Mean Field Game of Mutual Holding (MFG–MH)

Joint work with Nizar Touzi

Mao Fabrice Djete

École Polytechnique

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Motivation and Presentation

1. $N$–Player game
2. Intuition of the limit: Mean Field Game of Mutual Holding

MFG of Mutual Holding

1. Formulation of the problem
2. MFG of Mutual Holding
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Goal: Study optimal behavior of many agents who can hold each other
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2–Player game model

- Agents 1 and 2 hold each other through the dynamics $X^1$ and $X^2$:

Agent 1 holds part of Agent 2 via $\pi^{1,2}$ and Agent 2 holds part of Agent 1 via $\pi^{2,1}$
Motivation and Presentation

\textbf{N}-Player game

\textbf{Goal:} Study optimal behavior of many agents who can hold each other

\textbf{2-Player game model}

- Agents 1 and 2 hold each other through the dynamics $X^1$ and $X^2$:
  - Agent 1 holds part of Agent 2 via $\pi^{1,2}$ and Agent 2 holds part of Agent 1 via $\pi^{2,1}$

\[
\begin{align*}
    dX^1_t &= dP^1_t + \pi^{1,2}_t dX^2_t - \pi^{2,1}_t dX^1_t \\
    dX^2_t &= dP^2_t + \pi^{2,1}_t dX^1_t - \pi^{1,2}_t dX^2_t
\end{align*}
\]

with

\[
\begin{align*}
    dP^1_t &= b^1_t dt + \sigma^1_t dW^1_t + \sigma^{0,1}_t dW^0_t \\
    dP^2_t &= b^2_t dt + \sigma^2_t dW^2_t + \sigma^{0,2}_t dW^0_t
\end{align*}
\]
Goal: Study optimal behavior of many agents who can hold each other

2–Player game model

- Agents 1 and 2 hold each other through the dynamics $X^1$ and $X^2$:

  Agent 1 holds part of Agent 2 via $\pi^{1,2}$ and Agent 2 holds part of Agent 1 via $\pi^{2,1}$

  \[
  dX_t^1 = dP_t^1 + \pi_t^{1,2} dX_t^2 - \pi_t^{2,1} dX_t^1 \\
  \text{with} \quad dP_t^1 = b_t^1 dt + \sigma_t^1 dW_t + \sigma_t^{0,1} dW_t^0
  \]

  and

  \[
  dX_t^2 = dP_t^2 + \pi_t^{2,1} dX_t^1 - \pi_t^{1,2} dX_t^2 \\
  \text{with} \quad dP_t^2 = b_t^2 dt + \sigma_t^2 dW_t + \sigma_t^{0,2} dW_t^0
  \]

- $\pi^{1,2}$ and $\pi^{2,1}$ are the strategies/controls of players.

- Reward:

  \[
  J_1(\pi^{1,2}, \pi^{2,1}) := \mathbb{E}[U(X_T^1)] \quad \text{and} \quad J_2(\pi^{1,2}, \pi^{2,1}) := \mathbb{E}[U(X_T^2)]
  \]

- Objective: Find $(\pi^{*,1,2}, \pi^{*,2,1})$ a Nash equilibrium
N–Player game formulation

- Asset $X^i$ of agent $i = 1, \ldots, N$ follows:

$$\begin{align*}
\text{Part I hold} & \quad \text{Part owned by others} \\
P^i_t \text{ and } (\pi^i,j X^j) & \quad \text{d}X^i_t = dP^i_t - \sum_{j=1}^{N} \pi^i,j dX^j_t - \sum_{j=1}^{N} \pi^j,i dX^i_t
\end{align*}$$

with $dP^i_t = b^i_t dt + \sigma^i_t dW^i_t + \sigma^0,i dW^0_t$ and $\pi^i,j$ is the investment of agent $i$ in agent $j$.

