Propagation of chaos for a class of weakly interacting Snell envelopes

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Outline

1. Introduction

2. Mean-field reflected BSDEs with jumps

3. Weakly interacting nonlinear Snell envelopes

4. Convergence of the particle system and propagation of chaos
(i) Classical reflected BSDEs: introduced by El Karoui et al. (1997) in the Brownian setting, further developments by many authors e.g. Essaky, Ouknine, Hamadène, Matoussi etc.

(ii) Mean-field BSDEs: introduced by Buckdahn et al. (2009). Mean-field reflected BSDEs (Brownian setting): J. Li (2014)

Introduction

Aim:

- Prove existence and uniqueness of MF-BSDEs with jumps and obstacle which depends on the solution and its law, for general jumps by using a different approach compared with [DEH], which allows to weaken the assumptions (also in the Brownian setting)
- Wellposedness of the weakly interacting particle system of Snell envelopes
- Provide convergence and propagation of chaos results
Mean-field BSDEs and mean-field reflected BSDEs

(i) Mean-field BSDEs
A solution to a mean-field BSDE associated with \( \{f(t, \omega, y, z, \nu), \xi\} \) (\( \nu \) probability law on \( \mathbb{R} \times \mathbb{R}^{d \times d} \)) is a pair of \((\mathcal{F}_t)_{t \leq T}\)-adapted stochastic processes \((Y_t, Z_t)_{t \leq T}\) such that

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \mathbb{P}(Y_s, Z_s))ds - \int_t^T Z_s dB_s, \quad t \leq T.
\]

Example:

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \mathbb{E}[Y_s], \mathbb{E}[Z_s])ds - \int_t^T Z_s dB_s, \quad t \leq T.
\]

They are studied first in the papers Buckdahn-Li-Peng (2009) and Buckdahn-Djehiche-Li-Peng (2009).
Mean-field BSDEs and mean-field reflected BSDEs

Theorem (Buckdahn-Li-Peng (2009))

If $f$ is Lipschitz in $(y, z, \nu)$ and $\xi$ is square integrable then the MF – BSDE has a unique solution.
Mean-field BSDEs and mean-field reflected BSDEs

(ii) Mean-field reflected BSDEs

A specific type of MF-RBSDEs appears in Briand-Elie-Hu (2018) where they consider

\[
\begin{align*}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s; \\
\mathbb{E}[l(t, Y_t)] &\geq 0 \text{ and } \int_0^T \mathbb{E}[l(s, Y_s)]dK_s = 0.
\end{align*}
\]

Motivated by problems related to quantile hedging: instead of requiring a strong protection condition:

\[Y_t \geq L_t, \quad t \leq T,\]

we only require a weak protection condition such as

\[\mathbb{P}[Y_t \geq L_t] \geq \omega_t,\]

i.e. the barrier is of the form \(l(t, y) = 1_{\{y \geq L_t\}} - \omega_t.\)
Mean-field BSDEs and mean-field reflected BSDEs

Theorem (Briand et al. (2018))

Assume

(a) \( f \) is Lipschitz;
(b) \( \xi \) square integrable and \( \mathbb{E}[l(T, \xi)] \geq 0 \);
(c) (mainly) \( l \) is continuous, increasing with linear growth, bi-Lipschitz and \( \mathbb{E}[l(t, \infty)] \geq 0 \).

Then there exists a unique solution of (1) with \( K \) deterministic.

Remark. If we relax the fact that \( K \) is deterministic, then the solution is not necessarily unique.

Related works: Forward-Backward SDEs with constraints in law - Briand-Cardaliaguet- De Raynal - Hu (2020)
(ii) Mean-field reflected BSDEs

\[
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s]) ds + K_T - K_t - \int_t^T Z_s dB_s; \\
Y_t &\geq h(Y_t, \mathbb{E}[Y_t]), \quad t \leq T, \\
\int_0^T (Y_t - h(Y_t, \mathbb{E}[Y_t])) dK_t &= 0.
\end{aligned}
\]  

(2)

This RBSDE has the same structure as the usual one (i.e. strong reflection), the new feature being that the barrier depends on the solution and its marginal law. This type of RBSDEs have been introduced by Djehiche-Elie-Hamadène (2020) in the Brownian filtration.
(ii) Mean-field reflected BSDEs

