Modelling multi-period carbon markets using singular forward backward SDEs

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The 9th International Colloquium on BSDEs and Mean Field Systems
Annecy, 27 June - 1 July 2022

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Joint work with Jean-François Chassagneux (Paris Diderot), Hinesh Chotai (Citybank)

Jean-François Chassagneux, Hinesh Chotai, D.C., Modelling multi-period carbon markets using singular forward backward SDEs, to appear in Mathematics of Operations Research, 2022.

- Carbon markets are currently being implemented worldwide.
- Since 2005, the EU has had its own emissions trading system (ETS).
- China, whose carbon emissions make up approximately one quarter of the global total, is considering introducing a national emissions trading scheme for greenhouse gas emissions in the next few years.
- In 2013, seven pilots were introduced in seven different regions in China. The Chinese ETS will be the largest carbon market in the world.
- Carbon markets face many criticisms. Some critics claim that they reduce an industry's competitiveness, while others believe that the average carbon price today is not high enough to motivate substantial of greenhouse gas emissions reductions.
- Proponents of ETS claim that they lead to real emissions reductions when regulators operate them in an appropriate way.
- ETS are becoming increasingly important and prevalent. Scientific, particularly mathematical, studies of them are needed in order to expose their advantages, their shortcomings, and their efficient implementation.

Single period model:

Single compliance period of a carbon emissions market in isolation.

- the regulator releases Λ emissions allowances into the market at time 0
- the allowances are traded throughout the period $[0, T]$.
- The market participants only submit allowances at time T, and not earlier.
- At time T, each market participant must surrender one allowance for each unit of emissions made during this period.
- For any unit of emissions for which an emissions allowance was not surrendered, a market participant must pay the penalty ρ.

$$
\begin{aligned}
\mathrm{d}P_t &= b(P_t)\,\mathrm{d}t + \sigma(P_t)\cdot \mathrm{d}W_t, & P_0 &= p \in \mathbb{R}^d,\\
\mathrm{d}E_t &= \mu(P_t, Y_t)\,\mathrm{d}t, & E_0 &= e \in \mathbb{R},\\
Y_t &= E[Y_T|\mathcal{F}_t] & Y_T &= \phi(E_T).
\end{aligned}
$$

- $P_t = (S_t, D_t)$ S_t vector of fuel prices
- D_t an inelastic demand curve for electricity
- E_t is the level of cumulative emissions at t
- Y_t is the spot price of an allowance certificate at time t
- ϕ the penalty for over emission, $\phi(e) = \mathbf{1}_{[\Lambda,\infty)}(e)$
- Λ represents the cap on total emissions. Typically, we have $\Lambda > 0$.

- $\mu(P_t, Y_t)$ instantaneous emission rate, e.g.,

$$\mu(P_t, Y_t) = \tilde{c}(P_t - Y_t e_c)$$

- $\tilde{c}$ the inverse of the function giving the marginal costs of the goods production
- e_c the vector (with the same dimension as P_t) giving the rates of emission associated to the production of the various goods. Model as an FBSDEs

$$\begin{aligned} \mathrm{d}P_t &= b(P_t)\,\mathrm{d}t + \sigma(P_t)\cdot \mathrm{d}W_t, & P_0 &= p \in \mathbb{R}^d, \\ \mathrm{d}E_t &= \mu(P_t, Y_t)\,\mathrm{d}t, & E_0 &= e \in \mathbb{R}, \\ \mathrm{d}Y_t &= Z_t \cdot \mathrm{d}W_t, & Y_T &= \phi(E_T). \end{aligned} \tag{1}$$

- W is a standard d dimensional Wiener process
- b, σ and μ satisfy standard Lipschitz and linear growth conditions
- ϕ is monotone increasing and bounded but not continuous in general.
- $y \mapsto \mu(p, y)$ is strictly decreasing.

The FBSDE has the following special characteristics:

i. the forward and backward components are coupled
ii. the final condition is singular and
iii. the forward component of the model is degenerate.

The functions $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $\mu : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are s.t. there exist three constants $L \geq 1$, $l_1, l_2 > 0$, $1/L \leq l_1 \leq l_2 \leq L$

1. b and σ have L-linear growth (this holds uniformly in time for b):

$$|b(t,p)| + |\sigma(p)| \leq L(1 + |p|), \quad p \in \mathbb{R}^d, t \in [0,T], \tag{2}$$

and are L-Lipschitz continuous (also uniformly in time for b):

$$|b(t,p) - b(t,p')| + |\sigma(p) - \sigma(p')| \leq L|p - p'|, \quad p, p' \in \mathbb{R}^d, t \in [0,T]. \tag{3}$$

