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# REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH TIME-CHANGE LÉVY NOISES

joint work with Giulia di Nunno

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# Overview

- ⋆ Time change;

▷ Di Nunno, Sjursen, *BSDEs driven by time-changed Lévy noises and optimal control, Stochastic Processes and their Applications 124, 2014, 1679-1709.*

▷ G. Di Nunno, S. Sjursen, *On chaos representation and orthogonal polynomials for the doubly stochastic Poisson process, Robert C. Dalang, Marco Dozzi, Francesco Russo (Eds.), Seminar on Stochastic Analysis, Random Fields and Applications VII, in: Progress in Probability 67, Springer, Basel, 2013, 23–54.*

- ⋆ Existence & uniqueness of the solution of RBSDEs with time change and lower barrier;

▷ M. C. Quenez, A. Sulem, *Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps, Stochastic Processes and their Applications 5, 124, 2014, 3031–3054.*

- ⋆ Verification theorem;
- ⋆ Comparison theorem.

## Time-change process

- Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $X := [0, T] \times \mathbb{R}$, we will consider $X = ([0, T] \cup \{0\}) \cup ([0, T] \times \mathbb{R}_0)$, where $\mathbb{R}_0 = \mathbb{R} \backslash \{0\}$ and $T > 0$.
- Denote $\mathcal{B}_X$ the Borel $\sigma$-algebra on $X$. Throughout this presentation $\Delta \subset X$ denotes an element $\Delta$ in $\mathcal{B}_X$.
- Let $\lambda := (\lambda^B, \lambda^H)$ be a two dimensional stochastic process such that each component $\lambda^l$, $l = B, H$, satisfies
  (i) $\lambda^l_t \geq 0$ $\mathbb{P}$-a.s. for all $t \in [0, T]$,
  (ii) $\lim_{h \to 0} \mathbb{P}\left(\left|\lambda^l_{t+h} - \lambda^l_t\right| \geq \varepsilon\right) = 0$ for all $\varepsilon > 0$ and almost all $t \in [0, T]$,
  (iii) $\mathbb{E}\left[\int_0^T \lambda^l_t dt\right] < \infty$.
- The space of all processes $\lambda := (\lambda^B, \lambda^H)$ satisfying (i)-(iii) above is denoted by $\mathcal{L}$, and it is supplied with the random measure $\Lambda$ on $X$,

$$\Lambda(\Delta) := \int_0^T \mathbf{1}_{\{(t,0)\in\Delta\}}(t)\underline{\lambda^B_t dt} + \int_0^T \int_{\mathbb{R}_0} \mathbf{1}_\Delta(t,z)\nu(dz)\underline{\lambda^H_t dt}, \qquad (1)$$

  as the mixture of measures on disjoint sets. Here $\nu$ is a deterministic, $\sigma$-finite measure on the Borel sets of $\mathbb{R}_0$ satisfying $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$.

- We define the $\sigma$-algebra generated by the values of $\Lambda$ as $\mathcal{F}^{\Lambda}$, while $\Lambda^H$ denotes the restriction of $\Lambda$ to $[0, T] \times \mathbb{R}_0$ and $\Lambda^B$ the restriction of $\Lambda$ to $[0, T] \times \{0\}$.
- Hence for $\Delta \subseteq X$

$$\boxed{\Lambda(\Delta) = \Lambda^B(\Delta \cap [0, T] \times \{0\}) + \Lambda^H(\Delta \cap [0, T] \times \mathbb{R}_0)\,.}$$

- For $\Lambda$ It follow from (1) that

$$\Lambda(ds, \{0\}) = \lambda_s^B ds,$$
$$\Lambda(ds, dz) = \lambda_s^H ds, \; z \in \mathbb{R}_0.$$

## Definition ( Di Nunno, Sjursen, 2014)

**(A1)** $\mathbb{P}\left(B(\Delta) \leq x \mid \mathcal{F}^{\Lambda}\right) = \mathbb{P}\left(B(\Delta) \leq x \mid \Lambda^B(\Delta)\right) = \Phi\left(\frac{x}{\sqrt{\Lambda^B(\Delta)}}\right)$,
$x \in \mathbb{R}, \Delta \subseteq [0, T] \times \{0\}$,

**(A2)** $B(\Delta_1)$ and $B(\Delta_2)$ are conditionally independent given $\mathcal{F}^{\Lambda}$ whenever $\Delta_1$ and $\Delta_2$ are disjoint sets.

**(A3)** $\mathbb{P}\left(H(\Delta) = k \mid \mathcal{F}^{\Lambda}\right) = \mathbb{P}\left(H(\Delta) = k \mid \Lambda^H(\Delta)\right) = \frac{\Lambda^H(\Delta)^k}{k!} e^{-\Lambda^H(\Delta)}$,
$k \in \mathbb{N}, \Delta \subseteq, [0, T] \times \mathbb{R}_0$,

**(A4)** $H(\Delta_1)$ and $H(\Delta_2)$ are conditionally independent given $\mathcal{F}^{\Lambda}$ whenever $\Delta_1$ and $\Delta_2$ are disjoint sets.

**(A5)** $B$ and $H$ are conditionally independent given $\mathcal{F}^{\Lambda}$.

## Definition ( Di Nunno, Sjursen, 2014)

The random measure $\mu$ on the Borel subsets of $X$ is defined by

$$\boxed{\mu(\Delta) := B(\Delta \cap [0,T] \times \{0\}) + \tilde{H}\left(\Delta \cap [0,T] \times \mathbb{R}_0\right), \quad \Delta \subseteq X,} \tag{2}$$

where $\tilde{H} := H - \Lambda^H$ be the signed random measure given by

$$\tilde{H}(\Delta) = H(\Delta) - \Lambda^H(\Delta), \quad \Delta \subset [0,T] \times \mathbb{R}_0.$$

- <u>Properties</u>:

$\triangleright$ $$\left.\begin{array}{l} (A1) \Rightarrow \mathbb{E}\left[B(\Delta) \mid \mathcal{F}^\Lambda\right] = 0 \\ (A3) \Rightarrow \mathbb{E}\left[H(\Delta) \mid \mathcal{F}^\Lambda\right] = \Lambda^H(\Delta) \iff \mathbb{E}\left[\tilde{H}(\Delta) \mid \mathcal{F}^\Lambda\right] = 0 \end{array}\right\} \Rightarrow \mathbb{E}\left[\mu(\Delta) \mid \mathcal{F}^\Lambda\right] = 0$$

$\triangleright$ $$\left.\begin{array}{l} (A2),(A5) \Rightarrow \mathbb{E}\left[B(\Delta)^2 \mid \mathcal{F}^\Lambda\right] = \Lambda^B(\Delta) \\ (A4),(A5) \Rightarrow \mathbb{E}\left[\tilde{H}(\Delta)^2 \mid \mathcal{F}^\Lambda\right] = \Lambda^H(\Delta) \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} \mathbb{E}\left[\mu(\Delta)^2 \mid \mathcal{F}^\Lambda\right] = \Lambda(\Delta), \\ \mathbb{E}\left[\mu\left(\Delta_1\right)\mu\left(\Delta_2\right) \mid \mathcal{F}^\Lambda\right] = 0 \end{array}\right.$$

- The random measures $B$ and $H$ are related to a specific form of **time-change for Brownian motion and pure jump Lévy process**. More specifically define

$$B_t := B([0,t]\times\{0\}), \qquad \Lambda_t^B := \int_0^t \lambda_s^B ds,$$

$$\eta_t := \int_0^t \int_{\mathbb{R}_0} z\tilde{H}(ds,dz), \qquad \hat{\Lambda}_t^H := \int_0^t \lambda_s^H ds,\ t\in[0,T].$$