The component $Y$ of the solution of the MF-RBSDE (2) admits the representation

$$Y_t = \text{ess sup}_{\tau \geq t} \mathbb{E}\left[ \int_t^\tau f(s, Y_s, \mathbb{E}[Y_s])ds + h(Y_\tau, \mathbb{E}[Y_s]|_{s=\tau})\mathbf{1}_{\tau<T} + \xi \mathbf{1}_{\tau=T} \right].$$

(3)
(ii) Example: Guaranteed life endowment with a surrender/withdrawal option (Djehiche-Elie-Hamadène)

Consider a portfolio of a large number of $n$ of homogeneous life insurance policies $l$. Denote by $(Y^{l;n}, Z^{l;n})$ the characteristics of the prospective reserve of each policy $l = 1, 2, \ldots, n$. We consider nonlinear reserving where the driver $f$ depends on the reserve for the particular contract and on the average reserve characteristics over the $n$ contracts:

For each $l = 1, 2, \ldots, n$,

$$Y_t^{l;n} = \text{ess sup}_{\tau \in T_t} \mathbb{E} \left[ \int_t^\tau f(s, Y_s^{l;n}, (Y_s^{m;n})_{m \neq l}) ds + h(Y_{\tau}^{l;n}, Y_{\tau}^{m;n})_{m \neq l}) \mathbf{1}_{\tau < T} + \xi_{t}^{l;n} \right]$$

(4)
Mean-field BSDEs and mean-field reflected BSDEs

(ii) Example: Guaranteed life endowment with a surrender/withdrawal option (Djehiche-Elie-Hamadène)

\[ f(t, Y_{t}^{l,n}, (Y_{t}^{m,n})_{m\neq l}) := \alpha_t - \delta_t Y_{t}^{l,n} + \beta_t \max(\theta_t, Y_{t}^{l,n} - \frac{1}{n} \sum_{k=1}^{n} Y_{t}^{k,n}) ; \]
\[ L_t := h(Y_{t}^{l,n}, Y_{t}^{m,n})_{m\neq l} = u - c(Y_{t}^{l,n}) + \mu(\frac{1}{n} \sum_{k=1}^{n} Y_{t}^{k,n} - u)^+ , \]

where \( 0 < \mu < 1. \)

Sending \( n \) to infinity 'yields' the following forms of the driver and the obstacle:

\[ f(t, Y_{t}, \mathbb{E}[Y_{t}]) := \alpha_t - \delta_t Y_{t} + \beta_t \max(\theta, Y_{t} - \mathbb{E}[Y_{t}]) ; \]
\[ h(t, Y_{t}, \mathbb{E}[Y_{t}]) = u - c(Y_{t}) + \mu(\mathbb{E}[Y_{t}] - u)^+ , \]

of the prospective reserve of a representative life insurance contract, a.k.a. the model-point among actuaries.
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Mean-field reflected BSDEs with jumps

Spaces

(i) On the complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we consider

- \(B = (B_t)_{0 \leq t \leq T}\) is a standard one dimensional Brownian motion.
- \(N(dt, de)\) is a Poisson random measure, independent of \(B\), with compensator \(\nu(de)dt\) where \(\nu\) is a \(\sigma\)-finite measure on \(\mathbb{R}^* := \mathbb{R} - 0\).
- \(\tilde{N}(dt, de)\) is its compensated process.
- \(\mathbb{F} = \{\mathcal{F}_t\}\) the (completed) natural filtration associated with \(B\) and \(N\). Let \(\mathcal{P}\) be the \(\sigma\)-algebra on \(\Omega \times [0, T]\) of \(\mathcal{F}_t\)-progressively measurable sets.

(ii) \(L^p(\mathcal{F}_T)\) is the set of random variables \(\xi\) which are \(\mathcal{F}_T\)-measurable and \(\mathbb{E}[|\xi|^p] < \infty\).