2. μ also has L-linear growth,

$$|\mu(p,y)| \leq L(1 + |p| + |y|), \quad p \in \mathbb{R}^d, y \in \mathbb{R}, \tag{4}$$

and is L-Lipschitz continuous, satisfying

$$|\mu(p,y) - \mu(p',y')| \leq L\left(|p - p'| + |y - y'|\right), \quad p, p' \in \mathbb{R}^d, y, y' \in \mathbb{R}. \tag{5}$$

Finally, for any $p \in \mathbb{R}^d$, the real function $y \mapsto \mu(p,y)$ is strictly decreasing and μ satisfies the following monotonicity condition

$$l_1|y - y'|^2 \leq (y - y')\left(\mu(p,y') - \mu(p,y)\right) \leq l_2|y - y'|^2, \quad p \in \mathbb{R}^d, y, y' \in \mathbb{R}. \tag{6}$$

Consider a terminal condition $(p,e) \mapsto \phi(p,e)$ being L_ϕ-Lipschitz continuous in the p variable (uniformly in the e variable) and monotone increasing in the e variable. That is,

$$\begin{aligned} |\phi(p,e) - \phi(p',e)| \le L_\phi |p - p'|, &\quad p, p' \in \mathbb{R}^d,\ e \in \mathbb{R}, \\ \phi(p,e') \ge \phi(p,e) &\quad \text{if } e' \ge e. \end{aligned} \tag{7}$$

Also, assume that ϕ takes values in $[0,1]$, such that, for all $p \in \mathbb{R}^d$,

$$\inf_{e\in\mathbb{R}} \phi(p,e) = 0, \quad \sup_{e\in\mathbb{R}} \phi(p,e) = 1. \tag{8}$$

Spaces

- $\mathcal{S}^{2,k}([0,T]$ the set of $\mathbb{R}^k$-valued cadlag adapted processes Y, s.t.

$$\|Y\|^2_{\mathcal{S}^2} := \mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2\right] < \infty.$$

  $\mathcal{S}^{2,k}_{c}[0,T]$ is the subspace of continuous process.
- $\mathcal{H}^{2,k}[0,T]$ is the set of $\mathbb{R}^k$-valued progressively measurable processes Z, such that

$$\|Z\|^2_{\mathcal{H}^2} := \mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] < \infty.$$

Define ϕ_- and ϕ_+ to be the left and right continuous versions of ϕ:

$$\phi_-(p,e) = \sup \phi(p,e'), \;\; \phi_+(p,e) = \inf \phi(p,e').$$

Theorem

Given any initial condition $(p,e) \in \mathbb{R}^d \times \mathbb{R}$, there exists a unique progressively measurable 4-tuple of processes $(P_t, E_t, Y_t, Z_t)_{0 \le t \le T} \in \mathcal{S}_c^{2,d}[0,T] \times \mathcal{S}_c^{2,1}[0,T] \times \mathcal{S}_c^{2,1}[0,T] \times \mathcal{H}^{2,d}[0,T]$ satisfying the dynamics in (1) with $(P_0, E_0) = (p,e)$ and such that

$$\mathbb{P}\left[\phi_-(P_T, E_T) \le Y_T \le \phi_+(P_T, E_T)\right] = 1, \tag{9}$$

For any $a \ge 1$, there exists a constant $C' > 0$ depending on L, L_ϕ, T and a s.t.

$$\mathbb{E}\left[\sup_{t \in [0,T]} |Z_t|^a\right] \le C'(1 + |p|^a), \tag{10}$$

Moreover, if σ is bounded by L then there exists a constant C depending only on L, L_ϕ and T, such that $|Z_t| \le C$ for almost every $(t,\omega) \in [0,T] \times \Omega$.

Remark. Theorem proved by Carmona and Delarue for $\phi = \phi(e)$.

The value function

Let $Y_t^{t_0,p,e,\phi}$ be the solution of the FBSDE (1) with a starting point (t_0, p, e) and a terminal condition ϕ satisfying the conditions. We define

$$v(t_0, p, e) = Y_{t_0}^{t_0,p,e,\phi},$$

Then

$$Y_t^{t_0,p,e,\phi} = v(t, P_t^{t_0,p}, E_t^{t_0,p,e,\phi})$$

for every $(t_0, p, e) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}$ and every $t \in [t_0, T)$. Existence and uniqueness for FBSDE (1) also holds with a starting point (t_0, P_{t_0}, E_{t_0}) where $t_0 \in [0, T)$ and P_{t_0} and E_{t_0} are square integrable, $\mathcal{F}_{t_0}$-measurable random variables. In this case,

$$v(t, P_t^{t_0,P_{t_0}}, E_t^{t_0,P_{t_0},E_{t_0},\phi}) = Y_t^{t_0,P_{t_0},E_{t_0},\phi}.$$

Theorem

The value function defined above is the unique function s.t.

