9th International Colloquium on Backward Stochastic Differential Equations and Mean Field Systems

Mean Field Game of Mutual Holding (MFG–MH)

Joint work with Nizar Touzi

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2–Player game model

- Agents 1 and 2 hold each other through the dynamics X^1 and X^2 :

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$$\mathrm{d}X_t^1 = \mathrm{d}P_t^1 + \boxed{\pi_t^{1,2}\mathrm{d}X_t^2} - \boxed{\pi_t^{2,1}\mathrm{d}X_t^1} \quad \text{with} \quad \mathrm{d}P_t^1 = b_t^1\mathrm{d}t + \sigma_t^1\mathrm{d}W_t^1 + \sigma_t^{0,1}\mathrm{d}W_t^0$$

and

$$\mathrm{d}X_t^2 = \mathrm{d}P_t^2 + \boxed{\pi_t^{2,1}\mathrm{d}X_t^1} - \boxed{\pi_t^{1,2}\mathrm{d}X_t^2} \quad \text{with} \quad \mathrm{d}P_t^2 = b_t^2\mathrm{d}t + \sigma_t^2\mathrm{d}W_t^2 + \sigma_t^{0,2}\mathrm{d}W_t^0$$

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2–Player game model

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- $\pi^{1,2}$ and $\pi^{2,1}$ are the strategies/controls of players.
- Reward:

$$J_1(\pi^{1,2},\pi^{2,1}) := \mathbb{E}\big[U\big(X_T^1\big)\big] \text{ and } J_2(\pi^{1,2},\pi^{2,1}) := \mathbb{E}\big[U\big(X_T^2\big)\big]$$

- Objective: Find $(\pi^{\star,1,2},\ \pi^{\star,2,1})$ a Nash equilibrium

$N-$player game formulation

- Asset X^i of agent $i = 1, \dots, N$ follows:

Part I hold	Part owned by others
P^i_t and $(\pi^{i,j}_t X^j_t)_j$	$(\pi^{j,i}_t X^i_t)_j$

$$\rightarrow \mathrm{d}X^i_t = \mathrm{d}P^i_t + \sum_{j=1}^{N} \pi^{i,j}_t \mathrm{d}X^j_t - \sum_{j=1}^{N} \pi^{j,i}_t \mathrm{d}X^i_t$$

  with $\mathrm{d}P^i_t = b^i_t \mathrm{d}t + \sigma^i_t \mathrm{d}W^i_t + \sigma^{0,i}_t \mathrm{d}W^0_t$ and $\pi^{i,j}$ is the investment of agent i in agent j.

- The control of agent i is

$$\Pi^i := \big(\pi^{i,1}, \cdots, \pi^{i,N}\big).$$

- Reward of agent i is

$$J_i\big(\Pi^1, \cdots, \Pi^N\big) := \mathbb{E}\big[U(X^i_T)\big]$$

$N-$player game formulation

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$$\boxed{\begin{array}{ll} \text{Part I hold} & \text{Part owned by others} \\ P^i_t \text{ and } (\pi^{i,j}_t X^j_t)_j & (\pi^{j,i}_t X^i_t)_j \end{array}} \to \mathrm{d}X^i_t = \mathrm{d}P^i_t + \sum_{j=1}^N \pi^{i,j}_t \mathrm{d}X^j_t - \sum_{j=1}^N \pi^{j,i}_t \mathrm{d}X^i_t$$

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$$J_i\big(\Pi^1, \cdots, \Pi^N\big) := \mathbb{E}\big[U(X^i_T)\big]$$

- Goal: Find a Nash equilibrium $\big(\Pi^1, \cdots, \Pi^N\big)$ i.e. for each i

$$J_i\big(\Pi^1, \cdots, \Pi^N\big) \geq J_i\big(\Pi^1, \cdots, \Pi^{i-1}, \beta, \Pi^{i+1}, \cdots, \Pi^N\big), \text{ for all } \beta = (\beta^1, \cdots, \beta^N)$$