(iii) \(S^p_\beta\) is the set of real-valued càdlàg adapted processes \(y\) such that

\[
||y||_{S^p_\beta}^p := \mathbb{E}\left[ \sup_{0 \leq u \leq T} e^{\beta pu} |y_u|^p \right] < \infty.
\]

We set \(S^p = S^p_0\).
Mean-field reflected BSDEs with jumps

Spaces

(iv) $S^p_{\beta,i}$ is the subset of $S^p_{\beta}$ such that the process $k$ is non-decreasing and $k_0 = 0$. We set $S^p_i = S^p_{0,i}$.

(v) $\mathcal{H}^{p,d}$ is the set of $\mathcal{P}$-measurable, $\mathbb{R}^d$-valued processes such that $\mathbb{E}[(\int_0^T |v_s|^2 ds)^{p/2}] < \infty$.

(vi) $L^p_\nu$ is the set of measurable functions $l : \mathbb{R}^* \to \mathbb{R}$ such that $\int_{\mathbb{R}^*} |l(u)|^p \nu(du) < +\infty$. The set $L^2_\nu$ is a Hilbert space equipped with the scalar product $\langle \delta, l \rangle_\nu := \int_{\mathbb{R}^*} \delta(u) l(u) \nu(du)$ for all $(\delta, l) \in L^2_\nu \times L^2_\nu$, and the norm $|l|_{\nu,2} := \left(\int_{\mathbb{R}^*} |l(u)|^2 \nu(du)\right)^{1/2}$. 

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MF-RBSDEs with jumps

Spaces

(vii) $\mathcal{H}_\nu^{p,d}$ is the set of predictable processes $l$, i.e. measurable

$$l : ([0, T] \times \Omega \times \mathbb{R}^*, \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^*)) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)); \quad (\omega, t, u) \mapsto l_t(\omega, u)$$

such that $\|l\|_{\mathcal{H}_\nu^{p,d}}^p := \mathbb{E} \left[ \left( \int_0^T \sum_{j=1}^d |l_t^j|^2 d\nu_t \right)^p \right] < \infty$. For $d = 1$, we denote $\mathcal{H}_\nu^{p,1} := \mathcal{H}_\nu^p$.

(viii) $\mathcal{P}_p(\mathbb{R})$ is the set of probability measures on $\mathbb{R}$ with finite $p$th moment. We equip the space $\mathcal{P}_p(\mathbb{R})$ with the $p$-Wasserstein distance denoted by $\mathcal{W}_p$ and defined as

$$\mathcal{W}_p(\mu, \nu) := \inf \left\{ \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \pi(dx, dy) \right\}^{1/p},$$

where the infimum is over probability measures $\pi \in \mathcal{P}_p(\mathbb{R} \times \mathbb{R})$ with first and second marginals $\mu$ and $\nu$, respectively.
Definition

We say that the quadruple of processes \((Y_t, Z_t, U_t, K_t)_{t \leq T}\) is a solution of the mean-field reflected BSDE associated with \((f, \xi, h)\) if

\[
\begin{align*}
Y \in S^p, Z \in H^{p,1}, U \in H^p, \text{ and } K \in S^p_i \text{ and predictable,} \\
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s, U_s, P_{Y_s}) \, ds + K_T - K_t - \int_t^T Z_s \, dB_s - \int_t^T \int_{\mathbb{R}^*} U_s(e) \, \tilde{N}(ds, de), \\
Y_t &\geq h(t, Y_t, P_{Y_t}), \quad t \in [0, T], \\
\int_0^T (Y_t^- - h(t^-, Y_t^-, P_{Y_t^-})) \, dK_t &= 0 \quad \text{(Skorohod’s flatness condition)}.
\end{align*}
\]

(5)
MF-RBSDEs with jumps

Assumptions

The coefficients $f, h$ and $\xi$ satisfy

(i) $f$ is a mapping from $[0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times L_p^\nu \times \mathcal{P}_p(\mathbb{R})$ into $\mathbb{R}$ such that

(a) the process $(f(t,0,0,0,\delta_0))_{t \leq T}$ is $\mathcal{P}$-measurable and belongs to $\mathcal{H}_p^{1}$;

