]

\(\leftrightarrow\) better to consider the scalar conservation law to avoid issues at \(T\)

- Diffusion step: \(D_h(\psi) = \bar{v}(0, \cdot)\) with \(\bar{v}(t, p, e) = \mathbb{E}[\psi(P_h^{t, p}, e)]\)

- Then one sets: \(S_h = T_h \circ D_h\)
Theoretical splitting

- Recall the pde “satisfied” by the decoupling field ($r = 0$):

$$\partial_t v + \mu(p, v)\partial_e v + L_p v = 0, \quad \text{and} \quad v(T, e, p) = \phi(e)$$

and we know $v \in \mathcal{K} := \{(p, e) \mapsto \psi(p, e) : |\partial_p \psi| \leq L, \partial_e \psi \geq 0\}$

- We define operators from $(0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto \text{op}_h \in \mathcal{K}$.
  - Transport step: $T_h(\psi) = \tilde{v}(0, \cdot)$ with $\tilde{v}$ solution to
    $$\partial_t w + \mu(w, p)\partial_e w = 0 \quad \forall p \in \mathbb{R}^d$$
    $\leftarrow$ better to consider the scalar conservation law to avoid issues at $T$
  - Diffusion step: $D_h(\psi) = \bar{v}(0, \cdot)$ with $\bar{v}(t, p, e) = \mathbb{E}[\psi(P_{t}^{\cdot, p}, e)]$
  - Then one sets: $S_h = T_h \circ D_h$

- Splitting scheme $(u_{\pi}^n)$, solution to the backward induction on $\pi = (t_n)_{0 \leq n \leq N}$:
  - for $n = N$, set $u_{\pi}^N := \phi$,
  - for $n < N$, $u_{\pi}^n = S_{t_{n+1} - t_n}(u_{\pi}^{n+1})$. 

J-F Chassagneux
Theoretical splitting

- Recall the pde “satisfied” by the decoupling field \((r = 0)\):
  \[
  \partial_t v + \mu(p, v) \partial_e v + L_p v = 0, \quad \text{and} \quad v(T, e, p) = \phi(e)
  \]
  and we know \(v \in \mathcal{K} := \{(p, e) \mapsto \psi(p, e) : |\partial_p \psi| \leq L, \partial_e \psi \geq 0\}\)

- We define operators from \((0, \infty) \times \mathcal{K} \ni (h, \psi) \mapsto \text{op}_h \in \mathcal{K}\).
  - Transport step: \(T_h(\psi) = \tilde{v}(0, \cdot)\) with \(\tilde{v}\) solution to
    \[
    \partial_t w + \mu(w, p) \partial_e w = 0 \quad \forall p \in \mathbb{R}^d
    \]
    \(\leftarrow\) better to consider the scalar conservation law to avoid issues at \(T\)
  - Diffusion step: \(D_h(\psi) = \bar{v}(0, \cdot)\) with \(\bar{v}(t, p, e) = \mathbb{E}[\psi(P^{t, p}_{h}, e)]\)
  - Then one sets: \(S_h = T_h \circ D_h\)

- Splitting scheme \((u^n_\pi)\), solution to the backward induction on \(\pi = (t_n)_{0 \leq n \leq N}\):
  - for \(n = N\), set \(u^\pi_N := \phi\),
  - for \(n < N\), \(u^\pi_n = S_{t_{n+1} - t_n}(u^\pi_{n+1})\).

- We obtain, under minimal Lipschitz assumption + structure conditions
  \[
  \int_{\mathbb{R}} |v(0, p, e) - u^\pi_0(p, e)| de \leq C(1 + |p|^2) N^{-\frac{1}{2}},
  \]
Convergence ‘proof’

Classically, we study stability of the scheme and truncation error.

 ► Stability for the error $E\left[\int_{\mathbb{R}} |v(t_n, P_{t_n,e}) - u^\pi_n(P_{t_n}, e)|de\right]$, ok because:

$$|\mathcal{T}_h(\psi) - \mathcal{T}_h(\psi')|_{L_1} \leq |\psi - \psi'|_{L_1} \quad \text{and} \quad |\mathcal{D}_h(\psi) - \mathcal{D}_h(\psi')|_{L_\infty} \leq |\psi - \psi'|_{L_\infty}$$
Convergence ‘proof’

Classically, we study stability of the scheme and truncation error.