**Theorem 1. (Richard F. Serfozo, 1972, Bronius Grigelionis, 1972)**

Let $W_t, t\in[0,T]$ be a Brownian motion and $N_t, t\in[0,T]$ be a centered pure jump Lévy process with Lévy measure $\nu$. Assume that both $\underline{W \text{ and } N \text{ are independent of } \Lambda.}$ Then

- $B$ satisfies (A1) and (A2) if and only if, for any $t\geq 0$

$$B_t \overset{d}{=} W_{\Lambda_t^B},$$

- $\eta$ satisfies (A3) and (A4) if and only *if*, for any $t\geq 0$

$$\eta_t \overset{d}{=} N_{\hat{\Lambda}_t^H}.$$

## Filtrations

- Let us define $\mathbb{F}^\mu = \left\{\mathcal{F}_t^\mu, t \in [0,T]\right\}$ as the smallest filtration generated by $\mu(\Delta)$, $\Delta \subset [0,t] \times \mathbb{R}$. From the definition it follows that for any $t \in [0,T]$

$$\boxed{\mathcal{F}_t^\mu := \mathcal{F}_t^B \vee \mathcal{F}_t^H \vee \mathcal{F}_t^\Lambda,}$$

  where
  · $\mathcal{F}_t^B$ is generated by $B(\Delta \cap [0,T] \times \{0\})$,
  · $\mathcal{F}_t^H$ is generated by $H\left(\Delta \cap [0,T] \times \mathbb{R}_0\right)$,
  · $\mathcal{F}_t^\Lambda$ is generated by $\Lambda(\Delta)$, $\Delta \in [0,t] \times \mathbb{R}$.
  This follows from the application of the results from Winkel, 2001 & Di Nunno, Sjursen, 2013.
- Let us set $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$ where

$$\mathcal{F}_t = \bigcap_{r>t} \mathcal{F}_r^\mu.$$

- Furthermore, we set $\mathbb{G} = \{\mathcal{G}_t, t \in [0,T]\}$ where

$$\boxed{\mathcal{G}_t = \mathcal{F}_t^\mu \vee \mathcal{F}^\Lambda.}$$

  Remark that $\underline{\mathcal{G}_T = \mathcal{F}_T,\ \mathcal{G}_0 = \mathcal{F}^\Lambda, \text{ while } \mathcal{F}_0^\mu \text{ is trivial}}$.

  (In the sequel notation $\mathcal{F} = \mathcal{F}_T$ will be used.)

- Reference filtration $\mathbb{F}$ is the smallest right-continuous filtration to which $\mu$ is adapted.
- The filtration $\mathbb{G}$ is right-continuous.
- The random field $\mu$ is a martingale random field with respect to $\mathbb{F}$ (different representations hold under lack of informations) and $\mathbb{G}$, since:

  1. $\mu$ has a $\sigma$-finite variance measure;
  2. It is additive;
  3. Adapted;
  4. It has the martingale property;
  5. $\mu$ It has conditionally orthogonal values.

## Representation results

$\boxed{\mathbb{G}}$ Denote $\mathcal{I}^{\mathcal{G}}$ as the subspace of $L^2([0,T] \times \mathbb{R} \times \Omega, \mathcal{B}_X \times \mathbb{P}, \Lambda \times \mathbb{P})$ of the random fields admitting a $\mathbb{G}$-predictable modification, in particular

$$\|\phi\|_{\mathcal{I}^{\mathcal{G}}} := \left( \mathbb{E} \left[ \int_0^T \phi_s(0)^2 \lambda_s^B ds + \int_0^T \int_{\mathbb{R}_0} \phi_s(z)^2 \nu(dz) \lambda_s^H ds \right] \right)^{\frac{1}{2}} < \infty.$$

**Theorem 2.** (Di Nunno, Sjursen, 2014) Martingale representation under $\mathbb{G}$.

Assume $M_t$, t $\in [0, T]$, is a $\mathbb{G}$-**martingale**. Then there exists a unique $\phi \in \mathcal{I}^{\mathcal{G}}$ such that

$$M_t = \mathbb{E}\left[M_T \mid \mathcal{F}^{\Lambda}\right] + \int_0^t \int_{\mathbb{R}} \phi_s(z) \mu(ds, dz), \quad t \in [0, T],$$

**Theorem 3.** Version of Doob–Meyer decomposition under filtration $\mathbb{G}$.

Let $S_t$, $t \in [0, T]$, is **square integrable** $\mathbb{G}$-**supermartingale**. There exists a unique square-integrable martingal $M_t$, and a nondecreasing RCLL (right continuous with left limits) predictable process $A$, for which $E(A_T^2) < \infty$ and $A_0 = 0$, such that for every $t \in [0, T]$

$$S_t = M_t + A_t,$$

$\boxed{\mathbb{F}}$ Denote $\mathcal{I}^{\mathcal{F}}$ a set of random fields for which

$$\mathcal{I}^{\mathcal{F}} := \left\{ \varphi \in L^2(\Omega, \mathcal{F}, P), \mathbb{F} - predictable : \left( E\left[ \int_0^T \varphi_s^2 d\Lambda_s \right] \right)^{\frac{1}{2}} < \infty \right\}.$$

**Corollary 1. (Di Nunno, Sjursen, 2014) Martingale representation under $\mathbb{F}$.**

Assume $M_t$, t $\in [0, T]$, is a $\mathbb{F}$**-martingale** from $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a unique $\phi \in \mathcal{I}^{\mathcal{F}}$, and $\mathcal{F}$-martingale $N_t$ which are **orthogonal** to $\mu$, such that

$$M_t = N_t + \int_0^t \int_{\mathbb{R}} \phi_s(z)\mu(ds, dz), \quad t \in [0, T].$$

**Theorem 4. Version of Doob–Meyer decomposition under filtration $\mathbb{F}$.**

Assume $S_t^{\mathcal{F}}$,$t \in [0, T]$, is square integrable $\mathbb{F}$**-supermartingale**. Then there exist orthogonal martingale components $N_t$ and $\phi^{\mathcal{F}}$, a nondecreasing RCLL (right continuous with left limits) predictable process $A^{\mathcal{F}}$, for which $E[(A_T^{\mathcal{F}})^2] < \infty$ and $A_0^{\mathcal{F}} = 0$, such that for every $t \in [0, T]$

$$S_t^{\mathcal{F}} = N_t^{\mathcal{F}} + \int_0^t \int_{\mathbb{R}} \phi_s^{\mathcal{F}}(z)\mu(ds, dz) + A_t^{\mathcal{F}}.$$

# Reflected backward stochastic differential equations with time-change

## Lévy noises and lower barrier

For $\mathfrak{F} := \mathcal{G}, \mathcal{F}$ let us denote:

- $L^p(\mathfrak{F})$ is the set of random variables $\xi$ which are $\mathfrak{F}$-measurable and $p$-integrable, $p > 1$;
- $\mathcal{S}^{\mathfrak{F}_2}$ be the space of real-valued RCLL $\mathfrak{F}$-adapted stochastic processes $Y_t, t \in [0, T], \omega \in \Omega$, such that

$$\|Y\|_{\mathcal{S}^{\mathfrak{F}_2}} := \sqrt{\mathbb{E}\left[\sup_{0 \le t \le T} |Y_t|^2\right]} < \infty;$$

- $\mathcal{H}^{\mathcal{G}_2}$ is the space of $\mathfrak{F}$-predictable stochastic processes $f_t, t \in [0, T], \omega \in \Omega$, such that

$$\|f_s\|_{\mathcal{H}^{\mathfrak{F}:2}} := \mathbb{E}\left[\int_0^T f_s^2 ds\right] < \infty;$$

- $\mathcal{T}_0$ be the set of stopping times $\tau \in [0, T]$ a.s. .
- For $S$ in $\mathcal{T}_0$, let $\mathcal{T}_S$ be the set of stopping times $\tau$ such that $\tau \in [S, T]$ a.s. .