Literature: Bertucci–Touzi, Bassou–Touzi (*in preparation*)

- Important feature

Control of one agent i Π^i_t is s.t. $\Pi^i_t \in \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{N \text{ times}} \longrightarrow$ what happens when $N \to \infty$

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- Recall the formulation

$$\mathrm{d}X_t^i = \mathrm{d}P_t^i + \sum_{j=1}^{N} \pi_t^{i,j}\mathrm{d}X_t^j - \sum_{j=1}^{N} \pi_t^{j,i}\mathrm{d}X_t^i \;\text{ with }\; \mathrm{d}P_t^i = b_t^i\mathrm{d}t + \sigma_t^i\mathrm{d}W_t^i + \sigma_t^{0,i}\mathrm{d}B_t.$$

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control of agent i is $\Pi^i := \left(\pi^{i,1}, \cdots, \pi^{i,N}\right)$

- Need of symmetry and rescaling

$$\left(b_t^i, \sigma_t^i, \sigma_t^{0,i}\right) \xrightarrow{\text{replaced by}} (b, \sigma, \sigma^0)(t, X_t^i, \mu_t^N) \ \text{ and } \ \pi_t^{i,j} \xrightarrow{\text{replaced by}} \frac{1}{N}\pi_t^{i,j}$$

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$$\left(b_t^i, \sigma_t^i, \sigma_t^{0,i}\right) \xrightarrow{\text{replaced by}} (b,\sigma,\sigma^0)(t, X_t^i, \mu_t^N) \quad \text{and} \quad \pi_t^{i,j} \xrightarrow{\text{replaced by}} \frac{1}{N}\pi_t^{i,j}$$

- Intuition of the optimal control

$\pi^{i,j}$ the optimal investment of agent i in agent j has the shape

$$\pi_t^{i,j} = \pi\left(t, X_t^i, X_t^j, \mu_t^N\right)$$

- Following our guessing, the deviating player's dynamic is rewritten

$$\mathrm{d}X_t^i = \mathrm{d}P_t^i + \frac{1}{N}\sum_{j=1}^{N}\beta(t, X_t^i, X_t^j, \mu_t^N)\mathrm{d}X_t^j - \frac{1}{N}\sum_{j=1}^{N}\pi(t, X_t^j, X_t^i, \mu_t^N)\mathrm{d}X_t^i$$

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- Optimization problem (by propagation of chaos intuition) (π, μ) solves

$$\widehat{\mathbb{E}}^{\mu}\big[U\big(\widehat{X}_T\big)\big] = \mathbb{E}\big[U\big(X^{\pi,\pi,\mu}_T\big)\big] \geq \mathbb{E}\big[U\big(X^{\beta,\pi,\mu}_T\big)\big], \text{ for each } \beta$$

where $X^{\beta,\pi,\mu} := X^{\beta}$ with

$$X^{\beta}_{\cdot} = P^{\beta}_{\cdot} + \widehat{\mathbb{E}}^{\mu}\left[\int_0^{\cdot} \beta(t, X^{\beta}_t, \widehat{X}_t, \mu_t)\mathrm{d}\widehat{X}_t\right] - \int_0^{\cdot} \widehat{\mathbb{E}}^{\mu}\Big[\pi(t, \widehat{X}_t, X^{\beta}_t, \mu_t)\Big]\mathrm{d}X^{\beta}_t$$

and $\mathrm{d}P^{\beta}_t = b(t, X^{\beta}_t, \mu_t)\mathrm{d}t + \sigma(t, X^{\beta}_t, \mu_t)\mathrm{d}W_t + \sigma^0(t, X^{\beta}_t, \mu_t)\mathrm{d}W^0_t$

$\longrightarrow$ Two parameters are fixed !