(b) $f$ is Lipschitz w.r.t. $(y, z, u, \mu)$ uniformly in $(t, \omega)$, i.e. there exists a positive constant $C_f$ such that $\mathbb{P}$-a.s. for all $t \in [0, T],$

$$|f(t, y_1, z_1, u_1, \mu_1) - f(t, y_2, z_2, u_2, \mu_2)| \leq C_f(|y_1 - y_2| + |z_1 - z_2|$$

$$+ |u_1 - u_2|_\nu + \mathcal{W}_p(\mu_1, \mu_2))$$

for any $y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}, u_1, u_2 \in L_p^\nu$ and $\mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R})$.
MF-RBSDEs with jumps

Assumptions

(i) $f$ also satisfies

(c) Assume that $d\mathbb{P} \otimes dt$ a.e. for each

$$(y, z, u_1, u_2, \mu) \in \mathbb{R}^2 \times (L^2_\nu)^2 \times \mathcal{P}_p(\mathbb{R}),$$

$$f(t, y, z, u_1, \mu) - f(t, y, z, u_2, \mu) \geq \langle \gamma_t^{y,z,u_1,u_2,\mu}, l_1 - l_2 \rangle_{\nu},$$

with

$$\gamma : [0, T] \times \Omega \times \mathbb{R}^2 \times (L^2_\nu)^2 \times \mathcal{P}_p(\mathbb{R}) \mapsto L^2_\nu;$$

$$(\omega, t, y, z, u_1, u_2, \mu) \mapsto \gamma_t^{y,z,u_1,u_2,\mu}(\omega, \cdot)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L^2_\nu)^2) \otimes \mathcal{B}(\mathcal{P}_p(\mathbb{R}))$ measurable satisfying

$$\| \gamma_t^{y,z,u_1,u_2,\mu}(\cdot) \|_\nu \leq C$$

for all

$$(y, z, u_1, u_2, \mu) \in \mathbb{R}^2 \times (L^2_\nu)^2 \times \mathcal{P}_p(\mathbb{R}),$$

$d\mathbb{P} \otimes dt$-a.e., where $C$ is a positive constant, and such that $\gamma_t^{y,z,u_1,u_2,\mu}(e) \geq -1$, for all

$$(y, z, u_1, u_2, \mu) \in \mathbb{R}^2 \times (L^2_\nu)^2 \times \mathcal{P}_p(\mathbb{R}),$$

$d\mathbb{P} \otimes dt \otimes dv(e)$-a.e.
Assumptions

(ii) \( h \) is a mapping from \([0, T] \times \Omega \times \mathbb{R} \times \mathcal{P}_p(\mathbb{R})\) into \( \mathbb{R} \) such that
   (a) For all \((y, \mu) \in \mathbb{R} \times \mathcal{P}_p(\mathbb{R})\), \((h(t, y, \mu))_t \) is a right-continuous left-limited process;
   (b) the process \( \left( \sup_{(y, \mu) \in \mathbb{R} \times \mathcal{P}_p(\mathbb{R})} |h(t, y, \mu)| \right)_{0 \leq t \leq T} \) belongs to \( \mathcal{S}^p \);
   (c) \( h \) is Lipschitz w.r.t. \((y, \mu)\) uniformly in \((t, \omega)\), i.e. there exists two positive constants \( \gamma_1 \) and \( \gamma_2 \) such that \( \mathbb{P}\text{-a.s. for all } t \in [0, T], \)

\[
|h(t, y_1, \mu_1) - h(t, y_2, \mu_2)| \leq \gamma_1 |y_1 - y_2| + \gamma_2 \mathcal{W}_p(\mu_1, \mu_2)
\]

for any \( y_1, y_2 \in \mathbb{R} \) and \( \mu_1, \mu_2 \in \mathcal{P}_p(\mathbb{R}) \).

(iii) \( \xi \in L^p(\mathcal{F}_T) \) and satisfies \( \xi \geq h(T, \xi, \mathbb{P}_\xi) \).
Existence and uniqueness result

Theorem

Suppose that above Assumption is in force for some $p \geq 2$. Assume that $\gamma_1$ and $\gamma_2$ satisfy

$$\gamma_1^p + \gamma_2^p < 2^{2-\frac{3p}{2}}. \quad (6)$$

Then the system (5) has a unique solution in $S^p \times \mathcal{H}^{p,1} \times \mathcal{H}^p_V \times S^p_i$.