## Existence and uniqueness of the $\mathbb{G}$-solution

▷ For $T > 0$ let RBSDE driven by time-change Lévy noises and lower barrier under $\mathbb{G}$ be rewritten in following form

$$Y_t = Y_T + \int_t^T g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_t^T \phi_s(0)dB_s - \int_t^T\int_{\mathbb{R}_0} \phi_s(z)\tilde{H}(ds, dz) + \int_t^T dA_s,$$

which is equivalent to equation

$$Y_t = Y_T + \int_t^T g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dA_s,$$

where (*standard assumptions under* $\mathbb{G}$ *are*)

- $Y_T \in L^2(\Omega, \mathcal{G}, \mathbb{P})$ **(final value-reaching point of state process of BSDE).**
- **Function $g$ is a drift coefficient, ie *driver* which satisfies:**
  - $\cdot\ g(\lambda, Y, \phi, \cdot)$ **is $\mathbb{G}$-adapted for all** $\lambda \in \mathcal{L}$, $Y \in \mathcal{S}^{\mathcal{G}_2}$, $\phi \in \mathcal{I}^{\mathcal{G}}$,
  - $\cdot\ g(\lambda, 0, 0, \cdot) \in \mathcal{H}^{\mathcal{G}_2}$, **for all** $\lambda \in \mathcal{L}$.
- **Obstacle/barrier process $\xi$ is RCLL from** $S^{\mathcal{G}_2}$.

## Definition

Triple $(Y_t, \phi_t(\cdot), A_t)_{t\in[0,T]}$ is a $\mathbb{G}$-solution of RBSDE associated with triple $(Y_T, g_t, \xi_t)^{\mathbb{G}}_{t\in[0,T]}$ ($Y_T$ is final condition, obstacle $\xi$ is RCLL from $\mathcal{S}^{\mathcal{G}_2}$, and driver $g$ with introduced properties), if:

1. $(Y_t, \phi_t(\cdot), A_t)_{t\in[0,T]} \in S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}} \times S^{\mathcal{G}_2}$,
2. triple satisfies equation

$$Y_t = Y_T + \int_t^T g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dA_s, \quad (3)$$

   with $Y_T$ the terminal condition, $Y_T = \xi_T$,
3. $Y_t \geq \xi_t$ for every $t \in [0, T]$ a.s., where $\xi_t \in \mathcal{S}_2$ is a càdlàg $\mathcal{G}_t - \textit{adapted}$ process,
4. $A$ is a nondecreasing RCLL (right-continuous with left limits) predictable process with $A_0 = 0$, such that

   (i)

$$\int_0^T (Y_t - \xi_t)dA^c_t = 0 \quad a.s., \quad (4)$$

   (ii) $\Delta A^d_t = -\Delta Y_t 1_{\{Y_{t-}=\xi_{t-}\}} \quad a.s.$,

   where $A^c$ denotes the continuous part of $A$ and $A^d$ its discontinuous part.

**Remark 1.** Instead of saying that $(Y_t, \phi_t(\cdot), A_t)_{t\in[0,T]}$ progressively measurable processes is a solution of RBSDE, we can say that pair of progressively measurable processes $(Y_t, \phi_t(\cdot))_{t\in[0,T]}$ is a solution of RBSDE where:

– processes $Y_t, \phi_t(\cdot)$ are from $S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}}$;
– process $Y_t$ satisfies property 3. from the definition of the solution;
– process $A_t, t \in [0, T]$ has nonegative values and it is defined with

$$A_t = A_0 - Y_t - \int_0^t g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_0^t\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz),$$

In this sense we can prove that there exist unique pair of progressively measurable processes $(Y_t, \phi_t(\cdot))_{t\in[0,T]}$ which solves equation.

▷ We introduce Lipschitz driver".

**(H1)** If function $g : [0, T] \times [0, \infty)^2 \times \mathbb{R} \times \Phi \times \Omega \to \mathbb{R}$ satisfies (for some $L_g > 0$)

$$\begin{aligned}\left|g_t\left(\left(\lambda^B, \lambda^H\right), y_1, \phi^{(1)}\right) - g_t\left(\left(\lambda^B, \lambda^H\right), y_2, \phi^{(2)}\right)\right| &\le L_g \Big(||y_1 - y_2|| \\ + \left|\phi^{(1)}(0) - \phi^{(2)}(0)\right|\sqrt{\lambda^B} &+ \sqrt{\int_{\mathbb{R}_0}\left|\phi^{(1)}(z) - \phi^{(2)}(z)\right|^2 \nu(dz)}\sqrt{\lambda^H}\Big),\end{aligned}$$

for all $\left(\lambda^B, \lambda^H\right) \in [0, \infty)^2, y_1, y_2 \in \mathbb{R}$, and $\phi^{(1)}, \phi^{(2)} \in \Phi$, $dt \times d\mathbb{P}$ a.e.

▷ "Simple" RBSDE with time change & lower barrier

$$\boxed{Y_t = Y_T + \int_t^T g_s(\lambda_s)ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dA_s.}$$

**Lemma 1. (Di Nunno, Sjursen, 2014)] Estimate for the driver.**

Let $(Y, \phi), (U, \psi) \in S^{\mathcal{G}_2} \times \mathcal{I}^G$, and $g : [0, T] \times [0, \infty)^2 \times \mathbb{R} \times \Phi \times \Omega \to \mathbb{R}$, $g(\lambda, 0, 0, \cdot) \in \mathcal{H}^{\mathcal{G}_2}$, for all $\lambda \in \mathcal{L}$, satisfy $(H1)$. Then, for any $t \in [0, T]$,

$$\mathbb{E}\left[\left(\int_t^T g_s\left(\lambda_s, Y_s, \phi_s\right) - g_s\left(\lambda_s, U_s, \psi_s\right) ds\right)^2\right] \leq 3L_g^2(T-t)$$

$$\mathbb{E}\left[(T-t)\sup_{t\leq r\leq T}|Y_r - U_r|^2 + \int_t^T\int_{\mathbb{R}} |\phi_s(z) - \psi_s(z)|^2 \Lambda(ds, dz)\right],$$

and

$$\mathbb{E}\left[\left(\int_t^T |g_s\left(\lambda_s, U_s, \psi_s\right)| ds\right)^2\right] \leq (T-t)\mathbb{E}\left[2\int_t^T |g_s\left(\lambda_s, 0, 0\right)|^2 ds\right.$$

$$\left. +6L_g^2\left((T-t)\sup_{t\leq r\leq T}|U_r|^2 + \int_t^T\int_{\mathbb{R}} |\psi_s(z)|^2 \Lambda(ds, dz)\right)\right].$$

## Lemma 2. Estimate for the state process.

Let $U \in \mathcal{S}^{\mathcal{G}_2}$ and $\phi, \psi \in \mathcal{I}^{\mathcal{G}}$ and let $(Y_T, g_s, \xi_t)$ be standard parameters of RBSDEs with a lower barrier. We define stochastic process $Y_t$ for $t \in [0, T]$ in following way

$$Y_t := Y_T + \int_t^T g_s(\lambda_s, U_s, \psi_s)ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dA_s. \quad (5)$$

Then $Y \in \mathcal{S}^{\mathcal{G}_2}$ and

$$\mathbb{E}[\sup_{t \leq r \leq T} |Y_r|^2] \leq \mathbb{E}\left[4Y_T^2 + 4\left(\int_t^T |g_s(\lambda_s, U_s, \phi_s)|ds\right)^2 + 4A_T^2 + 40\int_0^T \int_{\mathbb{R}} \phi_s^2(z)\Lambda(ds, dz)\right].$$