- The control of agent $i$ is

$$\Pi^i := (\pi^{i,1}, \ldots, \pi^{i,N}).$$

- Reward of agent $i$ is

$$J_i(\Pi^1, \ldots, \Pi^N) := \mathbb{E}[U(X^i_T)]$$
$N$–Player game

$N$–player game formulation

- Asset $X^i$ of agent $i = 1, \ldots, N$ follows:

$$
\begin{align*}
\text{Part I hold} & \quad \text{Part owned by others} \\
P_t^i \text{ and } (\pi_t^{i,j}X_t^j)_j & \quad (\pi_t^{j,i}X_t^i)_j \\
\rightarrow dX_t^i = dP_t^i + \sum_{j=1}^{N} \pi_t^{i,j}dX_t^j - \sum_{j=1}^{N} \pi_t^{j,i}dX_t^i
\end{align*}
$$

with $dP_t^i = b_t^i dt + \sigma_t^i dW_t + \sigma_0^i dW_0$ and $\pi_t^{i,j}$ is the investment of agent $i$ in agent $j$.

- The control of agent $i$ is

$$
\Pi^i := (\pi_t^{i,1}, \ldots, \pi_t^{i,N}).
$$

- Reward of agent $i$ is

$$
J_i(\Pi^1, \ldots, \Pi^N) := \mathbb{E}[U(X_T^i)]
$$

- **Goal:** Find a Nash equilibrium $(\Pi^1, \ldots, \Pi^N)$ i.e. for each $i$

$$
J_i(\Pi^1, \ldots, \Pi^N) \geq J_i(\Pi^1, \ldots, \Pi^{i-1}, \beta, \Pi^{i+1}, \ldots, \Pi^N), \text{ for all } \beta = (\beta^1, \ldots, \beta^N)
$$

**Literature:** Bertucci–Touzi, Bassou–Touzi (*in preparation*)

- **Important feature**

Control of **one** agent $i \Pi_t^i$ is s.t. $\Pi_t^i \in \mathbb{R} \times \cdots \times \mathbb{R} \longrightarrow$ what happens when $N \rightarrow \infty$

$N$ times

Mao Fabrice Djeté

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9th International Colloquium on Backward Stochastic Differential Equations and Mean Field Systems
Motivation and Presentation

- N-Player game
- Intuition of the limit: Mean Field Game of Mutual Holding

MFG of Mutual Holding

- Formulation of the problem
- MFG of Mutual Holding
Recall the formulation

\[ dX_t^i = dP_t^i + \sum_{j=1}^{N} \pi_{t}^{i,j} dX_t^j - \sum_{j=1}^{N} \pi_{t}^{j,i} dX_t^i \quad \text{with} \quad dP_t^i = b_t^i dt + \sigma_t^i dW_t^i + \sigma_t^{0,i} dB_t. \]

control of agent \( i \) is \( \Pi^i := (\pi^{i,1}, \ldots, \pi^{i,N}) \)
Recall the formulation

$$dX_t^i = dP_t^i + \sum_{j=1}^{N} \pi_{t}^{i,j} dX_t^j - \sum_{j=1}^{N} \pi_{t}^{j,i} dX_t^i$$  
with  
$$dP_t^i = b_t^i dt + \sigma_t^i dW_t^i + \sigma_t^{0,i} dB_t.$$  

control of agent $i$ is $\Pi^i := (\pi^{i,1}, \cdots, \pi^{i,N})$

Need of symmetry and rescaling

$$(b_t^i, \sigma_t^i, \sigma_t^{0,i}) \xrightarrow{\text{replaced by}} (b, \sigma, \sigma^0)(t, X_t^i, \mu^N_t)$$  
and  
$$\pi_{t}^{i,j} \xrightarrow{\text{replaced by}} \frac{1}{N} \pi_{t}^{i,j}$$
Recall the formulation

\[ dX_t^i = dP_t^i + \sum_{j=1}^{N} \pi_t^{i,j} dX_t^j - \sum_{j=1}^{N} \pi_t^{j,i} dX_t^i \] with \[ dP_t^i = b_t^i dt + \sigma_t^i dW_t^i + \sigma_t^{0,i} dB_t. \]

control of agent \( i \) is \( \Pi^i := (\pi_i^1, \cdots, \pi_i^N) \)

Need of symmetry and rescaling

\( (b_t^i, \sigma_t^i, \sigma_t^{0,i}) \) replaced by \( (b, \sigma, \sigma^0)(t, X_t^i, \mu_t^N) \) and \( \pi_t^{i,j} \) replaced by \( \frac{1}{N} \pi_t^{i,j} \)