Remark: An existence and uniqueness result has been obtained in Djehiche-Elie-Hamadene (in the case of a Brownian filtration) under a different condition and using a penalization scheme technique which imposes two extra conditions: 'a domination condition' on $f$ and monotonicity conditions on $f$, $h$. 
Mean-field reflected BSDEs with jumps

Existence and uniqueness result

Our proof of solvability of MF-RBSDEs is based on the connection between \textit{nonlinear reflected BSDEs} and \textit{optimal stopping with nonlinear expectations}.

- Given a Lipschitz driver \( g(t, \omega, y, z, u) \) and a terminal condition \( \xi \in L^p(\mathcal{F}_T) \) there exists a unique solution

\[
(X^g(\xi, T), \pi^g(\xi, T), \theta^g(\xi, T)) \in S^p \times \mathcal{H}^{p,m} \times \mathcal{H}_\nu^p
\]

satisfying

\[
X^g_t = \xi + \int_t^T g(s, X^g_s, \pi^g_s, \theta^g_s) \, ds - \int_t^T \pi^g_s \, dB_s - \int_t^T \int_{\mathbb{R}^*} \theta^g_s(e) \tilde{N}(ds, de)
\]

(7)
Existence and uniqueness result

- Define the nonlinear operator $\mathcal{E}^g_{t,T}[\xi] := X_t^g(\xi, T)$.
- We have the following 'semigroup property': for any stopping time $\tau \in \mathcal{T}_t$,

$$\mathcal{E}^g_{t,T}[\xi] := \mathcal{E}^g_{t,\tau}[\mathcal{E}^g_{\tau,T}[\xi]]$$
Existence and uniqueness result

Recall that the solution to the reflected BSDE associated with \( \{g(t, \omega, y, z, u), \xi, (L_t)_{t \leq T}\} \) and the RCLL obstacle \((S_t)\) is a triple of \((\mathcal{F}_t)_{t \leq T}\) - adapted stochastic processes \((Y_t, Z_t, U_t, K_t)_{t \leq T}\) such that, for all \( t \leq T \),

\[
\begin{cases}
Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s)ds + K_T - K_t - \int_t^T Z_sdB_s \\
- \int_t^T \int_{\mathbb{R}} U_s(e)\tilde{N}(ds, de),
\end{cases}
\]

\[
Y_t \geq L_t, \quad \xi \geq L_T;
\]

\[
\int_0^T (Y_s^- - L_s^-)dK_s = 0.
\]

(8)
Mean-field reflected BSDEs with jumps

Existence and uniqueness result

Theorem

Let \((Y, Z, U, K) \in S^p \times \mathcal{H}^d \times \mathcal{H}_\nu^d \times S_i^p\) be the unique solution of the reflected BSDE (8). Then

\[ Y_t = \text{ess sup}_{\tau \in T_t} \mathcal{E}_{t, \tau}^g [\phi_\tau], \quad t \leq \tau \leq T, \text{ a.s.} \quad (9) \]

and

\[ \phi_\tau := L_\tau 1_{\tau < T} + \xi 1_{\tau = T}. \]

Note that the representation (9) is different from:

\[ Y_t = \text{ess sup}_{\tau \in T_t} \mathbb{E} \left[ \int_t^\tau f(s, Y_s, Z_s) ds + \phi_\tau 1_{\tau < T} + \xi 1_{\tau = T} \right]. \]
Mean-field reflected BSDEs with jumps

Existence and uniqueness result

Approach

Exploit the nonlinear optimal stopping representation and consider the map \( \hat{\phi} : \mathcal{S}^p \to \mathcal{S}^p \) which associates to a process \( Y \in \mathcal{S}^p \) another process \( \hat{\Phi}(Y) \) defined by:

\[
\hat{\Phi}(Y)_t := \text{ess sup}_{\tau \in \mathcal{T}_t} \mathcal{E}^{f \circ Y}_{t, \tau} \left[ h(\tau, Y_\tau, \mathbb{P}_{Y_s|s=\tau})1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} \right],
\]

where \((f \circ Y)(t, y, z, u) = f(t, y, z, u, \mathbb{P}_{Y_t})\). By using a priori estimates with universal constants, we show that existence of a fixed point on \([T - \delta, T]\) for some \(\delta > 0\) and then obtain the existence and uniqueness on \([0, T]\).
Complementary results