**Line of the proof.**
It follows

$$\mathbb{E}[\sup_{t\le r\le T}|Y_r|^2] \le 4\mathbb{E}\left[Y_T^2 + 4\left(\int_t^T |g_s(\lambda_s, U_s, \psi_s)|ds\right)^2 + 4\left(\int_t^T dA_s\right)^2\right]$$
$$+ 4\mathbb{E}\left[\sup_{t\le r\le T}\left(\int_r^T\int_{\mathbb{R}} \phi_s(z)\mu(ds,dz)\right)^2\right]$$

▷ Applying **elementary and Doob's inequality** ($\mathbb{E}\sup_{0\le t\le T}|M_t|^p \le \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_T|^p$),

$$\mathbb{E}\left[\sup_{t\le r\le T}\left(\int_r^T\int_{\mathbb{R}} \phi_s(z)\mu(ds,dz)\right)^2\right] \le 10\mathbb{E}\left(\int_0^T\int_{\mathbb{R}} \phi_s(z)\mu(ds,dz)\right)^2,$$

▷ Substituting last we obtain

$$\mathbb{E}[\sup_{t\le r\le T}|Y_r|^2] \le 4\mathbb{E}\left[Y_T^2 + 4\left(\int_t^T |g_s(\lambda_s, U_s, \psi_s)|ds\right)^2 + 4\left(\int_t^T dA_s\right)^2\right]$$
$$+ 40\mathbb{E}\left[\left(\int_0^T\int_{\mathbb{R}} \phi_s^2(z)\Lambda(ds,dz)\right)\right] < +\infty,$$

i.e., $Y \in \mathcal{S}^{\mathcal{G}_2}$. $\star$

## Theorem 5. Solution for simple RBSDE with time change & lower barrier

Let driver $g_s \in \mathcal{H}^{\mathcal{G}_2}$ be independent of processes of state and control, and let barrier process $\xi$ is RCLL from $S^{\mathcal{G}_2}$. Then, **simple RBSDE with time change & lower barrier** has a unique $\mathbb{G}$-solution $(Y_t, \phi_t(\cdot), A_t)_{t\in[0,T]} \in S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}} \times S^{\mathcal{G}_2}$, and for each $T_1 \in \mathcal{T}_0$,

$$Y_{T_1} = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{T_1}} \mathbb{E}\left[ Y_\tau + \int_{T_1}^{\tau} g_t(\lambda_t)dt \middle| \mathcal{G}_{T_1} \right] \ a.s. \ .$$

**Line of the proof.** *Existence.* Let us define

$$\tilde{Y}(T_1) := \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{T_1}} \mathbb{E}\left[ Y_\tau + \int_{T_1}^{\tau} g_t(\lambda_t)dt \middle| \mathcal{G}_{T_1} \right].$$

▷ By classical results of optimal control (Dellacherie, É. Lenglart 1981), there exists RCLL adapted process $\tilde{Y}_t$ such that for every $T_1 \in \mathcal{T}_0$

$$\tilde{Y}(T_1) = \tilde{Y}_{T_1} \qquad a.s. \ ,$$

▷ It follows that process $\tilde{Y}_t + \int_0^t g_s(\lambda_s)ds$ is a supermartingale, see Snell envelope.
▷ From Theorem 2 and Theorem 3, it follows that for $t \in [0, T]$ there exist $\phi_t(\cdot), A_t$, $\in \mathcal{I}^{\mathcal{G}} \times \mathcal{S}^{\mathcal{G}_2}$, such that $(\tilde{Y}_t, \phi_t(\cdot), A_t)_{t\in[0,T]}$ is a solution of the equation

$$Y_t = Y_T + \int_t^T g_s(\lambda_s)ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dA_s.$$

*Uniqueness.*

▷ Uniqueness of processes $\phi_t(\cdot)$ and $A_t$ follows from the uniqueness decomposition theorem.

▷ Let $(Y_t, \phi_t(\cdot), A_t)_{t\in[0,T]}$ be another solution. Since $Y_t \geq \xi_t$, $t \in [0, T]$ it follows that for every $T_1 \in \mathcal{T}_0$ and $\tau \in \mathcal{T}_1$ we have

$$Y_{T_1} \geq \mathbb{E}\left[Y_\tau + \int_{T_1}^{\tau} g_s(\lambda_s)ds \Big| \mathcal{G}_{T_1}\right] \geq \mathbb{E}\left[\xi_\tau + \int_{T_1}^{\tau} g_s(\lambda_s)ds \Big| \mathcal{G}_{T_1}\right] \quad a.s.$$

Applying supreme over $\tau \in \mathcal{T}_{T_1}$ on last expression, we obtain

$$Y_{T_1} \geq \tilde{Y}(T_1) \ a.s. \ .$$

▷ If we now define for $\varepsilon > 0$ and $T_1 \in \mathcal{T}_0$ stopping time $\tau^{\varepsilon}_{T_1} := \inf_{t\geq T_1}\{Y_t \leq \xi_t + \varepsilon\}$.
Direct consequence of this definition is that for every $t \in [T_1, \tau^{\varepsilon}_{T1}[$ following holds:
$Y_t > \xi_t + \varepsilon$ a.s. , $A^c$ is a constant, as well as that $A^d$ is constant a.s. Also,

$$Y_{(\tau^{\varepsilon}_{T_1})^-} > \xi_{(\tau^{\varepsilon}_{T_1})^-} + \varepsilon \Rightarrow \Delta A^d_{\tau^{\varepsilon}_{T_1}} = 0 \ \ a.s.$$

$\Rightarrow$ $Y_t + \int_0^t g_s(\lambda_s)ds$ is martingale on $[T_1, \tau^{\varepsilon}_{T_1}[$.

▷ By right continuity of process $\xi_t$ and $Y_t$ it follows

$$Y_{T_1} = \mathbb{E}\left[Y_{\tau^{\varepsilon}_{T_1}} + \int_{T_1}^{\tau^{\varepsilon}} g_s(\lambda_s)ds \Big| \mathcal{G}_{T_1}\right] \leq \tilde{Y}(T_1) + \varepsilon \qquad a.s.$$

As last holds for every $\varepsilon > 0$, it follows

$$Y_{T_1} \leq \tilde{Y}(T_1) \qquad a.s. \qquad \star$$

## Theorem 6. Solution for general RBSDE with time change & lower barrier

General RBSDEs with a lower barrier and associated triple $(Y_T, g_t, \xi_t)^{\mathbb{G}}_{t\in[0,T]}$, where driver $g$ satisfies hypothesis (H1), has a unique $\mathbb{G}$-solution.

**Line of the proof.** Let us introduce mapping $\Theta : S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}} \longrightarrow S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}}$ in following way

$$(Y, \phi) := \Theta(U, \psi),$$

such that $(Y, \phi)$ is a solution of eq.

$$Y_t := Y_T + \int_t^T g_s(\lambda_s, U_s, \psi_s)ds - \int_t^T \int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dA_s.$$

▷ From Lemma 2. and Theorem 3. it follows that mapping $\Theta$ is well defined. Let us prove that it is a **contraction**.

