Introduction: goals

- Newton method:
  - Iterative method.
  - Linearized critical point equation yields a new search direction (Newton step).
  - Globalization: choose appropriate step length (line-search).
  - Expected to converge faster than naive iterative method.

- Our purpose:
  - Design this principle for solving stochastic control.
  - Establish global convergence and quick local convergence.
  - Practical implementation.
1. Recall on Newton method for optimization in $\mathbb{R}^d$

2. Stochastic control: framework and Newton step computation

3. Theoretical convergence properties

4. Numerical implementation and results
1. Recall on Newton method for optimization in $\mathbb{R}^d$

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Newton method in $\mathbb{R}^d$: principle for $f \in C^2(\mathbb{R}^d, \mathbb{R})$

- Second-order iterative method for optimization problems

$$\min_{x \in \mathbb{R}^d} f(x).$$

- First order optimality condition (sufficient if $f$ is convex)

$$\nabla f(x^*) = 0.$$
Newton method in $\mathbb{R}^d$: principle for $f \in C^2(\mathbb{R}^d, \mathbb{R})$

- Second-order iterative method for optimization problems

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$$\nabla f(x^*) = 0.$$ 

- Successive linear approximations of the critical point equation

$$\nabla f(x^{(k)}) + \nabla^2 f(x^{(k)}) \Delta x = 0.$$ 

- New iterate

$$x^{(k+1)} = x^{(k)} + \Delta x.$$
Newton method in $\mathbb{R}^d$: convergence properties

If $f$ has a unique minimizer $x^*$ and is:
- twice differentiable with Lipschitz second-order derivatives,
- strongly convex

$$\exists \alpha > 0, \forall (x, y), \quad f(y) \geq f(x) + \nabla f(x) \cdot (y - x) + \frac{\alpha}{2} \|y - x\|^2.$$

Then $(x^{(k)})_{k \in \mathbb{N}}$ converges quadratically locally to $x^*$ [3]

$$x^{(0)} \in V \Rightarrow \forall k \in \mathbb{N}, \quad C\|x^{(k+1)} - x^*\| \leq \left(C\|x^{(k)} - x^*\|\right)^2.$$

Newton method converges in 1 iteration if $f$ is quadratic
Illustration of local convergence of Newton method

Strongly convex function $f$ with Lipschitz second-order derivative:

$$f : x \mapsto |x|^3 + x^2$$

**Figure:** Distance of current iterate $x^k$ to minimizer $x^* = 0$

Observe 2 convergence regimes.
Newton method may not converge globally (in $\mathbb{R}^d$)

Strongly convex function $f$ with Lipschitz second-order derivative:

$$f(x) := \begin{cases} 
\frac{x^2}{4} + \frac{4}{3}, & \text{if } |x| > 4, \\
\frac{2|x|^3}{3}, & \text{if } 1 \leq |x| \leq 4, \\
\frac{x^2}{2} + \frac{1}{6}, & \text{if } |x| < 1. 
\end{cases}$$

If $|x^{(0)}| \in (1, 4)$, then $x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})} = -x^{(k)}$.

No convergence, need to choose a stepsize.
Newton method with backtracking line-search (in $\mathbb{R}^d$)

- Take step $\sigma \Delta x$ instead of full Newton step $\Delta x$...
- ... with $\sigma$ largest value in $\{1, \beta, \beta^2, \beta^3, ...\}$ such that

$$f(x^{(k)} + \sigma \Delta x) \leq f(x^{(k)}) + \gamma \sigma \nabla f(x^{(k)}) \Delta x.$$
Newton method with backtracking line-search (in $\mathbb{R}^d$)

- Take step $\sigma \Delta x$ instead of full Newton step $\Delta x$...
- ... with $\sigma$ largest value in $\{1, \beta, \beta^2, \beta^3, \ldots\}$ such that
  \[ f(x^{(k)} + \sigma \Delta x) \leq f(x^{(k)}) + \gamma \sigma \nabla f(x^{(k)}) \Delta x. \]

**Theorem**

There is a $k^*$ s.t.