Definition

A progressive process \((\phi_t)\) is called *left-upper semicontinuous (in short l.u.s.c) along stopping times* if for all \(\tau \in \mathcal{T}_0\) and for each non-decreasing sequence of stopping times \((\tau_n)\) such that \(\tau_n \uparrow \tau\) a.s., \(\phi_\tau \geq \limsup_{n \to \infty} \phi_{\tau_n}\) a.s.
Complementary results

Theorem (Sufficient condition - continuity of the increasing process)

Suppose that $\gamma_1$ and $\gamma_2$ satisfy the condition (6). Assume that $h$ takes the form $h(t, \omega, y, \mu) := \xi_t(\omega) + \kappa(y, \mu)$, where $\xi$ belongs to $S^p$ and is a left upper semicontinuous process along stopping times, and $\kappa$ is a bounded and Lipschitz function with respect to $(y, \mu)$. Let $(Y, Z, U, K)$ be the unique solution of the mean-field reflected BSDE (5). Then $(Y_t)$ has jumps only at totally inaccessible stopping times (i.e. the predictable process $(K_t)$ is continuous).
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4. Convergence of the particle system and propagation of chaos
Weakly interacting nonlinear Snell envelopes

Framework

- For a given vector \( \mathbf{x} := (x^1, \ldots, x^n) \in \mathbb{R}^n \), denote the empirical measure associated to \( \mathbf{x} \) by

  \[
  L_n[\mathbf{x}] := \frac{1}{n} \sum_{k=1}^{n} \delta_{x^k};
  \]

- Given \( n \in \mathbb{N} \), let \( \{B_i^i\}_{1 \leq i \leq n}, \{\tilde{N}^i\}_{1 \leq i \leq n} \) be independent copies of \( B \) and \( \tilde{N} \);

- Let \( \mathbb{F}^n := \{\mathcal{F}^n_t\}_{t \in [0, T]} \) be the completion of the filtration generated by \( \{B^i\}_{1 \leq i \leq n} \) and \( \{\tilde{N}^i\}_{1 \leq i \leq n} \).
Particle system

Consider a family of weakly interacting processes $\mathbf{Y}^n := (Y^{1,n}, \ldots, Y^{n,n})$ evolving backward in time as follows: for $i = 1, \ldots, n$,

$$
\begin{cases}
Y^{i,n}_t = \xi^{i,n}_t + \int_t^T f(s, Y^{i,n}_s, Z^{i,j,n}_s, U^{i,j,n}_s, L_n[\mathbf{Y}^n_s])ds + K^{i,n}_T - K^{i,n}_t \\
- \int_t^T \sum_{j=1}^n Z^{i,j,n}_s dB^i_s - \int_t^T \int_{R^*} \sum_{j=1}^n U^{i,j,n}_s(e) \tilde{N}^j(ds, de);
Y^{i,n}_t \geq h(t, Y^{i,n}_t, L_n[\mathbf{Y}^n_t]);
\int_0^T (Y^{i,n}_{t^-} - h(t, Y^{i,n}_{t^-}, L_n[\mathbf{Y}^n_{t^-}]])dK^{i,n}_t = 0.
\end{cases}
$$

(10)

Remark: Note that if $(\xi^{1,n}, \ldots, \xi^{n,n})$ is exchangeable, then $(\mathbf{Y}^n, \mathbf{Z}^n, \mathbf{U}^n, \mathbf{K}^n)$ is exchangeable.
Weakly interacting nonlinear Snell envelopes

Assumptions

We assume that

- \( f \) and \( h \) are deterministic functions of \((t, y, z, u, \mu)\) and \((t, y, \mu)\), respectively.
- \( \xi^{i,n} \in L^p(\mathcal{F}_T^n) \), \( i = 1, \ldots, n \).
- \( \xi^{i,n} \geq h(T, \xi^{i,n}, L_n[\xi^n]) \) a.s. \( i = 1, \ldots, n \).