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and $\mathrm{d}P_t^{\beta} = b(t, X_t^{\beta}, \mu_t)\mathrm{d}t + \sigma(t, X_t^{\beta}, \mu_t)\mathrm{d}W_t + \sigma^0(t, X_t^{\beta}, \mu_t)\mathrm{d}W_t^0$

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- Optimal solution

$$X_{\cdot} = P_{\cdot} + \widehat{\mathbb{E}}^{\mu}\bigg[\int_0^{\cdot}\pi(t, X_t, \widehat{X}_t, \mu_t)\mathrm{d}\widehat{X}_t\bigg] - \int_0^{\cdot}\widehat{\mathbb{E}}^{\mu}\Big[\pi(t, \widehat{X}_t, X_t, \mu_t)\Big]\mathrm{d}X_t,$$

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and $\mathrm{d}P_t^{\beta} = b(t,X_t^{\beta},\mu_t)\mathrm{d}t + \sigma(t,X_t^{\beta},\mu_t)\mathrm{d}W_t + \sigma^0(t,X_t^{\beta},\mu_t)\mathrm{d}W_t^0$

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with $\mathrm{d}P_t = b(t,X_t,\mu_t)\mathrm{d}t + \sigma(t,X_t,\mu_t)\mathrm{d}W_t + \sigma^0(t,X_t,\mu_t)\mathrm{d}W_t^0$ and $\mu = \mathcal{L}(X|W^0)$.

Via $\widehat{\mathbb{E}}^{\mu}\big[\int_0^{\cdot}\cdots\mathrm{d}\widehat{X}_t\big]$, the (conditional) law of the differential appears in the dynamic !

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- Optimization problem X the optimal process is semi–martingale i.e.

$$\mathrm{d}X_t = B^{\mu}(t,X)\mathrm{d}t + \Sigma^{\mu}(t,X)\mathrm{d}W_t + \Sigma^{\mu,0}(t,X)\mathrm{d}W^0_t.$$

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$$\mathrm{d}X^{\beta}_t = \frac{\widehat{\mathbb{E}}^{\mu}\left[\beta(t,X^{\beta}_t,\widehat{X}_t,\mu_t)B^{\mu}(t,\widehat{X})\right] + b(t,X^{\beta}_t,\mu_t)}{1+\widehat{\mathbb{E}}^{\mu}\left[\pi(t,\widehat{X}_t,X^{\beta}_t,\mu_t)\right]}\mathrm{d}t + \frac{\sigma(t,X^{\beta}_t,\mu_t)}{1+\widehat{\mathbb{E}}^{\mu}\left[\pi(t,\widehat{X}_t,X^{\beta}_t,\mu_t)\right]}\mathrm{d}W_t$$
$$+ \frac{\widehat{\mathbb{E}}^{\mu}\left[\beta(t,X^{\beta}_t,\widehat{X}_t,\mu_t)\Sigma^{\mu,0}(t,\widehat{X})\right] + \sigma^0(t,X^{\beta}_t,\mu_t)}{1+\widehat{\mathbb{E}}^{\mu}\left[\pi(t,\widehat{X}_t,X^{\beta}_t,\mu_t)\right]}\mathrm{d}W^0_t$$

- Optimization problem X the optimal process is semi–martingale i.e.