- Until $k^*$: arithmetic convergence
  \[ \exists \eta > 0, \forall k \leq k^*, \quad \|x^{(k+1)} - x^*\| \leq \|x^{(k)} - x^*\| - \eta. \]

- After $k^*$: quadratic convergence
  \[ \exists C > 0, \forall k > k^*, \quad C \|x^{(k+1)} - x^*\| \leq \left(C \|x^{(k)} - x^*\| \right)^2. \]
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Mathematical formulation

- $u$: control variable (prog. mes. square integrable process)
- $X^u$: state of system (ODE controlled by $u$)
- Unconstrained convex problem: random linear dynamic (parameter $\alpha$), random convex running cost function $l$ and terminal cost function $\Psi$.

$$\min_u J(u) := \mathbb{E} \left[ \int_0^T l(t, u_t, X^u_t) \, dt + \Psi(X^u_T) \right]$$

s.t. $X^u_t := x_0 + \int_0^t \alpha_s u_s \, ds$.

- We focus on one-dimensional control/state processes.
- Usual regularity and growth conditions on $l$ and $\Psi$. 
Stochastic Pontryagin principle

Theorem

Under regularity assumptions, for any \( u \in \mathbb{H}^2 \), define \( Y^u \in \mathbb{H}^{\infty,2} \) by:

\[
Y^u_t = \mathbb{E}_t \left[ \Psi_x^\prime (X^u_T) + \int_t^T l_x^\prime (s, u_s, X^u_s) ds \right].
\]

Besides, \( J \) is Fréchet-differentiable with gradient at \( u \) denoted \( \nabla J (u) \in \mathbb{H}^2 \) given by:

\[
(\nabla J (u))_t = l_u^\prime (t, u_t, X^u_t) + \alpha_t Y^u_t,
\]

\[
\| \nabla J (u) - J (v) \|_{\mathbb{H}^2} \leq C \| u - v \|_{\mathbb{H}^2}.
\]
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Stochastic Pontryagina principle

Theorem

Under regularity assumptions, for any $u \in \mathbb{H}^2$, define $Y^u \in \mathbb{H}_{\infty,2}$ by:

$$Y^u_t = \mathbb{E}_t \left[ \psi'_x(X^u_T) + \int_t^T l'_x(s, u_s, X^u_s) \, ds \right].$$

Besides, $J$ is Fréchet-differentiable with gradient at $u$ denoted $\nabla J(u) \in \mathbb{H}^2$ given by:

$$(\nabla J(u))_t = l'_u(t, u_t, X^u_t) + \alpha_t Y^u_t,$$

$$\| \nabla J(u) - J(v) \|_{\mathbb{H}^2} \leq C \| u - v \|_{\mathbb{H}^2}.$$ 

Under additional convexity assumptions, $\nabla J(u) = 0$ characterizes the (unique) optimal control.
Characterization of Newton step

**Theorem**

▶ The mapping \( \Phi_Y : u \in H^2 \mapsto Y^u \in H^\infty,2 \) is Gateaux-differentiable, and its Gateaux derivative at \( u \) in direction \( v \) is given by

\[
D\Phi_Y(u)(v) = \dot{Y}^{u,v}, \text{ solution of the (linear) BSDE:}
\]

\[
\dot{Y}^u_t,v = E_t \left[ \psi''_{xx}(X_T^u)\dot{X}^v_T + \int_t^T \left( l''_{xx}(s, u_s, X_s^u)v_s + l''_{xx}(s, u_s, X_s^u)\dot{X}_s^v \right) ds \right].
\]

Besides, we have \( \|\dot{Y}^{u,v}\|_{H^{\infty,2}} \leq C\|v\|_{H^2} \).
Characterization of Newton step

**Theorem**

- The mapping $\Phi_Y : u \in \mathbb{H}^2 \mapsto Y^u \in \mathbb{H}^{\infty,2}$ is Gateaux-differentiable, and its Gateaux derivative at $u$ in direction $v$ is given by $D\Phi_Y(u)(v) = \dot{Y}^{u,v}$, solution of the (linear) BSDE:

$$
\dot{Y}^{u,v}_t = \mathbb{E}_t \left[ \psi''_{xx}(X^u_T) \dot{X}^v_T + \int_t^T \left( l''_{xx}(s, u_s, X^u_s) v_s + l''_{xx}(s, u_s, X^u_s) \dot{X}^v_s \right) ds \right].
$$

Besides, we have $\|\dot{Y}^{u,v}\|_{\mathbb{H}^{\infty,2}} \leq C \|v\|_{\mathbb{H}^2}$.