1. *For any $t \in [0, T)$, the function $v(t, \cdot, \cdot)$ is $1/(l_1(T - t))$-Lipschitz continuous wrt e.*
2. *For any $t \in [0, T)$, the function $v(t, \cdot, \cdot)$ is C-Lipschitz continuous with respect to p, where C is a constant depending on L, L_ϕ and T only.*

Theorem

1. *Let $(P_t)_{0\leq t\leq T}$ be the unique strong solution of the forward equation for P with initial value p. Then, the unique strong solution, $(E_t)_{0\leq t<T}$ of*
$$E_t = e + \int_0^t \mu(P_s, v(s,P_s,E_s))ds, \quad 0 \leq t < T,$$
is such that the process $(e^{-rt}v(s,P_t,E_t))_{0\leq t<T}$ is a $[0,1]$-valued martingale with respect to the complete filtration generated by W.
2. *The limit $\lim_{t\uparrow T} E_t$ exists almost surely and so does $\lim_{t\uparrow T}(e^{-rt}v(s,P_t,E_t))$, being the limit of a bounded martingale.*

The limit $\lim_{t\uparrow T} v(t,P_t,E_t)$ satisfies
$$\mathbb{P}\left[\phi_-(P_T,E_T) \leq \lim_{t\uparrow T} v(s,P_t,E_t) \leq \phi_+(P_T,E_T)\right] = 1.$$
For any $t\in[0,T)$, $p\in\mathbb{R}^d$ the function $e\mapsto v(t,p,e)$ is monotone increasing and satisfies
$$\lim_{e\to\infty} v(t,p,e) = e^{-r(T-t)}, \quad \lim_{e\to-\infty} v(t,p,e) = 0.$$

Multi-period period model:

- remove the constraint of compliance period in isolation
- add a discount factor: r is the instantaneous risk-free interest rate

$$
\begin{aligned}
dP_t &= b(t,P_t)dt + \sigma(P_t)dW_t, & P_0 &= p \in \mathbb{R}^d \\
dE_t &= \mu(P_t,Y_t)dt, & E_0 &= e \in \mathbb{R} \\
dY_t &= rY_t dt + Z_t \cdot dW_t, & Y_T &= \Phi(E_T),
\end{aligned} \tag{11}
$$

- Consider a market with $q \geq 2$ trading periods,

$$[T_0 = 0, T_1], \ [T_1, T_2], ..., [T_{q-1}, T_q = T].$$

- For every $k \geq 1$, at time T_k there is a penalty for non-compliance ρ_k if the cumulative emissions have exceeded a cap $\hat{\Lambda}_k$.
- For each k, $\hat{\Lambda}_k$ could be a constant or a function of the total registered emissions at T_{k-1}, the beginning of the period.

An example

- The regulator releases $c_{k+1} \geq 0$ allowances into circulation at each time T_k for $k = 0, 1, ..., q-1$.
- At time T_1, the level of cumulative emissions is equal to E_{T_1}, and there are c_1 allowances available for compliance. The cap on cumulative emissions at T_1 is therefore simply $\hat{\Lambda}_1 = c_1$.
- At time T_1, the regulator compares the value of E_{T_1} to the cap c_1. Suppose now that $E_{T_1} < c_1$. Then, at time T_1, a total of E_{T_1} allowances are surrendered by firms in the market for compliance.
- $c_1 - E_{T_1}$ allowances are left and are carried forward to the next period.
- At time T_1, the regulator releases c_2 allowances into circulation. Hence, the total number of allowances in circulation at the start of the $[T_1, T_2]$ period is $\Gamma_2(E_{T_1}) = c_1 + c_2 - E_{T_1}$.
- $\Gamma_2(E_{T_1})$, the number of allowances in circulation at T_1, can be thought of as the cap on emissions made during the $[T_1, T_2]$ period.
- At time T_2, the cumulative emissions are equal to E_{T_2} and the number of emissions made during the $[T_1, T_2]$ period is $E_{T_2} - E_{T_1}$.
- At T_2, the regulator checks whether $E_{T_2} - E_{T_1} < \Gamma_2(E_{T_1})$, or equivalently whether $E_{T_2} < \hat{\Lambda}_2(E_{T_1})$, where $\hat{\Lambda}_2(e) := \Gamma_2(e) + e$.

- $\hat{\Lambda}_2(E_{T_1})$ can be thought of as the cap on cumulative emissions at T_2.
- In summary, at T_2, compliance has occurred if and only if $E_{T_2} - E_{T_1} < \Gamma_2(E_{T_1})$, where $\Gamma_2(E_{T_1})$ is the number of allowances in circulation at T_1, or equivalently if $E_{T_2} < \hat{\Lambda}_2(E_{T_1})$.