Let $(Y', \phi')$ be another pair or process form $S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}}$ such that $(Y', \phi') = \Theta(U', \psi^{'})$ and it is a solution of general RBSDE associated with a driver $g_s(\lambda_s, U', \psi^{'})$. We will keep the notation

$$\begin{aligned} \tilde{U} &:= U - U', & \tilde{\psi} &:= \psi - \psi^{'}, \\ \tilde{Y} &:= Y - Y', & \tilde{\phi} &:= \phi - \phi^{'}, \\ \tilde{g} &:= g(\cdot, U, \psi) - g(\cdot, U', \psi^{'}). \end{aligned}$$

Ito formula for semimartingales on $e^{\beta t}\tilde{Y}_t^2$,

$$
\begin{aligned}
e^{\beta t}\tilde{Y}_t^2 = & -\beta\int_t^T e^{\beta s}\tilde{Y}_s^2 ds + 2\int_t^T e^{\beta s}\tilde{Y}_{s-} dA_{s,1} \\
& + 2\int_t^T e^{\beta s}\tilde{Y}_s[g_{s,1}(\lambda_s, Y_{s,1}, \phi_{s,1}(z)) - g_{s,2}(\lambda_s, Y_{s,2}, \phi_{s,2}(z))]ds \\
& - 2\int_t^T e^{\beta s}\tilde{Y}_s\tilde{\phi}_s(0)dB_s - 2\int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{Y}_s\tilde{\phi}_s(z)\tilde{H}(ds, dz) \\
& - 2\int_t^T e^{\beta s}\tilde{Y}_{s-} dA_{s,2} - \sum_{t\leq s\leq T} e^{\beta s}(\Delta\tilde{Y}_s)^2 - \int_t^T e^{\beta s}\tilde{\phi}_s^2(0)\lambda_s^B ds.
\end{aligned}
\tag{6}
$$

▷ Since the processes $A_i, i = 1, 2$ **jumps** only at predictable stopping times and $\mu(\cdot, dz)$ jumps only at totally inaccessible stopping times, it follows that

$$
\begin{aligned}
& \int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\nu(dz)\lambda_s^H ds - \sum_{t\leq s\leq T} e^{\beta s}(\Delta\tilde{Y}_s)^2 \\
= & \boxed{\int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\nu(dz)\lambda_s^H ds - \int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)H(ds, dz)} - \sum_{t\leq s\leq T} e^{\beta s}(\Delta A_{s,1} - \Delta A_{s,2})^2 \\
= & -\int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\tilde{H}(ds, dz) - \sum_{t\leq s\leq T} e^{\beta s}(\Delta A_{s,1} - \Delta A_{s,2})^2.
\end{aligned}
$$

▷ If we substitute last in (7), it follows

$$
\begin{aligned}
&e^{\beta t}\tilde{Y}_t^2+\beta\int_t^T e^{\beta s}\tilde{Y}_s^2 ds+\int_t^T e^{\beta s}\tilde{\phi}_s^2(0)\lambda_s^B ds+\int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\nu(dz)\lambda_s^H ds+\sum_{t\le s\le T} e^{\beta s}(\Delta A_{s,1}-\Delta A_{s,2})^2\\
&\le 2\int_t^T e^{\beta s}\tilde{Y}_s\underbrace{\left[g_{s,1}(\lambda_s,Y_{s,1},\phi_{s,1}(z))-g_{s,2}(\lambda_s,Y_{s,2},\phi_{s,2}(z))\right]}_{\le L_g\left(|\tilde{Y}_s|+|\tilde{\phi}_s(0)|\sqrt{\lambda^B}+\sqrt{\int_{\mathbb{R}_0}|\tilde{\phi}_s(z)|^2\nu(dz)\lambda^H}\right)+|\tilde{g}_s|} ds\\
&\quad -2\int_t^T e^{\beta s}\tilde{Y}_s\tilde{\phi}_s(0)dB_s-2\int_t^T\int_{\mathbb{R}_0} e^{\beta s}(2\tilde{Y}_s\tilde{\phi}_s(z)+\tilde{\phi}_s^2(z))\tilde{H}(ds,dz)\\
&\quad +\underbrace{2\int_t^T e^{\beta s}\tilde{Y}_{s-}dA_{s,1}}_{\le 0}-\underbrace{2\int_t^T e^{\beta s}\tilde{Y}_{s-}dA_{s,2}}_{\ge 0}
\end{aligned}
$$

▷ Apply conditional expectation with respect to $\mathcal{G}_t$ and elementary inequalities, it follows

$$
\begin{aligned}
&e^{\beta t}\tilde{Y}_t^2+E\left[\left(\beta\boxed{\int_t^T e^{\beta s}\tilde{Y}_s^2 ds}+\boxed{\int_t^T e^{\beta s}\tilde{\phi}_s^2(0)\lambda_s^B ds+\int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\nu(dz)\lambda_s^H ds}\right)\Big|\mathcal{G}_t\right]\\
&\le\mathbb{E}\left[\left(2L_g+\frac{1}{\varepsilon^2}\right)\boxed{\int_t^T e^{\beta s}\tilde{Y}_s^2 ds}+3\varepsilon^2L_g^2\left(\boxed{\int_t^T e^{\beta s}|\tilde{\phi}_s(0)|^2\lambda^B ds+\int_t^T\int_{\mathbb{R}_0}|\tilde{\phi}_s(z)|^2\nu(dz)\lambda_s^H ds}\right)\Big|\mathcal{G}_t\right]\\
&\quad +3\varepsilon^2\mathbb{E}\left[\int_t^T e^{\beta s}|\tilde{g}_s(\lambda_s)|^2 ds\Big|\mathcal{G}_t\right]. \qquad (\star)
\end{aligned}
$$

▷ If we denote $\eta := 3\varepsilon^2$, $\quad \beta > 2L_g + \frac{3}{\eta}$, (for $\varepsilon$ small enough $3\varepsilon^2 L_g^2 < 1$), taking expectation of sup over $t$, it follows

$$\Rightarrow ||\tilde{Y}_s||^2_{\mathcal{S}^{\mathcal{G}_2}} \leq \eta e^{\beta T} \mathbb{E}\left(\int_0^T |\tilde{g}_s(\lambda_s)|^2 ds\right) = \eta e^{\beta T} ||\tilde{g}_s||^2_{\mathcal{H}^{\mathcal{G}_2}}.$$

▷ From (★) applying expectation of $\sup_{t\in[0,T]}$, we obtain

$$\begin{aligned} ||\tilde{\phi}_s(z)||^2_{\mathcal{I}^{\mathcal{G}}} &= \mathbb{E}\left[\int_0^T |\tilde{\phi}_s(0)|^2 \lambda_s^B ds + \int_0^T \int_{\mathbb{R}_0} |\tilde{\phi}_s(z)|^2 \nu(dz) \lambda_s^H ds\right] \\ &\leq \dots \\ &\leq \eta L_g^2 e^{\beta T} \mathbb{E} \sup_{t\in[0,T]} \left[\int_t^T \tilde{\phi}_s^2(0) \lambda_s^B ds + \int_t^T \int_{\mathbb{R}_0} \tilde{\phi}_s^2(z) \nu(dz) \lambda_s^H ds\right] + \eta e^{\beta T} \mathbb{E}\left[\int_t^T \tilde{g}_s^2(\lambda_s) ds \middle| \mathcal{G}_t\right] \end{aligned}$$

▷ From last, if $\eta < \frac{1}{L_g^2}$, then

$$\Rightarrow ||\tilde{\phi}_s(z)||^2_{\mathcal{I}^{\mathcal{G}}} \leq \frac{\eta e^{\beta T}}{1 - \eta L_g^2 e^{\beta T}} ||\tilde{g}_s||^2_{\mathcal{H}^{\mathcal{G}_2}},$$