Intuition of the optimal control

\( \pi_t^{i,j} \) the optimal investment of agent \( i \) in agent \( j \) has the shape

\[ \pi_t^{i,j} = \pi(t, X_t^i, X_t^j, \mu_t^N) \]
Following our guessing, the deviating player’s dynamic is rewritten

\[ dX_t^i = dP_t^i + \frac{1}{N} \sum_{j=1}^{N} \beta(t, X_t^i, X_t^j, \mu_t^N) dX_t^j - \frac{1}{N} \sum_{j=1}^{N} \pi(t, X_t^j, X_t^i, \mu_t^N) dX_t^i \]
Motivation and Presentation

Intuition of the limit: Mean Field Game of Mutual Holding

- Following our guessing, the deviating player’s dynamic is rewritten

\[
dX^i_t = dP^i_t + \frac{1}{N} \sum_{j=1}^{N} \beta(t, X^i_t, X^j_t, \mu^N_t) dX^j_t - \frac{1}{N} \sum_{j=1}^{N} \pi(t, X^i_t, X^j_t, \mu^N_t) dX^i_t
\]

- **Optimization problem** (by propagation of chaos intuition) \((\pi, \mu)\) solves

\[
\hat{E}^\mu [U(\hat{X}_T)] = E[U(X_T^{\pi, \pi, \mu})] \geq E[U(X_T^{\beta, \pi, \mu})], \text{ for each } \beta
\]

where \(X^{\beta, \pi, \mu} := X^\beta\) with

\[
X^\beta = P^\beta + \hat{E}^\mu \left[ \int_0^T \beta(t, X^\beta_t, \hat{X}_t, \mu_t) d\hat{X}_t \right] - \int_0^T \hat{E}^\mu \left[ \pi(t, \hat{X}_t, X^\beta_t, \mu_t) \right] dX^\beta_t
\]

and

\[
dP^\beta_t = b(t, X^\beta_t, \mu_t) dt + \sigma(t, X^\beta_t, \mu_t) dW_t + \sigma^0(t, X^\beta_t, \mu_t) dW^0_t
\]

\[\rightarrow \text{Two parameters} \text{ are fixed!}\]
Following our guessing, the deviating player’s dynamic is rewritten

\[ dX^i_t = dP^i_t + \frac{1}{N} \sum_{j=1}^{N} \beta(t, X^i_t, X^j_t, \mu^N_t) dX^j_t - \frac{1}{N} \sum_{j=1}^{N} \pi(t, X^j_t, X^i_t, \mu^N_t) dX^i_t \]

Optimization problem (by propagation of chaos intuition) \((\pi, \mu)\) solves

\[ \hat{E}^{\mu} [U(\hat{X}_T)] = E[U(X^\pi_T, \pi, \mu)] \geq E[U(X^\beta_T, \pi, \mu)], \text{ for each } \beta \]

where \(X^{\beta, \pi, \mu} := X^\beta\) with

\[ X^\beta = P^\beta + \hat{E}^{\mu} \left[ \int_0^T \beta(t, X^\beta_t, \hat{X}_t, \mu_t) d\hat{X}_t \right] - \int_0^T \hat{E}^{\mu} \left[ \pi(t, \hat{X}_t, X^\beta_t, \mu_t) \right] dX^\beta_t \]

and \(dP^\beta_t = b(t, X^\beta_t, \mu_t) dt + \sigma(t, X^\beta_t, \mu_t) dW_t + \sigma^0(t, X^\beta_t, \mu_t) dW^0_t \)

\(\rightarrow\) Two parameters are fixed!