There are different mechanisms that link the trading periods. These affect the functional form of the cap functions $\hat{\Lambda}_k$ or Γ_k for $1 \leq k \leq q$:

- Banking: allowances that are not used in one period can be carried forward for compliance in the next period.
- Withdrawal: for any $1 \leq i \leq q-1$, if the cap on emissions is exceeded at T_i, then the regulator removes a quantity of allowances from the $[T_i, T_{i+1}]$ market allocation. The quantity of allowances removed is equal to the level of excess emissions at T_i.
- Borrowing: for any $1 \leq i \leq q-1$, firms may trade some of the allowances to be released at T_i during $[T_{i-1}, T_i]$. If each trading period represents a year, this means that firms can, in a particular year that is not the final year, use the following year's allowance allocation for compliance.

Banking, borrowing and withdrawal are all currently in place in the EU ETS (EU Emissions Trading System manual page 113).

For each $1 \leq k \leq q$, the value of $E^q_{T_{k-1}}$ represents the level of cumulative emissions at T_{k-1}. The cap on cumulative emissions at T_k will be $\hat{\Lambda}_k(E^q_{T_{k-1}})$; it will be $\mathcal{F}_{T_{k-1}}$ measurable and can also be expressed as

$$\hat{\Lambda}_k(E^q_{T_{k-1}}) = \Gamma_k(E^q_{T_{k-1}}) + E^q_{T_{k-1}},$$

where $\Gamma_k(E^q_{T_{k-1}})$ is the number of allowances in circulation at T_{k-1}.

Banking, borrowing and withdrawal

Suppose that, for each $1 \leq k \leq q$, the regulator has a quantity of allowances $c_k \geq 0$ set to be released into the market at T_{k-1}.

- For banking, borrowing and withdrawal, we will have
$$\Gamma_k(e) = \sum_{i=1}^{(k+1)\wedge q} c_i - e, \tag{12}$$
- For banking and withdrawal only, we will have
$$\Gamma_k(e) = \sum_{i=1}^{k} c_i - e, \tag{13}$$
for every $1 \leq k \leq q$.

- Consider a market consisting of firms whose activities cause emissions.
- q represents the number of trading periods being considered.
- Let $0 \leq T_0 < T_1 < ... < T_q$ and consider the q time intervals $[T_0 = 0, T_1]$, $[T_1, T_2]$,..., $[T_{q-1}, T_q = T]$.
- During any trading period $[T_{k-1}, T_k]$, for $1 \leq k \leq q$, emissions regulation is in effect.
- Let $(E_t^q)_{t \in [0,T]}$ be a real valued continuous process representing, at time t the cumulative emissions made in the market up to time t.
- We define $(E_t^q)_{t \in [0,T]}$ such that, for any $1 \leq k \leq q$, E^q is part of the solution of the FBSDE (1) on $[T_{k-1}, T_k]$.
- Assuming continuity at each compliance time T_k, for $1 \leq k \leq q-1$ allows us to stipulate that, the terminal value of E^q on the $[T_{k-1}, T_k]$ FBSDE, and the initial value of E^q on the $[T_k, T_{k+1}]$ FBSDE, for $1 \leq k \leq q-1$, are equal.
- This is required to specify the initial and terminal conditions for those FBSDEs, and it means that the solutions of the different FBSDEs defining the multi period model are coupled: A multi period model is different to several separate copies of a single period model.
- For every integer $0 \leq k \leq q$, at time T_k, the regulator records the level of cumulative emissions E_{T_k}, and, for each $0 \leq k \leq q-1$, a cap on emissions at T_{k+1} is defined.

- The cap on the emissions made during the $[T_k, T_{k+1}]$ trading period is equal to $\Gamma_k(E_{T_k})$, where Γ_k is a deterministic function which will be assumed to be monotone decreasing.
- We set $\hat{\Lambda}_k(e) = \Gamma_k(e) + e$ for every $e \in \mathbb{R}$ and every $1 \leq k \leq q$. At each time T_k, the regulator checks whether $E_{T_k} - E_{T_{k-1}} \geq \Gamma_k(E_{T_{k-1}})$ or equivalently whether $E_{T_k} \geq \hat{\Lambda}_k(E_{T_{k-1}})$.
- If so, it means that the $[T_{k-1}, T_k]$ period's emissions have exceeded the time T_k cap and market participants must pay a penalty ρ_k for each unit of emissions above the cap, where $\rho_k > 0$ is a deterministic constant.
- Since we are aggregating at the level of the market and not considering individual firms, we model this penalty by stipulating that the allowance price process should be equal to ρ_k at T_k if the emissions at T_k have exceeded the cap. Here, the functions $\hat{\Lambda}_k$ and Γ_k for every $1 \leq k \leq q$, are deterministic real valued, measurable functions. In applications, the functions Γ_k may also be bounded. For every $1 \leq k \leq q$, the cap at T_k is a $\mathcal{F}_{T_{k-1}}$ measurable random variable that is known to market participants at time T_{k-1}.
- In reality (e.g. in the EU ETS), there are many firms in the market, each of which has to surrender emissions allowances at the end of a trading period for each unit of emissions made during that trading period. Any unit of emissions that is not covered by an allowance incurs a penalty for the firm.