▷ Using result from Lemma 1 - estimate for the driver,

$$\Rightarrow ||\tilde{Y}_s||^2_{\mathcal{S}^{\mathcal{G}_2}} + ||\tilde{\phi}_s(z)||^2_{\mathcal{I}^{\mathcal{G}}} \leq 3L_g^2 \max\{T^2, 1\}\eta e^{\beta T}(||\tilde{Y}_s||^2_{\mathcal{S}^{\mathcal{G}_2}} + ||\tilde{\phi}_s(z)||^2_{\mathcal{I}^{\mathcal{G}}}).$$

▷ As $\eta < \frac{1}{3L_g^2 \max\{T^2,1\}}$, if we now take $\eta < \min\left\{\frac{1}{2L_g^2}, \frac{1}{3L_g^2 \max\{T^2,1\}}\right\}$, it follows from last that $\Theta$ is a contraction on $S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}}$. ⋆

## Existence and uniqueness of the $\mathbb{F}$-solution

▷ For $T > 0$ let RBSDE driven by time-change Lévy noises and lower barrier under $\mathbb{F}$ be rewritten in following form

$$Y_t = Y_T + \int_t^T g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_t^T \phi_s(0)dB_s - \int_t^T\int_{\mathbb{R}_0} \phi_s(z)\tilde{H}(ds, dz) + \int_t^T dM_s + \int_t^T dA_s,$$

which is equivalent to equation

$$Y_t = Y_T + \int_t^T g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dM_s + \int_t^T dA_s,$$

where $||M||_{\mathcal{M}_\beta^{\mathcal{F}_2}} := \int_t^T e^{\beta s} d[\tilde{M}]_s < +\infty, \beta > 0$ and (*standard assumptions under* $\mathbb{F}$ *are*)

- $Y_T \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ **(final value-reaching point of state process of BSDE).**
- **Function $g$ is a drift coefficient, ie *driver* which satisfies:**
  - $\cdot\ g(\lambda, Y, \phi, \cdot)$ **is $\mathbb{F}$-adapted for all $\lambda \in \mathcal{L}, Y \in \mathcal{S}^{\mathcal{F}_2}, \phi \in \mathcal{I}^{\mathcal{F}}$,**
  - $\cdot\ g(\lambda, 0, 0, \cdot) \in \mathcal{H}^{\mathcal{F}_2}$, **for all $\lambda \in \mathcal{L}$.**
- **Obstacle/barrier process $\xi$ is RCLL from $S^{\mathcal{F}_2}$.**

## Definition

Quartet $(Y_t, \phi_t(\cdot), M_t, A_t)_{t\in[0,T]}$ is a $\mathbb{F}$-solution of RBSDE associated with triple $(Y_T, g_t, \xi_t)^{\mathbb{F}}_{t\in[0,T]}$ ($Y_T$ is final condition, obstacle $\xi$ is RCLL from $\mathcal{S}^{\mathcal{F}_2}$, and driver $g$ with introduced properties), if:

1. $(Y_t, \phi_t(\cdot), M_t, A_t) \in S^{\mathcal{F}_2} \times \mathcal{I}^{\mathcal{F}} \times \mathcal{M}^{\mathcal{F}_2}_{\beta} \times S^{\mathcal{F}_2}$,
2. satisfies equation

$$Y_t = Y_T + \int_t^T g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dM_s + \int_t^T dA_s,$$

   with $Y_T$ the terminal condition, $Y_T = \xi_T$,
3. $Y_t \geq \xi_t$ for every $t \in [0, T]$ a.s., where $\xi_t \in \mathcal{S}_2$ is a càdlàg $\mathcal{F}_t - adapted$ process,
4. $A$ is a nondecreasing RCLL (right-continuous with left limits) predictable process with $A_0 = 0$, such that

   (i)
$$\int_0^T (Y_t - \xi_t)dA^c_t = 0 \quad a.s.,$$

   (ii) $\Delta A^d_t = -\Delta Y_t 1_{\{Y_{t-}=\xi_{t-}\}} \quad a.s.,$

   where $A^c$ denotes the continuous part of $A$ and $A^d$ its discontinuous part.

**Remark 1.** Instead of saying that $(Y_t, \phi_t(\cdot), M_t, A_t)_{t\in[0,T]}$ progressively measurable processes is a solution of RBSDE, we can say that triple of progressively measurable processes $(Y_t, \phi_t(\cdot), M_t)_{t\in[0,T]}$ is a solution of RBSDE where:

– processes $Y_t, \phi_t(\cdot), M_t$ are from $S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}} \times \mathcal{M}_\beta^{\mathcal{F}_2}$ – process $Y_t$ satisfies property 3. from the definition of the solution;
– process $A_t, t \in [0, T]$ has nonegative values and it is defined with

$$A_t = A_0 - Y_t - \int_0^t g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_0^t\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) - \int_t^T dM_s,$$

In this sense we can prove that there exist unique pair of progressively measurable processes $(Y_t, \phi_t(\cdot))_{t\in[0,T]}$ which solves equation.

▷ "Simple" RBSDE with time change & lower barrier

$$\boxed{Y_t = Y_T + \int_t^T g_s(\lambda_s)ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dM_s + \int_t^T dA_s.}$$

## Theorem 7. Solution for simple RBSDE with time change & lower barrier

Let driver $g_s \in \mathcal{H}^{\mathcal{F}_2}$ be independent of processes of state and control, and let barrier process $\xi$ is RCLL from $S^{\mathcal{F}_2}$. Then, **simple RBSDE with time change & lower barrier** has a unique $\mathbb{F}$-solution $(Y_t, \phi_t(\cdot), M_t, A_t)_{t\in[0,T]} \in S^{\mathcal{F}_2} \times \mathcal{I}^{\mathcal{F}} \times \mathcal{M}_{\beta}^{\mathcal{F}_2} \times S^{\mathcal{F}_2}$, and for each $T_1 \in \mathcal{T}_0$,

$$Y_{T_1} = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{T_1}} \mathbb{E}\left[ Y_\tau + \int_{T_1}^{\tau} g_t(\lambda_t)dt \Big| \mathcal{F}_{T_1} \right] \ a.s. \ .$$

**Line of the proof.**

▷ As in the framework under $\mathbb{G}$, let us define

$$\tilde{Y}(T_1) := \operatorname{ess\,sup}_{\tau\in\mathcal{T}_{T_1}} \mathbb{E}\left[ Y_\tau + \int_{T_1}^{\tau} g_t(\lambda_t)dt \Big| \mathcal{F}_{T_1} \right].$$

▷ As before, there exists RCLL adapted process $\tilde{Y}_t$ such that for every $T_1 \in \mathcal{T}_0$

$$\tilde{Y}(T_1) = \tilde{Y}_{T_1} \qquad a.s. \ ,$$

such that $\tilde{Y}_t + \int_0^t g_s(\lambda_s)ds$ is a supermartingale, see Snell envelope.

▷ From Theorem 4 - Doob Meyer's decomposition under $\mathbb{F}$, it follows that for $t \in [0, T]$ there exist $\phi_t(\cdot), M_t, A_t, \in \mathcal{I}^{\mathcal{F}} \times \mathcal{M}_{\beta}^{\mathcal{F}_2} \times S^{\mathcal{F}_2}$, such that $(\tilde{Y}_t, \phi_t(\cdot), M_t, A_t)_{t\in[0,T]}$ is a solution of the equation

$$Y_t = Y_T + \int_t^T g_s(\lambda_s)ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dM_s + \int_t^T dA_s. \qquad \star$$

## Theorem 8. Solution for general RBSDE with time change & lower barrier

General RBSDEs with a lower barrier and associated triple $(Y_T, g_t, \xi_t)^{\mathbb{F}}_{t\in[0,T]}$, where driver $g$ satisfies hypothesis (H1), has a unique $\mathbb{F}$-solution.