- $J$ is twice Gateaux-differentiable and $\nabla^2 J : \mathbb{H}^2 \mapsto \mathcal{L}(\mathbb{H}^2)$ is:

$$
(\nabla^2 J(u)(v))_t = l''_{uu}(t, u_t, X^u_t) v_t + l''_{ux}(t, u_t, X^u_t) \dot{X}^v_t + \alpha_t \dot{Y}^{u,v}_t.
$$

Besides, $\nabla^2 J(u) : \mathbb{H}^2 \mapsto \mathbb{H}^2$ is a continuous endomorphism $\|\nabla^2 J(u)(v)\|_{\mathbb{H}^2} \leq C \|v\|_{\mathbb{H}^2}$.
Newton step. We aim at computing $\Delta u$

$$\nabla^2 J(u)(\Delta u) = -\nabla J(u).$$

Crucial interpretation in terms of LQ stochastic control problem
(explicitly tractable using Riccati equations and LBSDE [14, 2]).

**Theorem**

Let $(u, w) \in \mathbb{H}^2 \times \mathbb{H}^2$. Consider $\min_{v \in \mathbb{H}^2} \tilde{J}^{LQ,u,w}(v)$ s.t. $\tilde{X}_t = \int_0^t \alpha_s v_s ds$,

where $\tilde{J}^{LQ,u,w}(v)$ is defined by:

$$\mathbb{E} \left[ \int_0^T \left\{ \frac{1}{2} l''_{uu} (t, u_t, X_t^u) v_t^2 + \frac{1}{2} l''_{xx} (t, u_t, X_t^u) \tilde{X}_t^2 + l''_{ux} (t, u_t, X_t^u) \tilde{X}_t v_t - w_t v_t \right\} dt + \frac{1}{2} \Psi''_{xx}(X_T^u) \tilde{X}_T^2 \right].$$
Newton step. We aim at computing $\Delta u$

$$\nabla^2 J(u)(\Delta u) = -\nabla J(u).$$

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Let $(u, w) \in H^2 \times H^2$. Consider $\min_{v \in H^2} \tilde{J}^{LQ,u,w}(v)$ s.t. $\tilde{X}_t = \int_0^t \alpha_s v_s ds$, where $\tilde{J}^{LQ,u,w}(v)$ is defined by:

$$E \left[ \int_0^T \left\{ \frac{1}{2} l''_{uu}(t, u_t, X_t^u) v_t^2 + \frac{1}{2} l''_{xx}(t, u_t, X_t^u) \tilde{X}_t^2 + l''_{ux}(t, u_t, X_t^u) \tilde{X}_t v_t - w_t v_t \right\} dt + \frac{1}{2} \psi''(X_T^u) \tilde{X}_T^2 \right].$$

Then $\tilde{J}^{LQ,u,w}$ has a unique minimizer $\tilde{u}^{u,w} \in H^2$ characterized by:

$$\begin{align*}
\tilde{X}_t^{u,w} &= \int_0^t \alpha_s \tilde{u}_s^{u,w} ds, \\
\tilde{Y}_t^{u,w} &= E_t \left[ \psi''(X_T^u) \tilde{X}_T^{u,w} + \int_t^T \left( l''_{xu}(s, u_s, X_s^u) \tilde{u}_s^{u,w} + l''_{xx}(s, u_s, X_s^u) \tilde{X}_s^{u,w} \right) ds \right], \\
l''_{uu}(t, u_t, X_t^u) \tilde{u}_t^{u,w} + l''_{ux}(t, u_t, X_t^u) \tilde{X}_t^{u,w} + \alpha_t \tilde{Y}_t^{u,w} &= w_t.
\end{align*}$$
Newton step. We aim at computing $\Delta u$

$$\nabla^2 J(u)(\Delta u) = -\nabla J(u).$$

Crucial interpretation in terms of LQ stochastic control problem (explicitly tractable using Riccati equations and LBSDE [14, 2]).

