Extended Mean Field Control Problems with Singular Controls

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We consider the state dynamics

$$dX_t = b(t, m_t, X_t, \xi_t)dt + \sigma(t, m_t, X_t, \xi_t)dW_t + \gamma(t)d\xi_t,$$

where $m_t := \mathbb{P}_{(X_t, \xi_t)}$ and ξ a non-decreasing, càdlàg control.

The goal is maximising the reward functional

$$J(\xi) := \int_0^T f(t, m_t)dt + g(m_T) - \mathbb{E}\left[\int_{[0,T]} c(t)d\xi_t\right].$$

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$\rightarrow$ How to extend this to $c(t, m_t, X_t, \xi_t)$?

Connection to Regular Controls

Suppose ξ is absolutely continuous with $\|\dot{\xi}_t\| \leq K$. $\rightarrow$ bounded velocity
We can view instead $u_t := \dot{\xi}_t$ as the control for the new dynamics

$$\begin{aligned} dX_t &= [b(t, X_t, \xi_t) + \gamma(t) u_t]\, dt + \sigma(t, X_t, \xi_t) dW_t \\ d\xi_t &= u_t dt. \end{aligned}$$

The reward functional can then be written as

$$J(u) = \mathbb{E}\left[\int_0^T f(t, X_t, \xi_t) dt + g(X_T, \xi_T) - \int_0^T c(t) \cdot u_t dt\right].$$

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$$

$\rightarrow$ Singular controls arise as the limit of bounded velocity controls.

The Extended Reward Functional

We define

$$J(t, m; \xi) := \int_t^T f(s, m_s) ds + g(m_T) - \mathbb{E}\left[\int_t^T c(s, m_s, X_s, \xi_s) d\xi_s\right],$$

for absolutely continuous controls.

For general singular controls we define the reward as the maximal reward we can get via absolutely continuous approximations

$$J(t, m, \xi) := \sup_{\xi^n \to \xi} \limsup_{n \to \infty} J(t, m, \xi^n).$$

The value function is defined as usual

$$V(t, m) := \sup_{\xi,\ \xi_{t-} = m} J(t, m, \xi).$$

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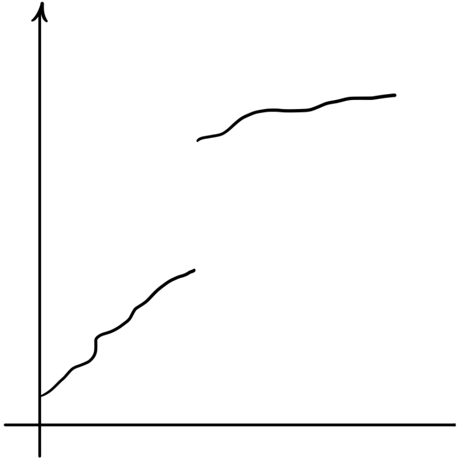
$\rightarrow$ Use the weak M_1 topology.

The M_1 topology in $\mathbb{R}$



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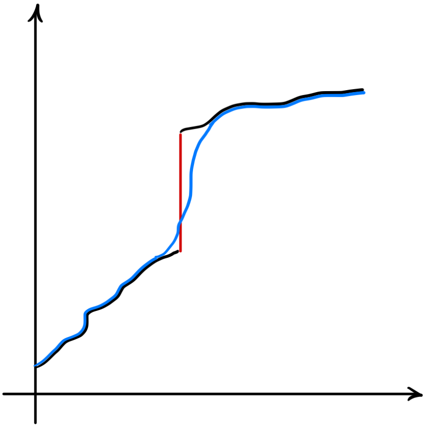

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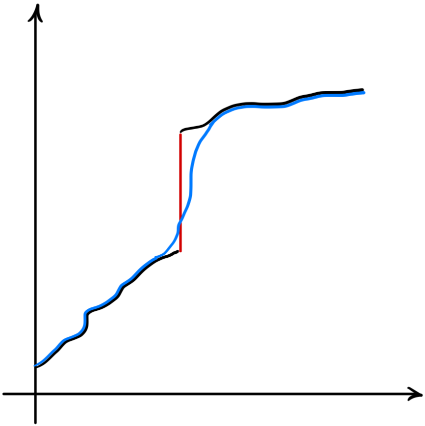
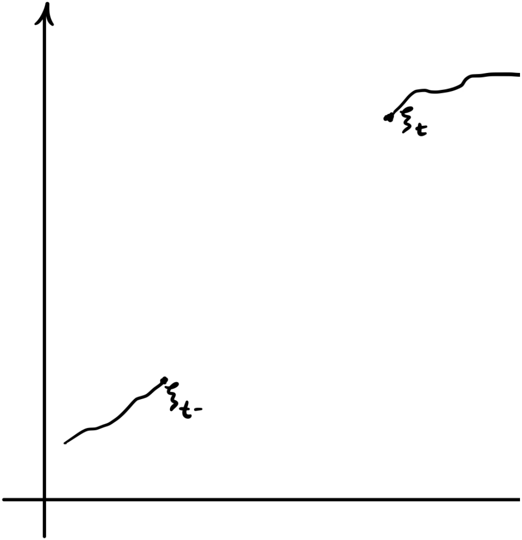