**Line of the proof.**
Itô formula for semimartingales $e^{\beta t}\tilde{Y}_t^2$, we obtain

$$
\begin{aligned}
& e^{\beta t}\tilde{Y}_t^2 + \beta\int_t^T e^{\beta s}\tilde{Y}_s^2 ds + \int_t^T e^{\beta s}\tilde{\phi}_s^2(0)\lambda_s^B ds + \int_t^T e^{\beta s} d\langle \tilde{M}_s\rangle \\
&\quad \le 2\int_t^T e^{\beta s}\tilde{Y}_s[g_{s,1}(\lambda_s, Y_{s,1}, \phi_{s,1}(z)) - g_{s,2}(\lambda_s, Y_{s,2}, \phi_{s,2}(z))]ds \\
&\qquad - 2\int_t^T e^{\beta s}\tilde{Y}_s\tilde{\phi}_s(0)dB_s - 2\int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{Y}_s\phi_s(z)\tilde{H}(ds,dz) - 2\int_t^T e^{\beta s}\tilde{Y}_s d\tilde{M}_s \\
&\qquad + 2\int_t^T e^{\beta s}\tilde{Y}_{s-}dA_{s,1} - 2\int_t^T e^{\beta s}\tilde{Y}_{s-}dA_{s,2} - \sum_{t\le s\le T} e^{\beta s}(\Delta\tilde{Y}_s)^2. \qquad (\triangle)
\end{aligned}
$$

▷ Since the processes $A_i, i = 1,2$ **jumps** only at predictable stopping times, while $\mu(\cdot, dz)$ and $M(\cdot)$ jump only at totally inaccessible stopping times, it follows that

$$
\begin{aligned}
&\int_t^T \int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\nu(dz)\lambda_s^H ds - \sum_{t\leq s\leq T} e^{\beta s}(\Delta \tilde{Y}_s)^2 \\
&= \boxed{\int_t^T \int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\nu(dz)\lambda_s^H ds - \int_t^T \int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)H(ds,dz)} - \sum_{t\leq s\leq T} e^{\beta s}(\Delta A_{s,1} - \Delta A_{s,2})^2 \\
&\quad - \sum_{0\leq s< T} e^{\beta s}(\Delta \tilde{M}_s)^2 \\
&= -\int_t^T \int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\tilde{H}(ds,dz) - \sum_{t\leq s\leq T} e^{\beta s}(\Delta A_{s,1} - \Delta A_{s,2})^2 - \sum_{0\leq s< T} e^{\beta s}(\Delta \tilde{M}_s)^2.
\end{aligned}
$$

▷ From (△), it follows

$$
\begin{aligned}
&e^{\beta t}\tilde{Y}_t^2 + \beta \int_t^T e^{\beta s}\tilde{Y}_s^2 ds + \int_t^T e^{\beta s}\tilde{\phi}_s^2(0)\lambda_s^B ds + \int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{\phi}_s^2(z)\nu(dz)\lambda_s^H ds \\
&+ \int_t^T e^{\beta s} d[\tilde{M}]_s + \sum_{t\leq s\leq T} e^{\beta s}(\Delta A_{s,1} - \Delta A_{s,2})^2 \\
&\qquad \leq 2\int_t^T e^{\beta s}\tilde{Y}_s[g_{s,1}(\lambda_s, Y_{s,1}, \phi_{s,1}(z)) - g_{s,2}(\lambda_s, Y_{s,2}, \phi_{s,2}(z))]ds \\
&\qquad\quad - 2\int_t^T e^{\beta s}\tilde{Y}_s\tilde{\phi}_s(0)dB_s - 2\int_t^T\int_{\mathbb{R}_0} e^{\beta s}\tilde{Y}_s\phi_s(z)\tilde{H}(ds,dz) - 2\int_t^T e^{\beta s}\tilde{Y}_s d\tilde{M}_s \\
&\qquad\quad + 2\int_t^T e^{\beta s}\tilde{Y}_{s-}dA_{s,1} - 2\int_t^T e^{\beta s}\tilde{Y}_{s-}dA_{s,2}. \qquad (7)
\end{aligned}
$$

$\triangleright$ Using similar arguments as in Theorem 6, applying conditional expectation with respect to $\mathcal{F}_t$, for the appropriate choice of constants, it follows that

$$\Rightarrow ||\tilde{Y}_s||^2_{\mathcal{S}^{\mathcal{F}_2}} + ||\tilde{\phi}_s(z)||^2_{\mathcal{I}^{\mathcal{F}}} + ||\tilde{M}_s||_{\mathcal{M}_1^{\mathcal{F}_2}} \leq \eta e^{\beta T} ||\tilde{g}_s||^2_{\mathcal{H}^{\mathcal{F}_2}}.$$

$\triangleright$ Let us introduce mapping $\Theta : S^{\mathcal{F}_2} \times \mathcal{I}^{\mathcal{F}} \times \mathcal{M}_1^{\mathcal{F}_2} \longrightarrow S^{\mathcal{F}_2} \times \mathcal{I}^{\mathcal{F}} \times \mathcal{M}_1^{\mathcal{F}_2}$ in following way

$$(Y, \phi, V) := \theta(U, \psi, W),$$

such that $(Y, \phi, V)$ is a solution of simple RBSDE associated with a driver $g_{s,1} := g_s(\lambda_s, U_s, \psi_s)$ and ortogonale martingale part $W$. From Theorem 7 it follows that mapping $\Theta$ is well defined.

$\triangleright$ If we choose that $\eta < \min\left\{\frac{1}{2L_g^2}, \frac{1}{3L_g^2 \max\{T^2,1\}}\right\}$,

$$\Rightarrow ||\tilde{Y}_s||^2_{\mathcal{S}^{\mathcal{F}_2}} + ||\tilde{\phi}_s(z)||^2_{\mathcal{I}^{\mathcal{F}}} + ||\tilde{M}_s||_{\mathcal{M}_\beta^{\mathcal{F}_2}} \leq \eta e^{\beta T} ||\tilde{g}_s||^2_{\mathcal{H}^{\mathcal{F}_2}}$$
$$\leq 3L_g^2 \max\{T^2, 1\} \eta e^{\beta T} (||\tilde{Y}_s||^2_{\mathcal{S}^{\mathcal{F}_2}} + ||\tilde{\phi}_s(z)||^2_{\mathcal{I}^{\mathcal{F}}}).$$

it follows that for a certain $\eta$ it follows that $\Theta$ is a contraction on $S^{\mathcal{F}_2} \times \mathcal{I}^{\mathcal{F}} \times \mathcal{M}_\beta^{\mathcal{F}_2}$. $\star$

## Characterization of the value function as the solution of an RBSDE with time change

$\triangleright$ Goal is to connect the state process $Y$ of the solution of RBSDE with time change (solution of this RBSDE is triple $(Y_t, \phi_t(\cdot), A_t)_{t\in[0,T]}$) characterized with $(Y_T, g_t, \xi_t)_{t\in[0,T]}$, i.e.

$$Y_t = Y_T + \int_t^T g_s(\lambda_s, Y_s, \phi_s(z))ds - \int_t^T\int_{\mathbb{R}} \phi_s(z)\mu(ds, dz) + \int_t^T dA_s \left( + \int_t^T dM_s \right),$$

$\triangleright$ with the state process $X$ of the solution of BSDE with time changed Lévy (solution of this BSDE is a pair $(X_t, \psi_t(\cdot))_{t\in[0,T]}$ characterized with $(X_T, g_t)_{t\in[0,T]}$ , i.e.

$$X_t(X_T, T) = X_T + \int_t^T g_s(\lambda_s, X_s, \psi_s(z))ds - \int_t^T \int_{\mathbb{R}} \psi_s(z)\mu(ds, dz) \left( + \int_t^T dM_s \right).$$

## Verification result for time changed Lévy under filtration $\mathbb{G}$

Proposition 1. (Di Nunno, Sjursen, 2014) Comparison for BSDE with time change

Let $\left(g^{(1)}, X_T^{(1)}\right)$ and $\left(g^{(2)}, X_T^{(2)}\right)$ be two sets of standard parameters for the BSDEs with solutions $\left(X^{(1)}, \psi^{(1)}\right), \left(X^{(2)}, \psi^{(2)}\right) \in S^{\mathcal{G}_2} \times \mathcal{I}^{\mathcal{G}}$. Let us introduce following assumption.