- Stability for the error \( \mathbb{E}\left[ \int \mathbb{R} |v(t_n, P_{t_n, e}) - u_{n}^{\pi}(P_{t_n, e})| \, de \right] \), ok because:

  \[
  |\mathcal{T}_h(\psi) - \mathcal{T}_h(\psi')|_{L_1} \leq |\psi - \psi'|_{L_1} \quad \text{and} \quad |\mathcal{D}_h(\psi) - \mathcal{D}_h(\psi')|_{L_\infty} \leq |\psi - \psi'|_{L_\infty}
  \]

- Truncation error: compare \( v(\cdot, \cdot) \) and one step of the scheme \( \tilde{v}(0, \cdot) \)
Convergence ‘proof’

Classically, we study stability of the scheme and truncation error.

- Stability for the error $\mathbb{E}\left[ \int_{\mathbb{R}} |v(t_n, P_{t_n}, e) - u_{\pi_n}^{P_{t_n}, e})| de \right]$, ok because:
  $$|T_h(\psi) - T_h(\psi')|_{L_1} \leq |\psi - \psi'|_{L_1} \quad \text{and} \quad |D_h(\psi) - D_h(\psi')|_{L_\infty} \leq |\psi - \psi'|_{L_\infty}$$

- Truncation error: compare $v(\cdot, \cdot)$ and one step of the scheme $\tilde{v}(0, \cdot)$
  - on $[0, h]$, $\tilde{v}(0, \cdot)$ given by $\tilde{v}(t, p, e) = \mathbb{E}\left[ \phi(P_{t_n}^{t, p, e}) \right]$ and $\tilde{v}(0, \cdot) = T_h(\tilde{v}(0, \cdot))$, etc.
Convergence ‘proof’

Classically, we study stability of the scheme and truncation error.

- Stability for the error $\mathbb{E}[\int_{\mathbb{R}} |v(t_n, P_{t_n}, e) - u_n^\pi(P_{t_n}, e)| de]$, ok because:

$$|\mathcal{T}_h(\psi) - \mathcal{T}_h(\psi')|_{L^1} \leq |\psi - \psi'|_{L^1} \quad \text{and} \quad |\mathcal{D}_h(\psi) - \mathcal{D}_h(\psi')|_{L^\infty} \leq |\psi - \psi'|_{L^\infty}$$

- Truncation error: compare $v(\cdot, \cdot)$ and one step of the scheme $\tilde{v}(0, \cdot)$
  - on $[0, h]$, $\tilde{v}(0, \cdot)$ given by $\tilde{v}(t, p, e) = \mathbb{E}[\phi(P_t^h, p, e)]$ and $\tilde{v}(0, \cdot) = \mathcal{T}_h(\tilde{v}(0, \cdot))$,
  - the associated FBSDE (the characteristics) is well defined: $\tilde{Y}_t := \tilde{v}(t, p, \tilde{E}_t)$

$$d\tilde{E}_t = \mu(\tilde{Y}_t, p)dt \quad d\tilde{Y}_t = 0 \quad \tilde{Y}_h = \tilde{v}(t, p, \tilde{E}_t)$$
Convergence ‘proof’

Classically, we study stability of the scheme and truncation error.

▶ Stability for the error \( \mathbb{E}[\int_{\mathbb{R}} |v(t_n, P_{t_n}, e) - u^\pi_n(P_{t_n}, e)|de] \), ok because:

\[
|T_h(\psi) - T_h(\psi')|_{L^1} \leq |\psi - \psi'|_{L^1} \quad \text{and} \quad |D_h(\psi) - D_h(\psi')|_{L^\infty} \leq |\psi - \psi'|_{L^\infty}
\]

▶ Truncation error: compare \( v(\cdot, \cdot) \) and one step of the scheme \( \tilde{v}(0, \cdot) \)

- On \([0, h]\), \( \tilde{v}(0, \cdot) \) given by \( \bar{v}(t, p, e) = \mathbb{E}[\phi(P^t_h, p, e)] \) and \( \tilde{v}(0, \cdot) = T_h(\bar{v}(0, \cdot)) \),
- The associated FBSDE (the characteristics) is well defined: \( \tilde{Y}_t := \tilde{v}(t, p, \tilde{E}_t) \)

\[
d\tilde{E}_t = \mu(\tilde{Y}_t, p)dt, \quad d\tilde{Y}_t = 0, \quad \tilde{Y}_h = \bar{v}(t, p, \tilde{E}_t)
\]

- Expand \( V_t = \tilde{v}(t, p, \tilde{E}_t) - v(t, P_t, \tilde{E}_t) \), to get \( |V_t|_\infty \leq C\sqrt{h} \).
Convergence ‘proof’

Classically, we study stability of the scheme and truncation error.