• We denote by $r \geq 0$ a risk free interest rate such that investment of 1 unit of currency at time 0 yields e^{rt} units of currency at time t.
• The evolution of the process $(P_t, E_t, Y_t, Z_t)_{0 \leq t \leq T}$ is governed by the FBSDE (1) on $[T_{k-1}, T_k)$. Moreover, for every $1 \leq k \leq q$,

$$\begin{aligned} \lim_{t \nearrow T_k} Y_t &= Y_{T_k}, && \text{if } E_{T_k} < \hat{\Lambda}_k(E_{T_{k-1}}), \\ \lim_{t \nearrow T_k} Y_t &= \rho_k, && \text{if } E_{T_k} > \hat{\Lambda}_k(E_{T_{k-1}}), \\ Y_{T_k} \leq \lim_{t \uparrow T_k} Y_t &\leq \rho_k, && \text{if } E_{T_k} = \hat{\Lambda}_k(E_{T_{k-1}}), \end{aligned} \tag{14}$$

for every $1 \leq k \leq q$.
• the rigurous terminal condition of the FBSDEs is given in terms of the corresponding value function.

Let $0 < t_0 < T$ and let Φ be a function satisfying (7) and

$$\begin{aligned} \inf_{e\in\mathbb{R}} \Phi(p,e) &= 0, \\ \sup_{e\in\mathbb{R}} \Phi(p,e) &= \rho, \end{aligned} \tag{15}$$

for some $\rho > 0$. Well-posedness holds for a FBSDE with terminal condition Φ. Additionally the terminal condition for the FBSDE on $[t_0, T]$ can depend on a $\mathcal{F}_{t_0}$-measurable random variable, and the initial values (p, e) can be replaced by a pair of square integrable $\mathcal{F}_{t_0}$ random variables $(P^*_{t_0}, E^*_{t_0})$.

We can define an operator mapping the parameters of FBSDE to the components of its solution. Given any $0 \le t_0 < T$ and any integer $k \ge 1$, we denote by $L^2(\mathcal{F}_{t_0}, \mathbb{R}^k)$ the set of all square integrable, $\mathcal{F}_{t_0}$-measurable random variables taking values in $\mathbb{R}^k$. Also, given any $X_{t_0} \in L^2(\mathcal{F}_{t_0}, \mathbb{R}^k)$ for some k, let $\oplus(X_{t_0})$ be the set of all random fields $\Phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$, measurable with respect to the sigma algebra generated by X_{t_0}, and satisfying both (7) and, for some $\rho > 0$, (15), almost surely. Given a T' with $0 < T' \le T$, let $\mathcal{D}_{T'}$ be the following set.

$$\begin{aligned} \mathcal{D}_{T'} :=&\{(t_0, P^*_{t_0}, E^*_{t_0}, T', \Phi) \in \mathbb{R} \times L^2(\mathcal{F}_{t_0}, \mathbb{R}^d) \times L^2(\mathcal{F}_{t_0}, \mathbb{R}) \times \mathbb{R} \times \\ &\oplus(E^*_{t_0}) : 0 \le t_0 < T'\}. \end{aligned}$$

For any such T', we can now define two operators $\mathcal{E} : \mathcal{D}_{T'} \to \mathcal{S}^2_{\mathrm{c}}([0, T'] : \mathbb{R})$ and $\mathcal{Y} : \mathcal{D}_{T'} \to \mathcal{S}^2_{\mathrm{c}}([0, T'] : \mathbb{R})$, mapping a tuple $(t_0, P^*_{t_0}, E^*_{t_0}, T', \Phi) \in \mathcal{D}_{T'}$ to the E and Y components, respectively, of the unique adapted solution of (1) on $[t_0, T']$ with initial parameters $(t_0, P^*_{t_0}, E^*_{t_0})$ and terminal condition Φ at T'. Precisely, for $(t_0, P^*_{t_0}, E^*_{t_0}, T', \Phi) \in \mathcal{D}_{T'}$ $(P^*_{t_0}, E^*_{t_0})$, we set

$$\begin{aligned} \mathcal{E}_t(t_0, P^*_{t_0}, E^*_{t_0}; T', \Phi) &= E^*_t, \quad t_0 \le t \le T', \\ \mathcal{Y}_t(t_0, P^*_{t_0}, E^*_{t_0}; T', \Phi) &= Y^*_t, \quad t_0 \le t \le T', \end{aligned} \tag{16}$$