**Theorem**

Let $(u, w) \in \mathbb{H}^2 \times \mathbb{H}^2$. Consider $\min_{v \in \mathbb{H}^2} \tilde{J}^{LQ,u,w}(v)$ s.t. $\tilde{X}_t = \int_0^t \alpha_s v_s ds$, where $\tilde{J}^{LQ,u,w}(v)$ is defined by:

$$E \left[ \int_0^T \left\{ \frac{1}{2} l''_{uu}(t, u_t, X_t^u) v_t^2 + \frac{1}{2} l''_{xx}(t, u_t, X_t^u) \tilde{X}_t^2 + l''_{ux}(t, u_t, X_t^u) \tilde{X}_t v_t - w_t v_t \right\} dt + \frac{1}{2} \psi''_{xx}(X_T^u) \tilde{X}_T^2 \right].$$

Then $\tilde{J}^{LQ,u,w}$ has a unique minimizer $\tilde{u}^{u,w} \in \mathbb{H}^2$ characterized by:

$$\begin{cases}
\tilde{X}_t^{u,w} = \int_0^t \alpha_s \tilde{u}_s^{u,w} ds,
\tilde{Y}_t^{u,w} = E_t \left[ \psi''_{xx}(X_T^u) \tilde{X}_T^{u,w} + \int_t^T \left( l''_{xx}(s, u_s, X_s^u) \tilde{u}_s^{u,w} + l''_{ux}(s, u_s, X_s^u) \tilde{X}_s^{u,w} \right) ds \right],
\tilde{l''}_{uu}(t, u_t, X_t^u) \tilde{u}_t^{u,w} + \tilde{l''}_{ux}(t, u_t, X_t^u) \tilde{X}_t^{u,w} + \alpha_t \tilde{Y}_t^{u,w} = w_t.
\end{cases}$$

Besides, for any $u \in \mathbb{H}^2$, $\nabla^2 J(u) \in \mathcal{L}(\mathbb{H}^2)$ is invertible and $(\nabla^2 J(u))^{-1}(w) = \tilde{u}^{u,w}$. 
1 Recall on Newton method for optimization in $\mathbb{R}^d$

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4 Numerical implementation and results
Insufficient regularity properties in $\mathbb{H}^2$

Problem:

$$\min_{u \in \mathbb{H}^2} J(u)$$

$\nabla^2 J$: Lipschitz-continuous $\Rightarrow$ local quadratic convergence.
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Insufficient regularity properties in $\mathcal{H}^2$

Problem:

$$\min_{u \in \mathcal{H}^2} J(u)$$

$\nabla^2 J$: Lipschitz-continuous $\Rightarrow$ local quadratic convergence.

$\mathcal{H}^2$: set of square integrable processes, $\|u\|_{\mathcal{H}^2} = \sqrt{\mathbb{E} \left[ \int_0^T u_t^2 dt \right]}$.

- $J : \mathcal{H}^2 \mapsto \mathbb{R}$ is twice differentiable.
- $\nabla J : \mathcal{H}^2 \mapsto \mathcal{H}^2$.
- $\nabla^2 J : \mathcal{H}^2 \mapsto \mathcal{L}(\mathcal{H}^2)$.

But $\nabla^2 J : \mathcal{H}^2 \mapsto \mathcal{L}(\mathcal{H}^2)$ not Lipschitz-continuous.
Counter-example

Example

Consider $\mathcal{J}$ given by:

$$\forall u \in H^2([0, 1] \times \Omega, \mathbb{R}) , \quad \mathcal{J}(u) := \mathbb{E} \left[ \int_0^1 l(u_t) dt \right] , \quad s.t. \quad X^u_t = 0, \forall t \in [0, 1],$$

where $l$ is such $l''(x) = \min(1 + |x|, 2)$; $l'(0) = 0$; $l(0) = 0$.

$\mathcal{J}$ is twice continuously differentiable, with second order-derivative $\nabla^2 \mathcal{J}$ given by $(\nabla^2 \mathcal{J}(u)(\nu))_t = l''(u_t)\nu_t$, for $u, \nu \in H^2$. 