Optimal solution

\[ X = P + \hat{E}^{\mu} \left[ \int_0^T \pi(t, X_t, \hat{X}_t, \mu_t) d\hat{X}_t \right] - \int_0^T \hat{E}^{\mu} \left[ \pi(t, \hat{X}_t, X_t, \mu_t) \right] dX_t, \]

with \(dP_t = b(t, X_t, \mu_t) dt + \sigma(t, X_t, \mu_t) dW_t + \sigma^0(t, X_t, \mu_t) dW^0_t\) and \(\mu = \mathcal{L}(X|W^0)\).
Following our guessing, the deviating player’s dynamic is rewritten

\[ dX_t^i = dP_t^i + \frac{1}{N} \sum_{j=1}^{N} \beta(t, X_t^i, X_t^j, \mu_t^N) dX_t^j - \frac{1}{N} \sum_{j=1}^{N} \pi(t, X_t^j, X_t^i, \mu_t^N) dX_t^i \]

Optimization problem (by propagation of chaos intuition) \((\pi, \mu)\) solves

\[ \hat{E}^\mu [U(\hat{X}_T)] = E[U(\hat{X}^\pi_T, \pi, \mu)] \geq E[U(X_T^\beta, \pi, \mu)] \]

where \(X^\beta, \pi, \mu := X^\beta\) with

\[ X_t^\beta = P_t^\beta + \hat{E}^\mu \left[ \int_0^t \beta(t, X_t^\beta, \hat{X}_t, \mu_t) d\hat{X}_t \right] - \int_0^t \hat{E}^\mu \left[ \pi(t, \hat{X}_t, X_t^\beta, \mu_t) \right] dX_t^\beta \]

and \(dP_t^\beta = b(t, X_t^\beta, \mu_t) dt + \sigma(t, X_t^\beta, \mu_t) dW_t + \sigma^0(t, X_t^\beta, \mu_t) dW_t^0\)

\[ \rightarrow \text{Two parameters are fixed!} \]

Optimal solution

\[ X_t = P_t + \hat{E}^\mu \left[ \int_0^t \pi(t, X_t, \hat{X}_t, \mu_t) d\hat{X}_t \right] - \int_0^t \hat{E}^\mu \left[ \pi(t, \hat{X}_t, X_t, \mu_t) \right] dX_t, \]

with \(dP_t = b(t, X_t, \mu_t) dt + \sigma(t, X_t, \mu_t) dW_t + \sigma^0(t, X_t, \mu_t) dW_t^0\) and \(\mu = \mathcal{L}(X|W^0)\).

Via \(\hat{E}^\mu [\int_0^t \cdots d\hat{X}_t]\), the (conditional) law of the differential appears in the dynamic!
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Optimization problem \( X \) the optimal process is semi–martingale i.e.

\[
dX_t = B^\mu(t, X)dt + \Sigma^\mu(t, X)dW_t + \Sigma^{\mu,0}(t, X)dW_t^0.
\]

Given \((\pi, \mu)\), the controlled process \( X^\beta \) is rewritten
Optimization problem $X$ the optimal process is semi–martingale i.e.
\[ dX_t = B^\mu(t, X_t)dt + \Sigma^\mu(t, X_t)dW_t + \Sigma^\mu,0(t, X_t)dW^0_t. \]

Given $(\pi, \mu)$, the controlled process $X^\beta$ is rewritten
\[ dX^\beta_t = \frac{\hat{\mathbb{E}}^\mu[\beta(t, X_t^\beta, \hat{X}_t, \mu_t) B^\mu(t, \hat{X})] + b(t, X_t^\beta, \mu_t)}{1 + \hat{\mathbb{E}}^\mu[\pi(t, \hat{X}_t, X_t^\beta, \mu_t)]} dt + \frac{\sigma(t, X_t^\beta, \mu_t)}{1 + \hat{\mathbb{E}}^\mu[\pi(t, \hat{X}_t, X_t^\beta, \mu_t)]} dW_t + \frac{\hat{\mathbb{E}}^\mu[\beta(t, X_t^\beta, \hat{X}_t, \mu_t) \Sigma^\mu,0(t, \hat{X})] + \sigma^0(t, X_t^\beta, \mu_t)}{1 + \hat{\mathbb{E}}^\mu[\pi(t, \hat{X}_t, X_t^\beta, \mu_t)]} dW^0_t. \]
Optimization problem \( X \) the optimal process is semi–martingale i.e.

\[
dX_t = B^\mu(t, X)dt + \Sigma^\mu(t, X)dW_t + \Sigma^{\mu,0}(t, X)dW^0_t.
\]