where we use the notation $\mathcal{E}_t(t_0, P^*_{t_0}, E^*_{t_0}; T', \Phi) = \big(\mathcal{E}(t_0, P^*_{t_0}, E^*_{t_0}; T', \Phi)\big)_t$, $\mathcal{Y}_t(t_0, P^*_{t_0}, E^*_{t_0}; T', \Phi) = \big(\mathcal{Y}(t_0, P^*_{t_0}, E^*_{t_0}; T', \Phi)\big)_t$, and E^*, P^* and Y^* satisfy the dynamics

$$\begin{aligned} t &\in [t_0, T'] \\ dP^*_t &= b(t, P^*_t)dt + \sigma(P^*_t)dW_t, \\ dE^*_t &= \mu(P^*_t, Y^*_t)dt \\ dY^*_t &= rY^*_t dt + Z^*_t \cdot dW_t, \end{aligned} \tag{17}$$

with

$$Y^*_{T'} = \Phi(P^*_{T'}, E^*_{T'}). \tag{18}$$

Remarks

- The terminal condition for (17), given by (18) should be understood in a relaxed sense, as described in Theorem 1 (see (8)).
- This means that Y^* as defined here will not, in general, satisfy $Y^*_{T'} = \Phi(P^*_{T'}, E^*_{T'})$ exactly.
- Note also that the Z^* process in (17) is, here and throughout the ensuing sections, the integrand in the martingale representation of the martingale $(e^{-rt} Y^*_t)_{t_0 \le t \le T'}$ as a stochastic integral with respect to W.
- Given a T', this process belongs to $\mathcal{H}^2([0, T'] : \mathbb{R}^d)$ and satisfies the properties of the Z process in Theorem 1.
- The Z^* process is less important here and can always be obtained by considering the integrand in the martingale representation of the process $(e^{-rt} Y^*_t)_{0 \le t \le T}$.

Now we will define a set of value functions $v_1^q, v_2^q, .., .v_q^q$ and terminal conditions $\Phi_1^q, \Phi_2^q, ..., \Phi_q^q$ for the q period pricing problem. We will drop the superscript q and write e.g. v_1 instead of v_1^q. The q period pricing problem started at time $(0, p, e)$ comes from the solution of the following q FBSDE: For every integer k such that $1 \leq k \leq q$, the period k dynamics are

$$\begin{aligned} &\text{for } t \in [T_{k-1}, T_k) : \\ &\quad E_t = \mathcal{E}_t(T_{k-1}, P_{T_{k-1}}, E_{T_{k-1}}; T_k, \Phi_k(\cdot, \cdot; E_{T_{k-1}})) \\ &\quad Y_t = \mathcal{Y}_t(T_{k-1}, P_{T_{k-1}}, E_{T_{k-1}}; T_k, \Phi_k(\cdot, \cdot; E_{T_{k-1}})) \\ &\quad\quad = v_k(t, P_t, E_t; E_{T_{k-1}}), \\ &\text{and } E_{T_k} := \lim_{t \uparrow T_k} E_t. \end{aligned} \tag{19}$$

where, for any $(p, e) \in \mathbb{R}^d \times \mathbb{R}$ and any $e_0 \in \mathbb{R}$, and any integer k such that $1 \leq k \leq q - 1$,

$$\Phi_k(p, e; e_0) = v_{k+1}(T_k, p, e; e), \quad \text{if } e < \hat{\Lambda}_k(e_0), \tag{20}$$

$$\Phi_k(p, e; e_0) = \rho_k, \quad \text{otherwise}, \tag{21}$$

and,

$$\Phi_q(p, e; e_0) = 0, \text{ if } e < \hat{\Lambda}_q(e_0), \tag{22}$$

$$\Phi_q(p, e; e_0) = \rho_q \quad \text{otherwise}. \tag{23}$$

Remarks.

- As stated before, $T_0 = 0$ and we set $P_0 = p$, $E_0 = e$ for some given $(p, e) \in \mathbb{R}^d \times \mathbb{R}$.
- Note that Φ_q is in fact independent of p.
- For each fixed $e_0 \in \mathbb{R}$, the function $v_k(\cdot, \cdot, \cdot; e_0)$ is simply the value function v for the FBSDE (1) with terminal time T_k and terminal condition $(p, e) \mapsto \Phi_k(p, e; e_0)$. For any $t \in [T_{k-1}, T_k)$, we have

$$v_k(t, p, e; e_0) = \mathcal{Y}_t(t, p, e; T_k, \Phi_k(\cdot, \cdot; e_0)), \tag{24}$$

  for any $(p, e) \in \mathbb{R}^d$ and $e_0 \in \mathbb{R}$.
- Similarly, with the notation set out above, let $v(t_0, P_{t_0}, E_{t_0}; T, \Phi) = Y^*_{t_0}$. When T and Φ are fixed, the solution to (17) satisfies $Y^*_t = v(t, P^*_t, E^*_t)$ for every $t_0 \leq t < T$.