However, let us define $u_n \in H^2$ the constant process with value $1$ with probability $1/n$ and $0$ else:

$$\|\nabla^2 \mathcal{J}(u_n) - \nabla^2 \mathcal{J}(0)\|_{L(H^2)} \|u_n - 0\|_{H^2} \geq \|\nabla^2 \mathcal{J}(u_n)(u_n) - \nabla^2 \mathcal{J}(0)(u_n)\|_{H^2} \|u_n\|_{H^2} = \sqrt{n} \rightarrow_{n \rightarrow +\infty} +\infty.$$ 

In particular $\nabla^2 \mathcal{J}: H^2 \rightarrow L(H^2)$ is not Lipschitz-continuous.
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Counter-example

Example

Consider $J$ given by:

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where $l$ is such $l''(x) = \min(1 + |x|, 2); l'(0) = 0; l(0) = 0$. $J$ is twice continuously differentiable, with second order-derivative $\nabla^2 J$ given by $(\nabla^2 J(u)(\nu))_t = l''(u_t)\nu_t$, for $u, \nu \in H^2$.

However, let us define $u^{(n)} \in H^2$ the constant process with value $1/n$ and $0$ else:

$$\frac{\| \nabla^2 J(u^{(n)}) - \nabla^2 J(0) \|_{\mathcal{L}(H^2)}}{\| u^{(n)} - 0 \|_{H^2}} \geq \frac{\| \nabla^2 J(u^{(n)})(u^{(n)}) - \nabla^2 J(0)(u^{(n)}) \|_{H^2}}{\| u^{(n)} \|_{H^2} \| u^{(n)} \|_{H^2}} = \sqrt{n} \xrightarrow{n \to +\infty} +\infty.$$

In particular $\nabla^2 J : H^2 \mapsto \mathcal{L}(H^2)$ is not Lipschitz-continuous.
Restriction to the space of essentially bounded processes

As a difference with $\mathbb{R}^d$, non-equivalence of norms in our setting $\Rightarrow$
Choose right space!

- Replace $H^2$ by $H^\infty$ endowed with $\|u\|_{H^\infty} = \sup_{t \in [0,T]} \text{essup} |u_t|$.
- Prove stability under restriction to $H^\infty$:

$$\nabla J(H^\infty) \subset H^\infty \ , \ \nabla^2 J(H^\infty) \subset L(H^\infty).$$

- $J : H^\infty \mapsto \mathbb{R}$ has bounded and Lipschitz second-order derivative if data regular.
- Has an impact on the backtracking line search algorithm.
Theorem (Stability of $\mathcal{H}\infty$)

Under regularity, convexity and boundedness Assumptions, for all $u, v, w \in \mathcal{H}\infty$, $X^u, Y^u, \nabla J(u), \dot{X}^v, \dot{Y}^{u,v}, \nabla^2 J(u)(v)$ and $(\nabla^2 J(u))^{-1}(w)$ are in $\mathcal{H}\infty$ and:

\[
\|X^u\|_{\mathcal{H}\infty} + \|Y^u\|_{\mathcal{H}\infty} + \|\nabla J(u)\|_{\mathcal{H}\infty} \leq C(1 + \|u\|_{\mathcal{H}\infty}),
\]
\[
\|X^u - X^v\|_{\mathcal{H}\infty} + \|Y^u - Y^v\|_{\mathcal{H}\infty} + \|\nabla J(u) - \nabla J(v)\|_{\mathcal{H}\infty} \leq C\|u - v\|_{\mathcal{H}\infty},
\]
\[
\|\dot{X}^v\|_{\mathcal{H}\infty} + \|\dot{Y}^{u,v}\|_{\mathcal{H}\infty} + \|\nabla^2 J(u)(v)\|_{\mathcal{H}\infty} \leq C\|v\|_{\mathcal{H}\infty},
\]
\[
\|\dot{Y}^{u,w} - \dot{Y}^{v,w}\|_{\mathcal{H}\infty} + \|\nabla^2 J(u)(w) - \nabla^2 J(v)(w)\|_{\mathcal{H}\infty} \leq C\|u - v\|_{\mathcal{H}\infty}\|w\|_{\mathcal{H}\infty},
\]
\[
\|(\nabla^2 J(u))^{-1}(w)\|_{\mathcal{H}\infty} \leq C\|w\|_{\mathcal{H}\infty}.
\]

- $\nabla^2 J$ defines a Lipschitz-continuous operator from $\mathcal{H}\infty \mapsto \mathcal{L}(\mathcal{H}\infty)$.
- $\nabla^2 J(u)$ and $(\nabla^2 J(u))^{-1}$ are bounded linear operators, uniformly in $u$.
- The Newton direction $\Delta_u$ at the point $u \in \mathcal{H}\infty$ is in $\mathcal{H}\infty$. 
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Backtracking line search

Alg. 1: Standard Backtracking line search

1: **Inputs:** Current point $u \in \mathbb{H}^\infty$, Current search direction $\Delta_u \in \mathbb{H}^\infty$, $\beta \in (0, 1)$, $\gamma \in (0, 1)$.