Given \((\pi, \mu)\), the controlled process \( X^\beta \) is rewritten

\[
dX^\beta_t = \frac{\hat{E}^\mu[\beta(t, X^\beta_t, \hat{X}_t, \mu_t)B^\mu(t, \hat{X})] + b(t, X^\beta_t, \mu_t)}{1 + \hat{E}^\mu[\pi(t, \hat{X}_t, X^\beta_t, \mu_t)]}dt + \frac{\sigma(t, X^\beta_t, \mu_t)}{1 + \hat{E}^\mu[\pi(t, \hat{X}_t, X^\beta_t, \mu_t)]}dW_t
\]

\[
+ \frac{\hat{E}^\mu[\beta(t, X^\beta_t, \hat{X}_t, \mu_t)\Sigma^{\mu,0}(t, \hat{X})] + \sigma^0(t, X^\beta_t, \mu_t)}{1 + \hat{E}^\mu[\pi(t, \hat{X}_t, X^\beta_t, \mu_t)]}dW^0_t
\]

Drift and Volatility at the equilibrium: \( \mu_t(dx)dt \) almost every \((t, x)\)

\[
\Sigma^\mu(t, x) = \frac{\sigma(t, x, \mu_t)}{1 + \int \pi(t, y, x, \mu_t)\mu_t(dy)}
\]

and

\[
(B^\mu, \Sigma^\mu)(t, x) = \frac{\int \pi(t, x, y, \mu_t)(B^\mu, \Sigma^\mu)(dy) + (b, \sigma^0)(t, x, \mu_t)}{1 + \int \pi(t, y, x, \mu_t)\mu_t(dy)}
\]

\( \rightarrow \) Optimization problem with control of volatility !

\( \rightarrow \) Equations over the drift and the volatility
Simple representation

- Let \((F, G, G^0)\) be known. Given \(\mu, \pi\) and \((B, \Sigma, \Sigma^0)\),

\[
dX_t^\beta = F_t(X_t^\beta, \mu, \beta, \pi, B)\,dt + G_t(X_t^\beta, \mu, \beta, \pi, \Sigma)\,dW_t + G^0_t(X_t^\beta, \mu, \beta, \pi, \Sigma^0)\,dW^0_t
\]

1- Optimization

\[
\sup_{\beta \in \mathcal{A}} \mathbb{E}[U(X_T^\beta)] \xrightarrow{\text{leading to}} \beta_t^* = \beta^*(t, X_t^\beta, \mu, \pi, (B, \Sigma, \Sigma^0))
\]

2- Consistency properties

\[
\begin{bmatrix}
B_t, \Sigma_t, \Sigma_t^0
\end{bmatrix} = \begin{bmatrix}
F_t(x, \mu, \pi, \pi, B),
G_t(x, \mu, \pi, \Sigma),
G^0_t(x, \mu, \pi, \Sigma^0)
\end{bmatrix}
\]

and

\[
\beta^*(t, X_t^\beta, \mu, \pi, (B, \Sigma, \Sigma^0)) = \pi_t \quad \text{and} \quad \mu_t = \mathcal{L}(X_t^\beta^* | W^0)
\]
Formulation of the problem

Simple representation

- Let \((F, G, G^0)\) be known. Given \(\mu, \pi\) and \((B, \Sigma, \Sigma^0)\),

\[
dX_t^\beta = F_t(X_t^\beta, \mu, \beta, \pi, B)dt + G_t(X_t^\beta, \mu, \beta, \pi, \Sigma)dW_t + G_t^0(X_t^\beta, \mu, \beta, \pi, \Sigma^0)dW_t^0
\]

1- **Optimization**

\[
\sup_{\beta \in A} \mathbb{E}[U(X_T^\beta)] \quad \text{leading to} \quad \beta_t^* = \beta^*(t, X_t^\beta, \mu, \pi, (B, \Sigma, \Sigma^0))
\]