So we define the costs for such a jump from $(t-, \xi_{t-})$ to (t, ξ_t) as

$$\int_0^{\xi_t - \xi_{t-}} c(t, m_t, X_{t-} + \gamma(t)\zeta, \xi_{t-} + \zeta) d\zeta.$$

The Weak M_1 topology in $\mathbb{R}^d$



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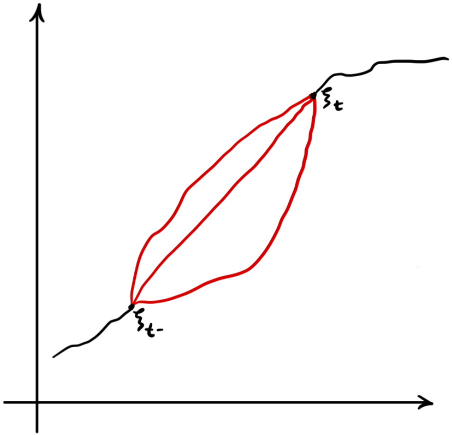

So we define

$$C_\xi(t, m, X_{t-}, \xi_{t-}, \xi_t) := \inf_{\zeta \in \Xi(\xi,\xi')} \int_0^1 c(t, m, X_{t-} + \gamma(t)(\zeta_\lambda - \xi_{t-}), \zeta_\lambda) d\zeta_\lambda,$$

over the set $\Xi(\xi, \xi')$ of all absolutely continuous and monotone paths from ξ_{t-} to ξ_t.

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$\rightarrow$ We can generalise this idea to jumps in $t \mapsto m_t \in \mathcal{P}_2$ and define

$$C_m(t, m, m')$$

as the minimal costs of an interpolating path from $(t-, m)$ to (t, m').

An Explicit Representation of J

Theorem

The reward functional J defined before has the following alternative characterisation

$$
\begin{aligned}
&J(t,m,\xi) \\
&= \int_t^T f(s,m_s)ds + g(m_T) - \sum_{J_{[t,T]}(m)} C_m(s,m_{s-},m_s) \\
&\quad - \mathbb{E}\Bigg[\sum_{J^c_{[t,T]}(m)\cap J_{[t,T]}(\xi)} C_\xi(s,m_s,X_{s-},\xi_{s-},\xi_s) \\
&\quad + \int_{J^c_{[t,T]}(m)\cap J^c_{[t,T]}(\xi)} c(s,m_s,X_s,\xi_s)d\xi_s \Bigg],
\end{aligned}
$$

where J and J^c denote the jump and continuity sets respectively.

Dynamic Programming Principle

Theorem

Let $(t, m) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^l)$. For all $s \in [t, T]$, we have the following dynamic programming principle

$$
\begin{aligned}
&V(t, m) \\
&= \sup_{\xi, \xi_{t-} = m} \Bigg[V(s, m_{s-}) + \int_t^s f(r, m_r) dr - \sum_{J_{[t,s]}(m)} C_m(r, m_{r-}, m_r) \\
&\quad - \mathbb{E}\Big[\sum_{J^c_{[t,s]}(m) \cap J_{[t,s]}(\xi)} C_\xi(r, m_r, X_{r-}, \xi_{r-}, \xi_r) \\
&\quad + \int_{J^c_{[t,s]}(m) \cap J^c_{[t,s]}(\xi)} c(r, m_r, X_r, \xi_r) d\xi_r \Big] \Bigg].
\end{aligned}
$$

Wasserstein Calculus

A function $u : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ admits a linear derivative if there exists a function $\delta_m u : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that

$$u(m') - u(m) = \int_0^1 \int_{\mathbb{R}^d} \delta_m u(\lambda m' + (1-\lambda)m, x)(m' - m)(dx) d\lambda.$$