2: $\sigma = 1$.

3: **while** $\mathcal{J}(u + \sigma \Delta_u) > \mathcal{J}(u) + \gamma \sigma \langle \nabla \mathcal{J}(u), \Delta_u \rangle_{\mathbb{H}^2}$ **do**

4: $\sigma \leftarrow \beta \sigma$.

5: **end while**

6: **return** $u + \sigma \Delta_u$.

The global convergence of the method is not guaranteed in our setting.

Open problem
Alg 2: Gradient Backtracking line search

1: **Inputs:** Current point $u \in \mathbb{H}^\infty$, Current search direction $\Delta_u \in \mathbb{H}^\infty$, $eta \in (0, 1)$, $\gamma \in (0, 1)$.

2: $\sigma = 1$.

3: while $\|\nabla J(u + \sigma \Delta_u)\|_{\mathbb{H}^\infty} > (1 - \gamma \sigma)\|\nabla J(u)\|_{\mathbb{H}^\infty}$ do

4: $\sigma \leftarrow \beta \sigma$.

5: end while

6: return $u + \sigma \Delta_u$. 

Theorem (global convergence and locally quadratic)

$\Delta$ Alg. 2 terminates in finitely many iterations.

$\Delta$ If Alg. 2 returns $\sigma = 1$, then the new point $u + \Delta u$ satisfies:

$$\|\nabla J(u + \Delta u)\|_{\mathbb{H}^\infty} \leq \min(1 - \gamma, C \|\nabla J(u)\|_{\mathbb{H}^\infty}) \|\nabla J(u)\|_{\mathbb{H}^\infty}.$$

$\Delta$ If Alg. 2 returns $\sigma < 1$, then the new iterate $u + \sigma \Delta u$ satisfies:

$$\|\nabla J(u + \sigma \Delta u)\|_{\mathbb{H}^\infty} \leq \|\nabla J(u)\|_{\mathbb{H}^\infty} - \beta \gamma (1 - \gamma) C.$$

$C$ is a constant depending on data.
Alg 2: Gradient Backtracking line search

1: **Inputs:** Current point $u \in \mathbb{H}^\infty$, Current search direction $\Delta u \in \mathbb{H}^\infty$, $\beta \in (0, 1)$, $\gamma \in (0, 1)$.
2: $\sigma = 1$.
3: **while** $\|\nabla J(u + \sigma \Delta u)\|_{\mathbb{H}^\infty} > (1 - \gamma \sigma)\|\nabla J(u)\|_{\mathbb{H}^\infty}$ **do**
4: $\sigma \leftarrow \beta \sigma$.
5: **end while**
6: **return** $u + \sigma \Delta u$.

Theorem (global convergence and locally quadratic)

- **Alg. 2 terminates in finitely many iterations.**
- If Alg. 2 returns $\sigma = 1$, then the new point $u + \Delta u$ satisfies:
  $$\|\nabla J(u + \Delta u)\|_{\mathbb{H}^\infty} \leq \min(1 - \gamma, C\|\nabla J(u)\|_{\mathbb{H}^\infty})\|\nabla J(u)\|_{\mathbb{H}^\infty}.$$  
- If Alg. 2 returns $\sigma < 1$, then the new iterate $u + \sigma \Delta u$ satisfies:
  $$\|\nabla J(u + \sigma \Delta u)\|_{\mathbb{H}^\infty} \leq \|\nabla J(u)\|_{\mathbb{H}^\infty} - \frac{\beta \gamma (1 - \gamma)}{C}.$$  

$C$ is a constant depending on data.