2- **Consistency properties**

\[
\begin{bmatrix}
B_t, \Sigma_t, \Sigma_t^0
\end{bmatrix} = \begin{bmatrix}
F_t(x, \mu, \pi, \pi, B), & G_t(x, \mu, \pi, \pi, \Sigma), & G_t^0(x, \mu, \pi, \pi, \Sigma^0)
\end{bmatrix}
\]

and

\[
\beta^*(t, X_t^\beta, \mu, \pi, (B, \Sigma, \Sigma^0)) = \pi_t \quad \text{and} \quad \mu_t = \mathcal{L}(X_{t^*}^\beta | W^0)
\]

- In our case: \(A = \{\text{All maps } \beta : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to [0, 1]\}\),

\[
G_t(x, \nu, \beta, \pi, \Sigma) = \frac{\sigma(t, x, \nu)}{1 + \int \pi(t, \hat{x}, x, \nu)d\hat{x}}
\]

and

\[
(F_t, G_t^0)(\cdots) = \frac{\int \beta(t, x, \hat{x}, \nu)(B, \Sigma^0)(t, \hat{x})\nu(d\hat{x}) + (b, \sigma^0)(t, x, \nu)}{1 + \int \pi(t, \hat{x}, x, \nu)\nu(d\hat{x})}
\]
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No common noise i.e. \( \sigma^0 = 0 \) Solving \( \sup_\beta \mathbb{E}[U(X_T^\beta)] \) leading to \( \beta^* \) where

\[
\begin{align*}
\text{d}X^\beta_t &= \frac{\widehat{\mathbb{E}}^\mu \left[ \beta(t, X_t^\beta, \hat{X}_t, \mu_t) B^\mu(t, \hat{X}) \right] + b(t, X_t^\beta, \mu_t)}{1 + \widehat{\mathbb{E}}^\mu \left[ \pi(t, \hat{X}_t, X_t^\beta, \mu_t) \right]} \text{d}t + \frac{\sigma(t, X_t^\beta, \mu_t)}{1 + \widehat{\mathbb{E}}^\mu \left[ \pi(t, \hat{X}_t, X_t^\beta, \mu_t) \right]} \text{d}W_t
\end{align*}
\]

and verifying

\[
\pi = \beta^* \quad + \quad \mu = \mathcal{L}(X^{\pi, \pi, \mu}) \quad + \quad \text{equation over the drift } B^\mu
\]
No common noise i.e. $\sigma^0 = 0$ Solving $\sup_\beta \mathbb{E}[U(X^\beta_T)]$ leading to $\beta^*$ where

$$dX^\beta_t = \frac{\widehat{\mathbb{E}}^\mu \left[ \beta(t, X^\beta_t, \hat{X}_t, \mu_t) B^\mu(t, \hat{X}) \right] + b(t, X^\beta_t, \mu_t)}{1 + \widehat{\mathbb{E}}^\mu \left[ \pi(t, \hat{X}_t, X^\beta_t, \mu_t) \right]} dt + \frac{\sigma(t, X^\beta_t, \mu_t)}{1 + \widehat{\mathbb{E}}^\mu \left[ \pi(t, \hat{X}_t, X^\beta_t, \mu_t) \right]} dW_t$$

and verifying

$$\pi = \beta^* + \mu = \mathcal{L}(X^{\pi, \pi, \mu}) + \text{equation over the drift } B^\mu$$

$\rightarrow$ No control of volatility!

$\rightarrow$ Not an obvious fact Indeed, remember $N$–player game

$$dX^i_t = dP^i_t + \frac{1}{N} \sum_{j=1}^N \beta^j_t dX^j_t - \frac{1}{N} \sum_{j=1}^N \pi^j,^i_t dX^i_t \rightarrow dX_t = \mathbf{M}(t, \beta, \Pi, \mu^N_t) \bullet \left[ b_t dt + \sigma_t \bullet dW_t \right] \text{.}$$

$N \times N$ matrix
Motivation and Presentation

No common noise i.e. $\sigma^0 = 0$ Solving $\sup_{\beta} \mathbb{E}[U(X_T^\beta)]$ leading to $\beta^*$ where

$$dX_t^\beta = \frac{\widehat{\mathbb{E}}^\mu \left[ \beta(t, X_t^\beta, \hat{X}_t, \mu_t) B^\mu(t, \hat{X}_t) \right]}{1 + \widehat{\mathbb{E}}^\mu \left[ \pi(t, \hat{X}_t, X_t^\beta, \mu_t) \right]} dt + \frac{\sigma(t, X_t^\beta, \mu_t)}{1 + \widehat{\mathbb{E}}^\mu \left[ \pi(t, \hat{X}_t, X_t^\beta, \mu_t) \right]} dW_t$$

and verifying

$$\pi = \beta^* + \mu = \mathcal{L}(X^\pi, \pi; \mu) + \text{ equation over the drift } B^\mu$$

$\rightarrow$ No control of volatility!