As example, for functions of the form

$$u(m) = \int_{\mathbb{R}^d \times \mathbb{R}^l} \psi(x) m(dx),$$

the linear derivative is given, up to an additive constant, by

$$\delta_m u(m, x) = \psi(x).$$

QVI, Continuation Region

Theorem (Itô formula without jumps by Cosso et al, 2022)

Let $u \in C_2^{1,2}$ and $\xi_r = \xi_{t-}$ for all $r \in [t, s]$. Then

$$u(s, m_s) = u(t, m_t) + \int_t^s \mathbb{L}u(r, m_r)dr,$$

where $\mathbb{L}$ is the infinitesimal generator of $(t, X_t)_{t\in[0,T]}$ defined by

$$\begin{aligned}\mathbb{L}u(t,m) \coloneqq \partial_t u(t,m) + \int_{\mathbb{R}^d\times\mathbb{R}^l} & b(t,m,x,\xi)\cdot\partial_x\delta_m u(t,m,x,\xi) \\ & + \frac{1}{2}\sigma\sigma^T(t,m,x,\xi) : \partial_{xx}^2\delta_m u(t,m,x,\xi)m(dx,d\xi).\end{aligned}$$

$\rightarrow$ In the continuation region the optimal control is $\xi_t = \xi_{t-}$, hence we can derive an HJB from the DPP and Itô's formula.

$$-\mathbb{L}u - f = 0.$$

QVI, Intervention Region

The intervention region should be characterised by

$$V(t,m) = \sup_{m' \preceq_t m, m' \neq m} [V(t,m') - C_m(t,m,m')],$$

where $\{m' \preceq_t m\}$ are the reachable states (t, m') starting from $(t-, m)$.

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$\to$ This is not a good characterisation since $m'_n \to m$ always gives equality.

Lemma

Assume that $u \in C_2^{1,2}$ and let $t \in [0,T]$ be fixed. Let $m \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^l)$, if we have

$$u(t,m) + C_m(t,m,m') \geq u(t,m') \qquad \text{for all } m' \preceq_t m,$$

then

$$\partial_{(x,\xi)} \delta_m u(t,m,x,\xi) \cdot (\gamma(t), 1) \leq c(t,m,x,\xi) \qquad \text{for all } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^l.$$

Quasi-Variational Inequality

Theorem

Suppose the value function V is continuous, then it is the minimal bounded viscosity supersolution to

$$\min\Big\{ -\mathbb{L}u(t,m) - f(t,m), \\ \inf_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^l} \big[c(t,m,x,\xi) - \partial_{(x,\xi)}\delta_m u(t,m,x,\xi)\cdot(\gamma(t),1)\big] \Big\} = 0,$$

$$\min\Big\{ u(T,m) - g(T,m), \\ \inf_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^l} \big[c(T,m,x,\xi) - \partial_{(x,\xi)}\delta_m u(T,m,x,\xi)\cdot(\gamma(t),1)\big] \Big\} = 0.$$

Proof Idea: Connection to Regular Controls (revisited)

Recall the bounded velocity control problem:
We consider a regular control u with $u \geq 0$ and $\|u\| \leq K$ and

$$\begin{aligned} dX_t &= [b(t, m_t, X_t, \xi_t) + \gamma(t) u_t]\, dt + \sigma(t, m_t, X_t, \xi_t) dW_t \\ d\xi_t &= u_t dt. \end{aligned}$$

We want to maximise

$$J(t, m; u) = \int_t^T f(t, m_t) dt + g(m_T) - \mathbb{E}\left[\int_{[t,T]} c(t, m_t, X_t, \xi_t) \cdot u_t dt\right].$$

Define the corresponding value function as

$$V_K(t, m) := \sup_{u \geq 0, \|u\| \leq K} J(t, m; u).$$

$\rightarrow$ We expect $V_K \uparrow V$ as $K \to \infty$.

Proof Idea: The Master Equation for V_K

The master equation for the bounded velocity problem is

$$
\begin{aligned}
&-\mathbb{L}v(t,m) - f(t,m) \\
&+ \int_{\mathbb{R}^d \times \mathbb{R}^l} \inf_{u \geq 0, \|u\| \leq K} \Big\{ \big[c(t,m,x,\xi) - \partial_{(x,\xi)} \delta_m v(t,m,x,\xi) \cdot (\gamma(t),1) \big] \cdot u \Big\} m(dx,d\xi) \\
&= 0, \\
&v(T,\cdot) = g.
\end{aligned}
$$

$\rightarrow$ This equation looks very similar to our QVI.

Quasi-Variational Inequality (revisited)

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$$\min\Big\{ u(T,m) - g(T,m),$$

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Thank you

preprint on arXiv soon