The rationale behind (19) is similar to the argument for the case $q = 1$ Assuming that there exist processes E and Y satisfying the dynamics (19) for any integer k such that $1 \leq k \leq q$, we will have

$$Y_{T_k} = v_{k+1}(T_k, P_{T_k}, E_{T_k}; E_{T_{k-1}}), \tag{25}$$

for every integer k such that $1 \leq k \leq q$. This means that the terminal condition in (19) at T_k will be

$$\Phi_k(P_{T_k}, E_{T_k}; E_{T_{k-1}}) = Y_{T_k}, \text{if } E_{T_k} < \hat{\Lambda}_k(E_{T_{k-1}}), \tag{26}$$

$$\Phi_k(P_{T_k}, E_{T_k}; E_{T_{k-1}}) = \rho_k, \quad \text{otherwise}, \tag{27}$$

for every integer k such that $1 \leq k \leq q - 1$ i.e. (26) is the terminal condition in (19) for every period before the final one. In the final period, the terminal condition reduces to

$$\Phi_q(P_{T_q}, E_{T_q}; E_{T_{q-1}}) = \rho_q \mathbf{1}_{[\hat{\Lambda}_q(E_{T_{q-1}}), +\infty)}(E_{T_q}), \tag{28}$$

which is similar to the terminal condition used in the single period model, but with an adjusted cap.

Additional Assumptions

- For every k, the function $\Gamma_k : \mathbb{R} \to \mathbb{R}$ is monotone decreasing and satisfies $\lim_{e \to -\infty} \Gamma_k(e) = +\infty$.
- The penalties $(\rho_k)_{k \geq 1}$ satisfy $0 < e^{-r(T_k - T_{k-1})}\rho_k \leq \rho_{k-1}$ for every $k \geq 2$ (See page 134 of the EU ETS handbook).

Theorem

The q period model as described above is well-posed. That is, there exists a unique progressively measurable 4-tuple of processes $(P_t, E_t, Y_t, Z_t)_{0 \leq t \leq T} \in \mathcal{S}_c^{2,d}[0,T] \times \mathcal{S}_c^{2,1}[0,T] \times \mathcal{S}^{2,1}[0,T] \times \mathcal{H}^{2,d}[0,T]$ *satisfying on each period* $[T_k, T_{k+1})$, $k \leq q$*:*

$$\begin{aligned} \mathrm{d}P_t &= b(P_t)\,\mathrm{d}t + \sigma(P_t)\,\mathrm{d}W_t, \\ \mathrm{d}E_t &= \mu(P_t, Y_t)\,\mathrm{d}t, \\ \mathrm{d}Y_t &= rY_t\,\mathrm{d}t + Z_t\,\mathrm{d}W_t. \end{aligned} \tag{29}$$

In addition, for each integer k such that $1 \leq k \leq q$*, the process* Y *is continuous on* $[T_{k-1}, T_k)$*; it can have a jump at* T_k *as follows: for every* $1 \leq k \leq q$, $\lim_{t \nearrow T_k} Y_t = Y_{T_k}$*, if* $E_{T_k} < \hat{\Lambda}_k(E_{T_{k-1}})$, $\lim_{t \nearrow T_k} Y_t = \rho_k$*, if* $E_{T_k} > \hat{\Lambda}_k(E_{T_{k-1}})$*, and* $Y_{T_k} \leq \lim_{t \nearrow T_k} Y_t \leq \rho_k$*, if* $E_{T_k} = \hat{\Lambda}_k(E_{T_{k-1}})$.

Theorem

For each q, we consider a q period model with penalties $(\rho_k)_{1\leq k\leq q}$ and cap functions $(\hat{\Lambda}_k)_{1\leq k\leq q}$ (or equivalently $(\Gamma_k)_{1\leq k\leq q}$). Under some assumptions on the penalties $(\rho_k)_{k\geq 1}$, and the cap functions, the corresponding decoupling fields satisfy

$$v_1^{q-1}(0,p,e;e_0) \leq v_1^q(0,p,e;e_0), \tag{30}$$

for every $(p,e) \in \mathbb{R}^d \times \mathbb{R}$ and $e_0 \in \mathbb{R}$. Since the sequence is upper bounded by the penalty ρ_1 it is also convergent.