$$dX_t = B^\mu(t,X)dt + \Sigma^\mu(t,X)dW_t + \Sigma^{\mu,0}(t,X)dW_t^0.$$

Given (π,μ), the controlled process X^β is rewritten

$$dX_t^\beta = \frac{\widehat{\mathbb{E}}^\mu\left[\beta(t,X_t^\beta,\widehat{X}_t,\mu_t)B^\mu(t,\widehat{X})\right] + b(t,X_t^\beta,\mu_t)}{1+\widehat{\mathbb{E}}^\mu\left[\pi(t,\widehat{X}_t,X_t^\beta,\mu_t)\right]}dt + \frac{\sigma(t,X_t^\beta,\mu_t)}{1+\widehat{\mathbb{E}}^\mu\left[\pi(t,\widehat{X}_t,X_t^\beta,\mu_t)\right]}dW_t$$
$$+\frac{\widehat{\mathbb{E}}^\mu\left[\beta(t,X_t^\beta,\widehat{X}_t,\mu_t)\Sigma^{\mu,0}(t,\widehat{X})\right] + \sigma^0(t,X_t^\beta,\mu_t)}{1+\widehat{\mathbb{E}}^\mu\left[\pi(t,\widehat{X}_t,X_t^\beta,\mu_t)\right]}dW_t^0$$

- Drift and Volatility at the equilibrium: $\mu_t(dx)dt$ almost every (t,x)

$$\Sigma^\mu(t,x) = \frac{\sigma(t,x,\mu_t)}{1+\int \pi(t,y,x,\mu_t)\mu_t(dy)}$$

and

$$\left(B^\mu,\Sigma^\mu\right)(t,x) = \frac{\int \pi(t,x,y,\mu_t)\left(B^\mu,\Sigma^\mu\right)(dy) + (b,\sigma^0)(t,x,\mu_t)}{1+\int \pi(t,y,x,\mu_t)\mu_t(dy)}$$

$\longrightarrow$ Optimization problem with control of volatility !

$\longrightarrow$ Equations over the drift and the volatility

Simple representation

- Let (F, G, G^0) be known. Given μ, π and (B, Σ, Σ^0),

$$\mathrm{d}X_t^\beta = F_t(X_t^\beta, \mu, \beta, \pi, B)\mathrm{d}t + G_t(X_t^\beta, \mu, \beta, \pi, \Sigma)\mathrm{d}W_t + G_t^0(X_t^\beta, \mu, \beta, \pi, \Sigma^0)\mathrm{d}W_t^0$$

1- Optimization

$$\sup_{\beta \in A} \mathbb{E}[U(X_T^\beta)] \xrightarrow{\text{leading to}} \beta_t^\star = \beta^\star(t, X^\beta, \mu, \pi, (B, \Sigma, \Sigma^0))$$

2- Consistency properties

$$\Big[B_t,\ \Sigma_t,\ \Sigma_t^0\Big] = \Big[F_t(x, \mu, \pi, \pi, B),\ \ G_t(x, \mu, \pi, \pi, \Sigma),\ \ G_t^0(x, \mu, \pi, \pi, \Sigma^0)\Big]$$

and

$$\beta^\star(t, X^\beta, \mu, \pi, (B, \Sigma, \Sigma^0)) = \pi_t \quad \text{and} \quad \mu_t = \mathcal{L}(X_t^{\beta^\star} | W^0)$$

Simple representation

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- In our case: $A = \big\{\text{All maps } \beta : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to [0, 1]\big\}$,

$$G_t(x, \nu, \beta, \pi, \Sigma) = \frac{\sigma(t, x, \nu)}{1 + \int \pi(t, \widehat{x}, x, \nu)\mathrm{d}\widehat{x}}$$

and

$$(F_t, G_t^0)(\cdots) = \frac{\int \beta(t, x, \widehat{x}, \nu)(B, \Sigma^0)(t, \widehat{x})\nu(\mathrm{d}\widehat{x}) + (b, \sigma^0)(t, x, \nu)}{1 + \int \pi(t, \widehat{x}, x, \nu)\nu(\mathrm{d}\widehat{x})}$$