$\rightarrow$ Not an obvious fact Indeed, remember $N$–player game

$$dX_t^i = dP_t^i + \frac{1}{N} \sum_{j=1}^N \beta_t^j dX_t^j - \frac{1}{N} \sum_{j=1}^N \pi_t^{i,j} dX_t^i \quad \longrightarrow \quad dX_t = M(t, \beta, \Pi, \mu_t^N) \bullet \left[ b_t dt + \sigma_t \bullet dW_t \right].$$

**Theorem (D. and Touzi (2021))** Under technical conditions over $(b, \sigma, U)$ and $U \nearrow$, there is at least one MFG-MH equilibrium $(\pi^*, \mu)$ with $\pi^*(t, x, y) = 1_{\left\{ b(t, y, \mu_t) \geq -c(t, \mu_t) \right\}}$

$$B(t, x, m) := \left( \frac{1}{2} (b + c)^+ - (b + c)^- \right)(t, x, m) \quad \text{and} \quad \Sigma(t, x, m) := \frac{\sigma(t, x, m)}{1 + 1_{\left\{ B(t, x, m) \geq 0 \right\}}}$$

$c(t, m) \geq 0$ is the unique solution of the equation $c = \frac{1}{2} \int_{\mathbb{R}} \left( c + b(t, y, m) \right)^+ m(dy)$.
Non-negative drift i.e. \( b \geq 0 \quad \rightarrow \quad c(t, m) = \int_{\mathbb{R}} b(t, y, m)m(dy) \)

- **Optimal control** \( \pi^*(t, x, y) = 1 \)

- **Equilibrium dynamics** \( B(t, x, m) := \frac{1}{2} (b(t, x, m) + c(t, m)) \) and \( \Sigma(t, x, m) := \frac{1}{2} \sigma(t, x, m) \)
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- **Example** When \((b, \sigma)\) constant

\[
\begin{aligned}
\left\{ \begin{array}{l}
dP_t = b dt + \sigma dW_t \\
P_t \sim \mathcal{N}(bt, \sigma^2 t)
\end{array} \right. &\quad \implies \quad \left\{ \begin{array}{l}
dX^*_t = b dt + \frac{1}{2} \sigma dW_t \\
X^*_t \sim \mathcal{N}(bt, \frac{1}{4} \sigma^2 t)
\end{array} \right.
\end{aligned}
\]
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**Example** When \( (b, \sigma) \) constant

\[
\begin{align*}
\begin{cases}
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\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
dX^*_t &= bdt + \frac{1}{2} \sigma dW_t \\
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\end{cases}
\end{align*}
\]

Negative drift i.e. \( b < 0 \longrightarrow c(t, m) = 0 \)

- **Optimal control** \( \pi^*(t, x, y) = 0 \)

- **Equilibrium dynamics** \( B(t, x, m) := b(t, x, m) \) and \( \Sigma(t, x, m) := \sigma(t, x, m) \)
Non-negative drift i.e. $b \geq 0 \rightarrow c(t, m) = \int_{\mathbb{R}} b(t, y, m) m(dy)$

- **Optimal control** $\pi^*(t, x, y) = 1$

- **Equilibrium dynamics** $B(t, x, m) := \frac{1}{2} (b(t, x, m) + c(t, m))$ and $\Sigma(t, x, m) := \frac{1}{2} \sigma(t, x, m)$