Remark: The results are valid for more general \( f \) and \( g \), e.g.

\[
\begin{align*}
    f(t, \omega, y, z, \mu) &:= \hat{f}(t, (B_s(\omega), \tilde{N}_s(\omega))_{0 \leq s \leq t}, y, z, \mu) \\
    h(t, \omega, y, \mu) &:= \hat{h}(t, (B_s(\omega), \tilde{N}_s(\omega))_{0 \leq s \leq t}, y, \mu),
\end{align*}
\]

with \( \hat{f} \) and \( \hat{h} \) measurable and Lipschitz w.r.t. \( y, \mu \).
Wellposedness of the particle system

Theorem

Suppose that Assumptions are in force for some $p \geq 2$. Suppose further that $\gamma_1$ and $\gamma_2$ satisfy (6) i.e.

$$\gamma_1^p + \gamma_2^p < 2^{2-\frac{3p}{2}}.$$ 

Then the system of nonlinear weakly interacting Snell envelopes (10) has a unique solution in $S^p \otimes n \otimes \mathcal{H}^p \otimes n \otimes \mathcal{H}^p \otimes n \otimes S^p \otimes n$. 

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4. Convergence of the particle system and propagation of chaos
Convergence of the particle system and propagation of chaos

- The theory of propagation of chaos initiated by Kac with the purpose of investigating particle system approximations of some nonlocal PDEs.
- **Main idea**: Consider a large number of \( n \) of particles. If the interaction between the particles is *sufficiently weak* and the particles are *symmetric*, then as the size of the system increases, there is less and less interaction and in the limit the particles 'become independent'.
- **References**: e.g. McKean, Sznitman, Gartner. Further developments and applications: e.g. Jabin, Lacker, Shkolnikov, Briand et al.
- **Propagation of chaos for interacting backward particles**: Buckdahn et al, Hu et al., Laurière and Tangpi, Briand et al.
Convergence of the particle system and propagation of chaos

Assumptions

(i) The sequence $\xi^n = (\xi_1^n, \xi_2^n, \ldots, \xi_n^n)$ is exchangeable i.e. the sequence of probability laws $\mu^n$ of $\xi^n$ on $\mathbb{R}^n$ is symmetric.

(ii) For each $i \geq 1$, $\xi_i^n$ converges in $L^p$ to $\xi_i$, i.e.

$$\lim_{n \to \infty} \mathbb{E}[|\xi_i^n - \xi_i|^p] = 0,$$

where the random variables $\xi_i \in L^p(F^i_T)$ are independent and equally distributed (iid) with probability law $\mu$.

(iii) $h(t, y, \mu) = \xi_t + h_1(y, \mu)$, with $\xi \in S^p$ and l.u.s.c. along stopping times and $h_1$ Lipschitz bounded.
Convergence of the particle system and propagation of chaos

Framework

- Consider the (Polish) Skorohod space \( \mathbb{D} := D([0, T], d^0) \), with
  \[
d^0(x, y) = \inf_{\lambda \in \Lambda} \{ \| \lambda \|^0 \lor \sup_{0 \leq t \leq T} |x(t) - y(\lambda(t))| \},
\]
  where \( \Lambda \) is the space of increasing bijections from \([0, T]\) into itself.

- Set
  \[
  \mathbb{H}^{2,n}_\nu := L^2([0, T], \mathbb{R}^n, dt), \quad \mathbb{H}^{2,n}_\nu := L^2([0, T] \times \mathbb{R}^*, \mathbb{R}^n, dt \otimes d\nu).
  \]

- Consider the product space \( G := \mathbb{D} \times \mathbb{H}^{2,n} \times \mathbb{H}^{2,n}_\nu \times \mathbb{D} \) endowed with the product metric
  \[
  \delta(((y, z, u, k), (y', z', u', k')) := \left( d^0(y, y')^p + \| z - z' \|^{p}_{\mathbb{H}^{2,n}} +
  \| u - u' \|^{p}_{\mathbb{H}^{2,n}} + d^0(k, k')^p \right)^{\frac{1}{p}}.
  \]
Convergence of the particle system and propagation of chaos

Framework

- Define the Wasserstein metric on \( \mathcal{P}_p(G) \) by

\[
D_G(P, Q) = \inf \left\{ \left( \int_{G \times G} \delta((y, z, u, k), (y', z', u', k'))^p R(dx, dy) \right)^{1/p} \right\},
\]

over \( R \in \mathcal{P}(G \times G) \) with marginals \( P \) and \( Q \). Since \( (G, \delta) \) is a Polish space, \( (\mathcal{P}_p(G), D_G) \) is a Polish space and induces the topology of weak convergence.
Convergence of the particle system and propagation of chaos