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No common noise i.e. $\boxed{\sigma^0 = 0}$ Solving $\sup_\beta \mathbb{E}[U(X_T^\beta)]$ leading to $\beta^\star$ where

$$\mathrm{d}X_t^\beta = \frac{\widehat{\mathbb{E}}^{\mu}\left[\beta(t, X_t^\beta, \widehat{X}_t, \mu_t) B^{\mu}(t, \widehat{X})\right] + b(t, X_t^\beta, \mu_t)}{1 + \widehat{\mathbb{E}}^{\mu}\left[\pi(t, \widehat{X}_t, X_t^\beta, \mu_t)\right]} \mathrm{d}t + \frac{\sigma(t, X_t^\beta, \mu_t)}{1 + \widehat{\mathbb{E}}^{\mu}\left[\pi(t, \widehat{X}_t, X_t^\beta, \mu_t)\right]} \mathrm{d}W_t$$

and verifying

$$\pi = \beta^\star \quad + \quad \mu = \mathcal{L}(X^{\pi,\pi,\mu}) \quad + \quad \text{equation over the drift } B^{\mu}$$

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$\longrightarrow$ No control of volatility !

$\longrightarrow$ Not an obvious fact Indeed, remember N–player game

$$\mathrm{d}X_t^i = \mathrm{d}P_t^i + \frac{1}{N}\sum_{j=1}^N \beta_t^j \mathrm{d}X_t^j - \frac{1}{N}\sum_{j=1}^N \pi_t^{j,i}\mathrm{d}X_t^i \longrightarrow \mathrm{d}\mathbf{X_t} = \underbrace{\mathbf{M}(t, \beta, \Pi, \mu_t^N)}_{N\times N \text{ matrix}} \bullet \left[\mathbf{b}_t \mathrm{d}t + \boldsymbol{\sigma}_t \bullet \mathrm{d}\mathbf{W}_t\right].$$

No common noise i.e. $\boxed{\sigma^0 = 0}$ Solving $\sup_\beta \mathbb{E}[U(X_T^\beta)]$ leading to $\beta^\star$ where

$$\mathrm{d}X_t^\beta = \frac{\widehat{\mathbb{E}}^\mu\left[\beta(t, X_t^\beta, \widehat{X}_t, \mu_t) B^\mu(t, \widehat{X})\right] + b(t, X_t^\beta, \mu_t)}{1 + \widehat{\mathbb{E}}^\mu\left[\pi(t, \widehat{X}_t, X_t^\beta, \mu_t)\right]}\mathrm{d}t + \frac{\sigma(t, X_t^\beta, \mu_t)}{1 + \widehat{\mathbb{E}}^\mu\left[\pi(t, \widehat{X}_t, X_t^\beta, \mu_t)\right]}\mathrm{d}W_t$$

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$$\mathrm{d}X_t^i = \mathrm{d}P_t^i + \frac{1}{N}\sum_{j=1}^N \beta_t^j \mathrm{d}X_t^j - \frac{1}{N}\sum_{j=1}^N \pi_t^{j,i}\mathrm{d}X_t^i \longrightarrow \mathrm{d}\mathbf{X_t} = \underbrace{\mathbf{M}(t, \beta, \Pi, \mu_t^N)}_{N \times N \text{ matrix}} \bullet \left[\mathbf{b}_t \mathrm{d}t + \boldsymbol{\sigma}_t \bullet \mathrm{d}\mathbf{W}_t\right].$$

Theorem (D. and Touzi (2021)) Under technical conditions over (b, σ, U) and $U \nearrow$, there is at least one MFG-MH equilibrium $(\pi^\star, \mu)$ with $\pi^\star(t, x, y) = \mathbf{1}_{\{b(t,y,\mu_t) \geq -c(t,\mu_t)\}}$

$$B(t, x, m) := \left(\frac{1}{2}(b + c)^+ - (b + c)^-\right)(t, x, m) \quad \text{and} \quad \Sigma(t, x, m) := \frac{\sigma(t, x, m)}{1 + \mathbf{1}_{\{B(t,x,m) \geq 0\}}}$$

$c(t, m) \geq 0$ is the unique solution of the equation $c = \frac{1}{2}\int_{\mathbb{R}} \left(c + b(t, y, m)\right)^+ m(\mathrm{d}y)$.