- **Example** When $(b, \sigma)$ constant

  $\begin{aligned}
  \left\{ \begin{aligned}
  \mathrm{d}P_t &= b \mathrm{d}t + \sigma \mathrm{d}W_t \\
  P_t &\sim \mathcal{N}(bt, \sigma^2 t)
  \end{aligned} \right.
  \rightarrow
  \left\{ \begin{aligned}
  \mathrm{d}X^*_t &= b \mathrm{d}t + \frac{1}{2} \sigma \mathrm{d}W_t \\
  X^*_t &\sim \mathcal{N}(bt, \frac{1}{4} \sigma^2 t)
  \end{aligned} \right.
  \end{aligned}$

Negative drift i.e. $b < 0 \rightarrow c(t, m) = 0$

- **Optimal control** $\pi^*(t, x, y) = 0$

- **Equilibrium dynamics** $B(t, x, m) := b(t, x, m)$ and $\Sigma(t, x, m) := \sigma(t, x, m)$

**General drift** $\rightarrow$ No explicit $c(t, m)$! But, a combination of the two previous situations occurs
O–U dynamics i.e. $b(t, x, m) = \theta(m - x)$ and $\sigma(t, x, m) = \bar{\sigma}$
Approximate solution for the $N$–player game

Given $\Gamma := (\gamma^{i,j})_{1 \leq i,j \leq N}$,

$$dX_t^i = dP_t^i + \frac{1}{N} \sum_{i=1}^{N} \gamma_t^{i,j} dX_t^j - \frac{1}{N} \sum_{j=1}^{N} \gamma_t^{j,i} dX_t^j \rightarrow dX_t = B(t, X_t, \Gamma_t) dt + \Sigma(t, X_t, \Gamma_t) \cdot dW_t$$

Let $\pi(t, x^i, m^N) = \pi^i(t, x) := 1\{B(t, x^i, m^N) \geq 0\}$,

$$\Sigma^{i,j}(t, x) := \frac{\sigma(t, x^i, m^N) 1\{i=j\} + \frac{1}{N} A^j(t, x) \sigma(t, x^q, m^N)}{1 + \pi(t, x^i, m^N)} , \quad A^j(t, x) := \frac{\frac{\pi^j(t, x)}{1 + \pi^j(t, x)}}{1 - \frac{1}{N} \sum_{k=1}^{N} \frac{\pi^k(t, x)}{1 + \pi^k(t, x)}}$$

$$dX_t^i = B(t, X_t^i, \mu_t^N) dt + \sum_{j=1}^{N} \Sigma^{i,j}(t, X_t^i, \mu_t^N) dW_t^j \rightarrow dX_t = B(t, X_t, \Pi_t) dt + \Sigma(t, X_t, \Pi_t) \cdot dW_t$$

where $\Pi_t^N := (\pi^{i,j}_t)_{1 \leq i,j \leq N}$ with $\pi^{i,j}_t = \pi^j_t := 1\{B(t, X_t^j, \mu_t^N) \geq 0\}$

For $\beta := (\beta^1, \ldots, \beta^N)^T$, $\Gamma^{-i}(\beta) := \left((\gamma^1, \ldots, \gamma^{i-1}, \beta_t, \gamma^{i+1}, \ldots, \gamma^N)^T\right)^T$

**Theorem (D. and Touzi (2022))** For all $N \geq 1$,

$$| \Sigma^{k,k}(t, \Pi^{-i}_t(\beta), x) - \Sigma^{k,k}(t, \Pi_t, x) | + | B^{k}(t, \Pi^{-i}_t(\beta), x) - B^{k}(t, \Pi_t, x) | \leq \frac{C}{N} \quad \text{for all } k \neq i,$$

$$\sup_{1 \leq q \neq e \leq N} | \Sigma^{e,q}(t, \Pi_t, x) | + \sup_{1 \leq k \leq N} \left| \Sigma^{k,k}(t, \Pi_t, x) - \frac{\sigma(t, x^k, m^N_t(x))}{1 + \pi^k_t} \right| \leq \frac{C}{N}$$

and the mutual holding strategy $\Pi^N$ is an $\varepsilon_N$–Nash equilibrium with $\lim_{N \to \infty} \varepsilon_N = 0$. 

Mao Fabrice Djete

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THANK YOU FOR YOUR ATTENTION