- Stability for the error $\mathbb{E}\left[\int_{\mathbb{R}} |v(t_n, P_{t_n}, e) - u^\pi_n(P_{t_n}, e)| de \right]$, ok because:

$$|\mathcal{T}_h(\psi) - \mathcal{T}_h(\psi')|_{L_1} \leq |\psi - \psi'|_{L_1} \text{ and } |\mathcal{D}_h(\psi) - \mathcal{D}_h(\psi')|_{L_\infty} \leq |\psi - \psi'|_{L_\infty}$$

- Truncation error: compare $v(\cdot, \cdot)$ and one step of the scheme $\tilde{v}(0, \cdot)$

  - on $[0, h]$, $\tilde{v}(0, \cdot)$ given by $\tilde{v}(t, p, e) = \mathbb{E}\left[\phi(P^t_{h}, p, e)\right]$ and $\tilde{v}(0, \cdot) = \mathcal{T}_h(\tilde{v}(0, \cdot))$,

  - the associated FBSDE (the characteristics) is well defined: $\tilde{Y}_t := \tilde{v}(t, p, \tilde{E}_t)$

$$d\tilde{E}_t = \mu(\tilde{Y}_t, p)dt, \quad d\tilde{Y}_t = 0, \quad \tilde{Y}_h = \tilde{v}(t, p, \tilde{E}_t)$$

- Expand $V_t = \tilde{v}(t, p, \tilde{E}_t) - v(t, P_t, \tilde{E}_t)$, to get $|V_t|_\infty \leq C\sqrt{h}$.

- We want to control $\int |V_0| de$: study $t \mapsto \int |V_t \partial_e \tilde{E}_t| de$.

  $\leftrightarrow$ we get terms like

$$\int \partial_y \mu(\tilde{v}(t, p, \tilde{E}_t), p) \partial_e \tilde{v}(\cdot) \partial_e \tilde{E} de = \int \partial_e M(\tilde{v}(t, p, \tilde{E}_t), p) de$$

that we can control...
Outline

Introduction
- Emission Trading Scheme
- FBSDEs approach
- One-period model

Approximation schemes
- Classical FBSDE schemes
- Splitting scheme

Numerical results
- Numerical schemes
- Examples
Backward scheme

- For the $E$-direction:
  - $J$ be a positive integer and $\mathcal{E} = (e_j)_{1 \leq j \leq J}$ a discrete grid of $\mathbb{R}$.
  - $\mathcal{T}_h^\mathcal{E}$ an approximation of the transport operator on $\mathcal{E}$:
    \[
    \mathbb{R}^d \times \mathbb{R}^J \ni (p, \theta) \mapsto \mathcal{T}_h^\mathcal{E}(p, \theta) \in \mathbb{R}^J
    \]
Backward scheme

- For the $E$-direction:
  - $J$ be a positive integer and $\mathcal{E} = (e_j)_{1 \leq j \leq J}$ a discrete grid of $\mathbb{R}$.
  - $\mathcal{T}_h^\mathcal{E}$ an approximation of the transport operator on $\mathcal{E}$:
    $$\mathbb{R}^d \times \mathbb{R}^J \ni (p, \theta) \mapsto \mathcal{T}_h^\mathcal{E}(p, \theta) \in \mathbb{R}^J$$

- Euler scheme associated to $P$ on $\pi$, namely, for $n \geq 0$,
  $$\hat{P}_{t_{n+1}}^\pi = \hat{P}_{t_n}^\pi + b(\hat{P}_{t_n}^\pi)h + \sigma(\hat{P}_{t_n}^\pi)\Delta \hat{W}_n \quad \text{and} \quad \hat{P}_{0}^\pi = p.$$
Backward scheme

- For the $E$-direction:
  - $J$ be a positive integer and $E = (e_j)_{1 \leq j \leq J}$ a discrete grid of $\mathbb{R}$.
  - $T^E_h$ an approximation of the transport operator on $E$:
    \[
    \mathbb{R}^d \times \mathbb{R}^J \ni (p, \theta) \mapsto T^E_h(p, \theta) \in \mathbb{R}^J
    \]