**Theorem (Convergence result I)**

Assume that, for some $p > 2$, $\gamma_1$ and $\gamma_2$ satisfy

$$2^{\frac{5p}{2}} - 2(\gamma_1 + \gamma_2) < \left(\frac{p - \kappa}{2p}\right)^{p/\kappa}$$

for some $\kappa \in [2, p)$. Then, under the above assumptions, we have

$$\lim_{n \to \infty} \left(\|Y_i^n - Y^i\|_{S^p} + \|Z_i^n - Z^i e_i\|_{\mathcal{H}^p} + \|U_i^n - U^i e_i\|_{\mathcal{H}_V^p} + \|K_i^n - K^i\|_{S^p}\right) = 0.$$
Theorem (Propagation of chaos result I)

Under the assumptions of the previous theorem, the particle system (10) satisfies the propagation of chaos property, i.e. for any fixed positive integer $k$,

$$\lim_{n \to \infty} \text{Law}(\Theta_{1}^{n}, \Theta_{2}^{n}, \ldots, \Theta_{k}^{n}) = \text{Law}(\Theta_{1}, \Theta_{2}, \ldots, \Theta_{k}).$$

For any fixed $1 \leq k \leq n$, $\Theta_{i}^{n} := (Y_{i}^{n}, Z_{i}^{n}, U_{i}^{n}, K_{i}^{n}), i = 1, \ldots, k$ and $\Theta_{i} := (Y_{i}, Z_{i}, U_{i}, K_{i}), i = 1, \ldots, k$, with independent terminal values $Y_{i}^{T} = \xi_{i}, i = 1, \ldots, k$, are independent copies of $\Theta := (Y, Z, U, K)$. 
The proof of the propagation of chaos property follows from the inequalities

\[
D^p_G(\mathbb{P}^k, \mathbb{P}^k_\Theta) \leq \sum_{i=1}^{k} \left( \| Y^{i,n} - Y^i \|_{S^p}^p + \| Z^{i,n} - Z^i e_i \|_{\mathcal{H}^p,n}^p + \| U^{i,n} - U^i e_i \|_{\mathcal{H}^p,n}^p \right),
\]

where for each \( i = 1, \ldots, n \), \( e_i := (0, \ldots, 0, 1, 0, \ldots, 0) \).
Convergence of the particle system and propagation of chaos

Theorem (Convergence result II)

Assume that for some $p \geq 2$, $\gamma_1$ and $\gamma_2$ satisfy

$$2^{\frac{5p}{2}} - 2(\gamma_1^p + \gamma_2^p) < 1.$$ 

Then, under the above Assumptions, we have

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} E[|Y_{i,n}^i - Y_i^i|^p] = 0.$$ 

In particular,

$$\lim_{n \to \infty} \|Y_{i,n}^i - Y_i^i\|_{H^{p,1}} = 0.$$
Theorem (Propagation of chaos result II)

*Under the assumptions of previous Theorem, the solution $Y^{i,n}$ of the particle system (10) satisfies the propagation of chaos property, i.e. for any fixed positive integer $k$,*

$$\lim_{n \to \infty} \text{Law}(Y^{1,n}, Y^{2,n}, \ldots, Y^{k,n}) = \text{Law}(Y^1, Y^2, \ldots, Y^k).$$
Convergence of the particle system and propagation of chaos

An ingredient of the proof is the following convergence result:

**Theorem (Law of Large Numbers)**

Let $Y^1, Y^2, \ldots, Y^n$ with terminal values $Y^i_T = \xi^i$ be independent copies of the solution $Y$ of (5). Then, we have

$$
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \mathcal{W}_p(L_n[Y_t], \mathbb{P}_{Y_t}) \right] = 0.
$$
Related works

- **Preprint**: R.D., B. Djehiche, Zero-sum mean-field Dynkin games: characterization and convergence, February 2022
- **Work in progress**: R.D., B. Djehiche, I. Kharroubi, H. Pham, Viscosity solutions for an extended class of variational inequalities on the Wasserstein space
Thank you for your attention!