Emmanuel GOBET - Maxime GRANGEREAU

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Numerical example: Set of heterogeneous consumers, with different flexibility - Supply-demand balance

- Common noise (weather...)
- Independent individual noise (agent consumption...)
- General filtration (not necessarily Brownian) to allow jumps in exogenous factors
Control/dynamics: multi-category with common noise

- $M \in \mathbb{N}$: number of agent categories, indexed by $k$ or $l$
- Each category $k \in \{1, ..., M\}$ (with same characteristics) has $N_k$ agents indexed by $i$ or $j$
Control/dynamics: multi-category with common noise

- \( M \in \mathbb{N} \): number of agent categories, indexed by \( k \) or \( l \)
- Each category \( k \in \{1, \ldots, M\} \) (with same characteristics) has \( N_k \) agents indexed by \( i \) or \( j \)
- Dynamics for storage \( i \) in category \( k \)
  \[
  X_t^{(k,i)} = x_0^{(k,i)} + \int_0^t \left( \alpha_s^{(k)} u_s^{(k,i)} + \beta_s^{(k)} X_s^{(k,i)} + \gamma_s^{(k,i)}(\omega) \right) ds,
  \]
- Controls: \( (u^{(k,i)})_{1 \leq k \leq M, 1 \leq i \leq N_k} \) power consumptions
Control/dynamics: multi-category with common noise

- \( M \in \mathbb{N} \): number of agent categories, indexed by \( k \) or \( l \)
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  \]
- Controls: \( (u^{(k,i)})_{1 \leq k \leq M, 1 \leq i \leq N_k} \) power consumptions
- Examples of storage:
  - Battery state of charge: \( X_t = x_0 + \int_0^t \frac{u_s}{\varepsilon_{\text{max}}} \, ds \).
  - Temperature thermal storage:
    \[
    X_t = x_0 + \int_0^t (\alpha u_s - \beta (X_s - T_{\text{out}}(s)) + \gamma_s) \, ds.
    \]
- Processes impacting individual consumers (consumption, solar production...) are independent given \( \mathcal{G}_T \) (weather noise...)

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Objective function (to be minimised): a cooperative approach

$$\mathbb{E} \left[ \frac{1}{N} \sum_{k=1}^{M} \sum_{i=1}^{N_k} \left\{ \int_0^T Q^{(k,i)}(t, u_t^{(k,i)}, X_t^{(k,i)}) \, dt + \Psi^{(k,i)}(X_T^{(k,i)}) \right\} \right]$$

management cost for storage $i$ of category $k$

$$+ \mathbb{E} \left[ \int_0^T \mathcal{L} \left( t, \frac{1}{N} \sum_{l=1}^{M} \sum_{j=1}^{N_l} (u_t^{(l,j)} + P_{t}^{\text{load,(l,j)}} - P_{t}^{\text{prod}}) \right) \, dt \right]$$

instantaneous overall imbalance

$$Q^{(k,i)}(t, u, x) = \mu_t^{(k)} \left( u - u_t^{\text{ref,(k,i)}} \right)^2 / 2 + \nu_t^{(k)} \left( x - x_t^{\text{ref,(k,i)}} \right)^2 / 2,$$

$$\Psi^{(k,i)}(x) = \rho(k) \left( x - x_T^{f,(k,i)} \right)^2 / 2.$$
The control problem

\[
\min_{(u^{(k,i)})_{k,i}} \mathbb{E} \left[ \frac{1}{N} \sum_{k=1}^{M} \sum_{i=1}^{N_k} \left\{ \int_0^T Q^{(k,i)} \left( t, u^{(k,i)}_t, X^{(k,i)}_t \right) dt + \psi^{(k,i)} \left( X^{(k,i)}_T \right) \right\} \right]
\]

\[
+ \mathbb{E} \left[ \int_0^T \mathcal{L} \left( t, \frac{1}{N} \sum_{l=1}^{M} \sum_{j=1}^{N_l} (u^{(l,j)}_t + P^\text{load}_{t,(l,j)} - P^\text{prod}_{t}) \right) dt \right],
\]

s.t. \( X^{(k,i)}_t = x^{(k,i)}_0 + \int_0^t \left( \alpha^{(k)}_s u^{(k,i)}_s + \beta^{(k)}_s X^{(k,i)}_s + \gamma^{(k,i)}_s \right) ds, \forall k, i. \)

- Semi Linear-Quadratic (\( \mathcal{L}(t,.) \) not necessarily quadratic) and strongly convex stochastic control problem...