Non–negative drift i.e. $b \geq 0 \longrightarrow c(t,m) = \int_{\mathbb{R}} b(t,y,m)m(\mathrm{d}y)$

- Optimal control $\pi^*(t,x,y) = 1$
- Equilibrium dynamics $B(t,x,m) := \frac{1}{2}\big(b(t,x,m) + c(t,m)\big)$ and $\Sigma(t,x,m) := \frac{1}{2}\sigma(t,x,m)$

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- Example When (b,σ) constant

$$\boxed{\begin{cases} \mathrm{d}P_t = b\mathrm{d}t + \sigma \mathrm{d}W_t \\ P_t \sim \mathcal{N}(bt, \sigma^2 t) \end{cases}} \longrightarrow \boxed{\begin{cases} \mathrm{d}X^\star_t = b\mathrm{d}t + \frac{1}{2}\sigma \mathrm{d}W_t \\ X^\star_t \sim \mathcal{N}(bt, \frac{1}{4}\sigma^2 t) \end{cases}}$$

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Negative drift i.e. $b < 0 \longrightarrow c(t,m) = 0$

- Optimal control $\pi^*(t,x,y) = 0$
- Equilibrium dynamics $B(t,x,m) := b(t,x,m)$ and $\Sigma(t,x,m) := \sigma(t,x,m)$

Non–negative drift i.e. $b \geq 0 \longrightarrow c(t,m) = \int_{\mathbb{R}} b(t,y,m)m(\mathrm{d}y)$

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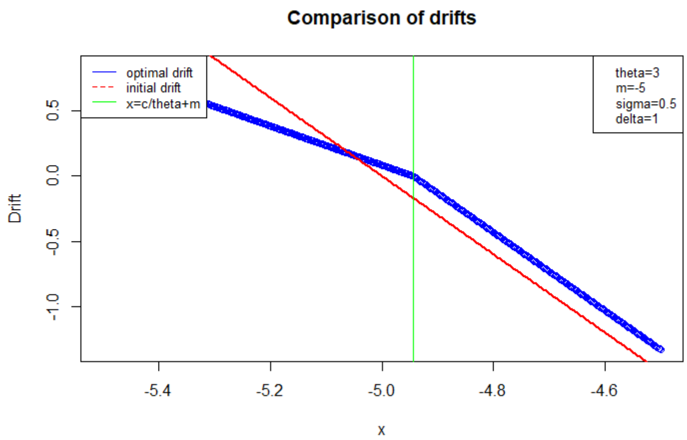
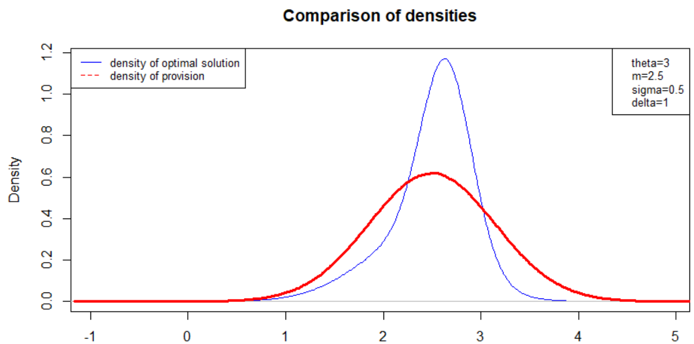
Negative drift i.e. $b < 0 \longrightarrow c(t,m) = 0$

- Optimal control $\pi^*(t,x,y) = 0$
- Equilibrium dynamics $B(t,x,m) := b(t,x,m)$ and $\Sigma(t,x,m) := \sigma(t,x,m)$

General drift $\longrightarrow$ No explicit $c(t,m)$! But, a combination of the two previous situations occurs

O–U dynamics i.e. $b(t,x,m) = \theta(\overline{m} - x)$ and $\sigma(t,x,m) = \overline{\sigma}$