- Euler scheme associated to $P$ on $\pi$, namely, for $n \geq 0$,
  \[
  \hat{P}^\pi_{t_{n+1}} = \hat{P}^\pi_{t_n} + b(\hat{P}^\pi_{t_n}) h + \sigma(\hat{P}^\pi_{t_n}) \Delta \hat{W}_n \quad \text{and} \quad \hat{P}^\pi_0 = p.
  \]

- Scheme:
  1. For $n = N$, $\Gamma^j_N = \phi(\hat{P}^\pi_{t_N}, e_j)$ for $1 \leq j \leq J$.
  2. Then, compute for $n < N$
    \[
    \tilde{\Gamma}^j_n = \mathbb{E}\left[\Gamma^j_{n+1} | \hat{P}^\pi_{t_n}\right] \quad \text{for all} \quad 1 \leq j \leq J,
    \]
    \[
    \Gamma^j_n = T^E_h(\hat{P}^\pi_{t_n}, \tilde{\Gamma}^j_n).
    \]
Implementation

- The transport operator is implemented using finite difference schemes: Upwind scheme or Lax-Friedrichs scheme, with $J$ steps in space.
Implementation

- The transport operator is implemented using finite difference schemes: Upwind scheme or Lax-Friedrichs scheme, with $J$ steps in space.
- The regression to estimate functions from $\mathbb{R}^d \to \mathbb{R}^J$ is computed using NN. (simple version of HPW scheme)
The transport operator is implemented using finite difference schemes: Upwind scheme or Lax-Friedrichs scheme, with $J$ steps in space.

The regression to estimate functions from $\mathbb{R}^d \to \mathbb{R}^J$ is computed using NN. (simple version of HPW scheme)

We develop also an alternative scheme: the regression is computed on a tree and transport operator approximated by a particles system. This works well for $d \leq 4$ and $P_t := f(t, W_t)$. (Convergence with a rate can be proven).
The transport operator is implemented using finite difference schemes: Upwind scheme or Lax-Friedrichs scheme, with $J$ steps in space.

The regression to estimate functions from $\mathbb{R}^d \to \mathbb{R}^J$ is computed using NN. (simple version of HPW scheme)

We develop also an alternative scheme: the regression is computed on a tree and transport operator approximated by a particles system. This works well for $d \leq 4$ and $P_t := f(t, W_t)$. (Convergence with a rate can be proven).

We test also a multiplicative model:

$$dP_t^\ell = \mu P_t^\ell dt + \sigma P_t^\ell dW_t^\ell, \quad P_0^\ell = 1, \quad \text{and} \quad dE_t = \tilde{\mu}(Y_t, P_t)dt$$

with $\tilde{\mu}(y, p) = \left(\prod_{\ell=1}^d p^\ell\right)^{\frac{1}{\sqrt{d}}} e^{-\theta y}$, for some $\theta > 0$ and $\phi(p, e) = 1_{\{e \geq 0\}}$.

$\Rightarrow$ it can be reduced to a 2-dimensional model!
Some numerics on the Toy model

(m) $\sigma = 0.01$

(n) $\sigma = 0.3$

(o) $\sigma = 1.0$

Figure: Linear Toy Model: Comparison of the three methods:
- Neural Nets & Lax-Friedrichs (NN&LF) with $d = 10$
- an alternative scheme (BT&SPD) with $d = 4$
- The Proxy solution given by particle method.

Lax-Friedrichs scheme implemented with discretization of space $J = 1500, 1000, 500$, for $\sigma = 0.01, 0.3, 1$ respectively and number of time step $K = 30$. The number of time step for the splitting is $N = 64$. For $BT&SPD$, the number of particles is $M = 3500$ and the number of time steps $N = 20$. 

J-F Chassagneux
On the multiplicative model

Figure: A multiplicative model in dimension \(d = 10\). Comparison of two methods:
- Neural nets & Upwind scheme
- the alternative scheme on equivalent 4-dimensional model (BT&SPD).

The Upwind scheme used discretization of space \(J = 100, 400, 500\) respectively for \(\sigma = 1, 0.3, 0.01\) and number of time step \(K = 20\). The number of time step for the splitting is \(N = 32\). For \(BT&SPD\), the number of particles is \(M = 3500\), and the number of time steps \(N = 20\).