**(A1)**

1. $g_t^{(2)}(\lambda, x, \psi, \omega) = f_t\left(x, \psi(0)\kappa_t(0)\sqrt{\lambda^B}, \int_{\mathbb{R}_0} \psi(z)\kappa_t(z)v(dz)\sqrt{\lambda^H}, \omega\right)$ where $\kappa \in \mathcal{I}^{\mathcal{G}}$ satisfies following condition

   **(c)** there exists $K_E > 0$ such that $0 \leq \kappa_t(z) < K_E z$ for $z \in \mathbb{R}_0$, and $|\kappa_t(0)| < K_E \; dt \times d\mathbb{P} - a.e.$

2. $f$ is a function $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ which satisfies, for some $K_f > 0$,

$$\left|f_t(x, b, h) - f_t\left(x', b', h'\right)\right| \leq K_f\left(\left|x - x'\right| + \left|b - b'\right| + \left|h - h'\right|\right)$$

   $dt \times d\mathbb{P}$ a.e. and $\mathbb{E}\left[\int_0^T |f_t(0,0,0)|^2\, dt\right] < \infty$.

**(A2)** $X_T^{(1)} \leq X_T^{(2)}$ $\mathbb{P}$-a.s. and $g_s^{(1)}\left(\lambda_s, X_s^{(1)}, \psi_s^{(1)}\right) \leq g_s^{(2)}\left(\lambda_s, X_s^{(1)}, \psi_s^{(1)}\right)$ $dt \times d\mathbb{P}$-a.e., then

$$X_t^{(1)} \leq X_t^{(2)} \quad dt \times d\mathbb{P} - a.e.$$

## Theorem 9. Verification result under $\mathbb{G}$.

Let $T > 0$ be the terminal time. Let $(\xi_t, 0 \leq t \leq T)$ be an RCLL process in $S^{\mathcal{G}_2}$ and let $g$ is a Lipschitz driver satisfying (H1) with a constant $L_g$, such that it also satisfies assumptions from Proposition 1. Suppose that $(Y_t, \phi_t(\cdot), A_t)_{t\in[0,T]}$ is the solution of the RBSDE with time change. Then for each stopping time $S \in \mathcal{T}_0$, it follows that

$$Y_S = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_S} X_S(\xi_\tau, \tau) \ a.s. \ ,$$

where $\tau \in \mathcal{T}_S$, $X(\xi_\tau, \tau)$ is the state process of the solution for the BSDE with time changed Lévy associated with terminal time $\tau$,terminal condition $\xi_\tau$, and driver $g$.

**Line of the proof.** "$\geq$"

- As $A_s$ is nondecreasing process it follows that

$$g(\lambda_s, y, \phi_s) + dAs \geq g(\lambda_s, y, \phi_s).$$

- For $\tau \in \mathcal{T}_S$, $\quad Y_\tau \geq \xi_\tau \ a.s.$

$$\Rightarrow \qquad Y_S \geq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_S} X_s(\xi_\tau, \tau) \ a.s$$

"$\leq$"

<u>*Step I*</u> Let $\varepsilon > 0$. By definition of $\tau_S^\varepsilon$ and by right-continuity property of processes $Y_t$ and $\xi_t$, it follows that $Y_{\tau_S^\varepsilon} \leq \xi_{\tau_S^\varepsilon} + \varepsilon \ a.s.$, so

$$Y_S = X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) \leq X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \ a.s.$$

<u>*Step II*</u> **Lemma 3.** Let $T > 0$, $\xi_i \in \mathcal{S}^{\mathcal{G}_2}$, $i = 1, 2$ and let $g_{s,i}$, $i = 1, 2$ drivers, such that $g_{s,1}$ is a Lipschitz driver satisfying $(H1)$ with a constant $L_g$, and $\beta, \eta$ are such that $\beta > 2L_g + \frac{3}{\eta}$ and $\eta < \frac{1}{L_g^2}$, then for each $t \in [0, T]$ we have that a.s.

$$e^{\beta t}|X_{t,1} - X_{t,2}|^2 \leq \mathbb{E}\left[e^{\beta T}|\xi_1 - \xi_2|^2 \Big| \mathcal{G}_t\right] + \eta \mathbb{E}\left[\int_t^T e^{\beta s}|g_{s,1}(\lambda_s, X_{s,1}, \psi_{s,1}) - g_{s,2}(\lambda_s, X_{s,2}, \psi_{s,2})|^2 ds \Big| \mathcal{G}_t\right]$$

le

$$\Rightarrow |X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) - X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon)|^2 \leq e^{\beta(T-S)}\varepsilon^2 \ a.s.$$

<u>*Step III*</u> From Step I and II

$$Y_S \leq X_S(\xi_{\tau_S^\varepsilon} + \varepsilon, \tau_S^\varepsilon) \leq X_S(\xi_{\tau_S^\varepsilon}, \tau_S^\varepsilon) + e^{\beta(T-S)}\varepsilon^2 \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) + e^{\beta(T-S)}\varepsilon^2 \ a.s.$$

So for every $\varepsilon > 0$ it follows that

$$Y_S \leq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} X_S(\xi_\tau, \tau) \ a.s. \qquad \star$$

## Comparison result under $\mathbb{G}$

**Theorem 10. Comparison theorem under $\mathbb{G}$**

Let $\xi_i, i = 1, 2$ be two RCLL obstacle processes in $S^{\mathcal{G}_2}$. Let $g_i, i = 1, 2$ be such that they satisfy assumption $(H1)$ and let assumptions of Proposition 1 be satisfied. Furthermore, suppose that for $t \in [0, T]$ following assumption holds:

**(A3)** $\xi_{t,1} \leq \xi_{t,2}, t \in [0, T]$ a.s., and $g_{t,1}(\lambda, y, \phi) \leq g_{t,2}(\lambda, y, \phi)$ for all $\lambda \in [0, \infty)^2$, $y \in \mathbb{R}$, $\phi \in \mathcal{I}^{\mathcal{G}}$, $dt \times dP$-a.s. .

Let $(Y^i_t, \phi^i_t(\cdot), A^i_t)_{t\in[0,T]}$ be a $\mathbb{G}$-solution of RBSDE associated with triple $(Y^i_T, g^i_t, \xi^i_t)^{\mathbb{G}}_{t\in[0,T]}$, $i = 1, 2$. Then we have that

$$Y^1_t \leq Y^2_t, \quad t \in [0, T] \quad a.s.$$

**Line of the proof.** From comparison theorem for BSDE with time change Levy and verification theorem

$$X^1_t(\xi^1_\tau, \tau) \leq X^2_t(\xi^2_\tau, \tau) \Rightarrow Y^1_t = \operatorname{ess\,sup}_{\tau\in\mathcal{T}_t} X^1_t(\xi^1_\tau, \tau) \leq \operatorname{ess\,sup}_{\tau\in\mathcal{T}_t} X^2_t(\xi^2_\tau, \tau) = Y^2_t \ \ a.s. \qquad \star$$

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**$\star$Thank you for your attention.$\star$**