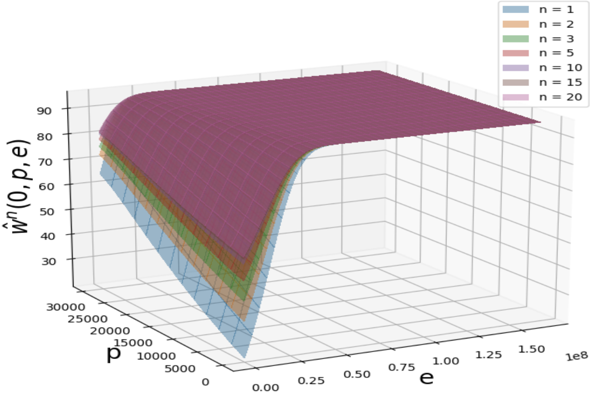

Figure: Plot of $(p, e) \mapsto v^n(0, p, e)$

Case study: Sam Howison and Daniel Schwarz. Risk-neutral pricing of financial instruments in emission markets: a structural approach. SIAM Journal on Financial Mathematics, 3(1):709739, 2012.

- The q period model introduced in this work is more realistic and applicable than a single period model because it allows one to model multiple times at which compliance occurs and a new allowance allocation is released into a carbon market.
- One disadvantage of this, however, is that, for a q period model, one must specify the end date T_q. This is important because, at T_q, the terminal condition is different to the terminal condition specified at every prior time T_k, for $1 \leq k \leq q - 1$,
- The time T_q is the time at which all emissions regulation ceases and this is why the terminal condition at this time specifies that allowances at T_q will have price 0 if the time T_q cumulative emissions are below the time T_q cap.
- For an more realistic model, one can consider a model for a carbon market with no specified end date. In the setting of the EU ETS, there is currently no time at which one can say with certainty that emissions regulation will cease or the banking of allowances will be prohibited.
- We introduce a model for a carbon market in operation over the time period $[0, \infty)$ with no end date and show that it is well posed (under certain condition).

Theorem

Set $T_k = k\tau$ *and* $\Lambda_k = k\lambda$, $\lambda, \tau > 0$. *Under certain assumptions, there exist a 4-tuple* (P, E, Y, Z) *s.t. for any* $[T_k, T_{k+1})$:

$$\begin{aligned} \mathrm{d}P_t &= b(P_t)\,\mathrm{d}t + \sigma(P_t)\,\mathrm{d}W_t \\ \mathrm{d}E_t &= \mu(P_t, Y_t)\,\mathrm{d}t \\ \mathrm{d}Y_t &= rY_t\,\mathrm{d}t + Z_t\,\mathrm{d}W_t. \end{aligned}$$

The process Y *is continuous on* $[T_{k-1}, T_k)$. *It can jump at* $(T_k)_{k\geq 1}$, *where*

$$\begin{aligned} Y_{T_k-} &= Y_{T_k}, && \text{if } E_{T_k} < \Lambda_k, \\ Y_{T_k-} &= 1, && \text{if } E_{T_k} > \Lambda_k, \qquad (31) \\ Y_{T_k} \leq Y_{T_k-} &\leq 1, && \text{if } E_{T_k} = \Lambda_k. \end{aligned}$$

Moreover, there exists a continuous function $w : [0, \tau) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ *such that* $Y_t = w(t - T_{k-1}, P_t, E_t - \Lambda_{k-1})$, $t \in [T_{k-1}, T_k)$, $k \geq 1$. *Setting, for* $e \in \mathbb{R}$,

$$\Phi(p, e) = w(0, p, e - \lambda), \qquad \text{if } e < \lambda, \qquad (32)$$

$$\Phi(p, e) = 1, \text{otherwise}. \qquad (33)$$

the function w *satisfies,*
$\Phi_-(p, e) \leq \liminf_{t\uparrow\tau} w(t, p_t, e_t) \leq \limsup_{t\uparrow\tau} w(t, p_t, e_t) \leq \Phi_+(p, e)$, *for any family* $(p_t, e_t)_{0\leq t<\tau}$ *converging to* (p, e) *as* t *tends to* τ.

- In this paper we study a risk-neutral models for pricing carbon allowances with multiple/infinite trading periods.
- The main result of the paper gives a characterization of the pair of processes emission rates/carbon allowance price as the unique solution of a set of FBSDEs that are linked through their transition values at times T_k, $k = 0, \ldots, q - 1$ and terminal conditions at times T_k, $k = 1, \ldots, q$.
- The study of this FBSDE is closely linked to the study of the associated value function (known as decoupling field in the FBSDE literature). This decoupling field can be considered to be the solution solution to a degenerate quasilinear elliptic PDE.
- We also introduce a model for a carbon market in operation over the time period $[0, \infty)$ with no end date and show that it is well posed under certain conditions.
- We characterize the pair of processes emission rates/carbon allowance price as the unique solution of an infinite sequence of FBSDEs that are linked through their transition values at times T_k, $k \geq 1$.
- We show that the spot price Y_t^q of an allowance certificate for the q-period model converges, as the number of periods q increases, to the spot price Y_t^∞ of an allowance certificate for the infinite period model.