Approximate solution for the $N-$player game Given $\Gamma := (\gamma^{i,j})_{1\le i,j\le N}$,

$$\mathrm{d}X_t^i = \mathrm{d}P_t^i + \frac{1}{N}\sum_{i=1}^N \gamma_t^{i,j}\mathrm{d}X_t^j - \frac{1}{N}\sum_{j=1}^N \gamma_t^{j,i}\mathrm{d}X_t^j \longrightarrow \mathrm{d}\mathbf{X}_t = \mathbf{B}(t,\mathbf{X}_t,\Gamma_t)\mathrm{d}t + \mathbf{\Sigma}(t,\mathbf{X}_t,\Gamma_t)\bullet\mathrm{d}\mathbf{W}_t$$

Let $\pi(t,x^i,m^N) = \pi^i(t,\mathbf{x}) := \mathbf{1}_{\{B(t,x^i,m^N)\ge 0\}}$,

$$\Sigma^{i,j}(t,\mathbf{x}) := \frac{\sigma(t,x^i,m^N)\mathbf{1}_{\{i=j\}} + \frac{1}{N}A^j(t,\mathbf{x})\sigma(t,x^q,m^N)}{1+\pi(t,x^i,m^N)},\quad A^j(t,\mathbf{x}) := \frac{\frac{\pi^j(t,\mathbf{x})}{1+\pi^j(t,\mathbf{x})}}{1-\frac{1}{N}\sum_{k=1}^N \frac{\pi^k(t,\mathbf{x})}{1+\pi^k(t,\mathbf{x})}}$$

$$\mathrm{d}X_t^i = B(t,X_t^i,\mu_t^N)\mathrm{d}t + \sum_{j=1}^N \Sigma^{i,j}(t,X_t^i,\mu_t^N)\mathrm{d}W_t^j \longrightarrow \mathrm{d}\mathbf{X}_t = \mathbf{B}(t,\mathbf{X}_t,\Pi_t)\mathrm{d}t + \mathbf{\Sigma}(t,\mathbf{X}_t,\Pi_t)\bullet\mathrm{d}\mathbf{W}_t$$

where $\boxed{\Pi_t^N := (\pi_t^{i,j})_{1\le i,j\le N} \text{ with } \pi_t^{i,j} = \pi_t^j := \mathbf{1}_{\{B(t,X_t^j,\mu_t^N)\ge 0\}}}$

For $\beta := (\beta^1,\cdots,\beta^N)^\intercal$, $\Gamma^{-i}(\beta) := \Big((\gamma^{1,\cdot})^\intercal,\cdots,(\gamma^{i-1,\cdot})^\intercal,\beta_t,(\gamma^{i+1,\cdot})^\intercal,\cdots,(\gamma^{N,\cdot})^\intercal\Big)^\intercal$

Theorem (D. and Touzi (2022)) For all $N\ge 1$,

$$\left|\mathbf{\Sigma}^{k,k}(t,\Pi_t^{-i}(\beta),\mathbf{x}) - \mathbf{\Sigma}^{k,k}(t,\Pi_t,\mathbf{x})\right| + \left|\mathbf{B}^k(t,\Pi_t^{-i}(\beta),\mathbf{x}) - \mathbf{B}^k(t,\Pi_t,\mathbf{x})\right| \le \frac{C}{N} \text{ for all } k\ne i,$$

$$\sup_{1\le q\ne e\le N}\left|\mathbf{\Sigma}^{e,q}(t,\Pi_t,\mathbf{x})\right| + \sup_{1\le k\le N}\left|\mathbf{\Sigma}^{k,k}(t,\Pi_t,\mathbf{x}) - \frac{\sigma(t,x^k,m^N(\mathbf{x}))}{1+\pi_t^k}\right| \le \frac{C}{N}$$

and the mutual holding strategy Π^N is an ε_N-Nash equilibrium with $\lim_{N\to\infty}\varepsilon_N = 0$.

THANK YOU FOR YOUR ATTENTION