- ... in high dimension (\( N := \sum_{k=1}^M N_k \)) with coupling.
Approximation for the aggregator and consumers

From EG, MG: ”Federated stochastic control of numerous heterogeneous energy storage systems” https://hal.archives-ouvertes.fr/hal-03108611

- **Theorem**: the $N$-dimensional control problem is approximately equivalent to a leader-follower control problem:
  - 1 control problem for the aggregator in dimension $M$ (number of categories)
  - for each consumer, a 1-dimensional control problem

Aggregator gives a coordination signal, to be used by all consumers in parallel ⇒ Solves the privacy preserving issue
Approximation for the aggregator and consumers

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- Theorem: the $N$-dimensional control problem is approximately equivalent to a leader-follower control problem:
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Aggregator gives a coordination signal, to be used by all consumers in parallel $\Rightarrow$ Solves the privacy preserving issue

- Theorem: error bounds
  - $O(1/\sqrt{M})$ for control, in $L_2$
  - $O(1/M)$ for functional cost
The aggregator control problem

$$\min \quad \mathbb{E} \left[ \sum_{k=1}^{M} \pi^{(k)} \left\{ \int_0^T \bar{Q}^{(k,N)} \left( t, u_t^{(k)}, X_t^{(k)} \right) \, dt + \bar{\Psi}^{(k,N)} \left( X_T^{(k)} \right) \right\} \right]$$

$$+ \mathbb{E} \left[ \int_0^T \mathcal{L} \left( t, \sum_{l=1}^{M} \pi^{(l)} u_t^{(l)} + \bar{P}_{t}^{\text{load},(N)} - \bar{P}_{t}^{\text{prod}} \right) \, dt \right],$$

where $\bar{\Psi}^{(k,N)} (x) := \rho^{(k)} \frac{(x - \bar{X}_T^{f,(k,N)})^2}{2}$. Can be solved in the $\mathbb{G}$-filtration. Depends on statistics of consumers. Reduced-size problem.
Solving the aggregator problem

Description of Newton iteration:

- Each iteration requires solving one ODE and three BSDEs.

- **Numerical solution of BSDEs.** Many conditional expectations \( \Rightarrow \) Empirical Least-Squares Regression [8].

**Markovian framework:**

\[
Y_t = \mathbb{E} \left[ \int_t^T f(s, X_s, Y_s) ds + G(X_T) \mid \mathcal{F}_t \right] = \phi_t(X_t),
\]

\[
\phi_t(.) = \arg \min_{h \text{ meas.}} \mathbb{E} \left[ \left( \int_t^T f(s, X_s, Y_s) ds + G(X_T) - h(X_t) \right)^2 \right].
\]

- Solve backwards in \( t \) (after time discretization \( \simeq \) Euler).
- Choose \( h \) in finite-dimensional functional vector space \( V \) or non linear space (e.g. NN).
- Expectation \( \simeq \) empirical mean over the simulations
- \( H^\infty \) norm computed over the simulations
Recall on Newton method for optimization in $\mathbb{R}^d$

Stochastic control: framework and Newton step computation
Theoretical convergence properties
Numerical implementation and results

**Numerical performance**

(a) Computation time (in seconds)

(b) Number of step size reductions

(c) $\|\nabla J(u^{(k)})\|_{H_\infty}$ along iterations

(d) $\|\nabla \bar{J}(u^{(k)})\|_{H_2}$ along iterations

(e) Cost $\bar{J}(u^{(k)})$ along iterations $k$

(f) Suboptimality gap $\bar{J}(u^{(k)}) - \min_j \bar{J}(u^{(j)}) + 10^{-9}$ (log scale)

Figure: Performance of Newton method with the two line search methods along iterations

Emmanuel GOBET - Maxime GRANGEREAU

Newton method for stochastic control problems
Conclusion and perspectives

- Design of Newton algorithm for stochastic control
- Careful choice of norms
- Proof of convergence (global, and locally quadratic). A few iterations are sufficient.
- Iterative method made of simple BSDEs, fast to solve.
- On-going works:
  - extension to more general control problems
  - analysis of numerical errors
References I


References II


References III

