On mean field control and mean field games in infinite dimension

Fausto Gozzi
Luiss University, Roma, Italy

two related joint works with:
A. Cosso, I. Kharroubi, H. Pham, M. Rosestolato (MFC);
S. Federico, D. Ghilli, M. Rosestolato (MFG).

BSDE2022, Annecy (FR), June 28, 2022

Outline

1. Motivation and examples

2. Our results on Mean Field Control
   - Our Mean Field Control setting and the law invariance
   - Differential calculus on Wasserstein space and HJB equation
   - Comparison theorem and uniqueness

3. Ongoing results on Mean Field Games

Motivation 1

Key problem in macroeconomics: understand the behavior of an economy where many forward looking agents optimize their own preferences (described by a *utility function*) and interact each other. In this context two main approaches are considered:

- "Planner's problem": a single representative agent maximizes a linear combination of the objectives of the agents
  $\longrightarrow$ Pareto optimal cooperative equilibrium;
- "Agent's problem": every agent maximizes its objective
  $\longrightarrow$ Nash non-cooperative equilibrium.

Treatable case: Limit with a continuum of homogeneous agents each of which sees the other agents only through their distribution. This gives rise to two related but different mathematical topics:

Motivation 2

- "Planner's problem": Control of McKean-Vlasov type SDEs or Mean Field Control (MFC) where one studies optimally controlled stochastic dynamical systems where *the dynamics of the state equation (and, possibly, also the objective functional) depends on the state/control law*.

- "Agent's problem": Mean Field Games (MFG) where one studies the Nash non-cooperative equilibria of differential games among a continuum of homogeneous agents whose *state dynamics depend on the distribution of the other agents*.

Motivation 3

- Many papers recently have studied mean field control problems in the case when the state equation is finite dimensional (see e.g. the book [Carmona-Delarue '18] and various other papers) and possibly path-dependent [Wu-Zhang '20].
- Up to our knowledge, no paper studied the infinite dimensional case (Suggestions are welcome). However such case arises naturally in applications as we see below.
- For Mean Field Games, we are only aware of one papers which consider infinite dimensional problems (Fouque-Zhang 2020) in a special case. Suggestions are welcome, too.
- We then aim to develop both theories in the infinite dimensional (possibly path-dependent) case trying also to clarify some issues left in previous papers. Our papers are the first step in this direction.

Example 1 (MFC): spatial economic growth models

Spatial growth models (see e.g. [Boucekkine-Camacho-Fabbri '16], [G.-Leocata '21]) are written as Optimal Control Problems whose state equation is a PDE, possibly Stochastic, such as (here the variable x is the location)

$$\frac{\partial k(t,x)}{\partial t} = \frac{\partial^2 k(t,x)}{\partial x^2} + ak(t,x) - \delta k(t,x) - c(t,x) + \xi(t,x).$$

Here the state variable is the capital stock k, the control variable is the consumption rate c, the (generic) noise term is ξ, while the data a and δ are, respectively, the productivity and the depreciation of the capital (here constants for simplicity).

A typical issue in such problems is the fact that the productivity depends on the capital distribution (see e.g. [Penalosa-Turnovsky '06]) i.e. $a = a\left(\mathbb{P}_{k(t,\cdot)}\right)$.

Ex. 1: the SPDE as an SDE in infinite dimension

The above state equation can be rewritten as an SDE in an infinite dimensional space H (here we take $H = L^2(S^1)$). The result, under reasonable assumptions on the noise ξ, is an SDE in H like

$$dk(t) \;=\; \Big[Ak(t) + a\Big(\mathbb{P}_{k(t,\cdot)}\Big)k(t) - \delta k(t) - c(t)\Big]\,dt + \sigma(k(t))dB(t),$$

where now, for each $t \geq 0$, the $k(t)$, $c(t)$ are elements of $L^2(S^1)$, hence functions of the location x. Here A is the Laplace operator, B is a cylindrical Wiener process, and is $\sigma(\cdot)$ a given operator, possibly linear or constant.

It is also reasonable to include, in such type of models, delay/path-dependent features like time-to build or vintage capital.

Example 2 (MFC): Lifecycle portfolio with "sticky" wages

In such problems (as it is done in [Djeiche-G.-Zanco-Zanella '22]), it is natural to model the dynamics of the labor income "$y(\cdot)$" (one of the state equations of the optimal portfolio problem) using one-dimensional delay SDEs of McKean-Vlasov type as follows (here $\phi \in L^2(-d,0;\mathbb{R})$ is a given datum and Z is a one-dimensional Brownian motion).

$$dy(t) = \left[b_0(\mathbb{P}_{y(t)}) + \int_{-d}^{0} y(t+s)\phi(s)\, ds \right] dt + \sigma\, y(t)\, dZ(t).$$

Here ϕ gives the weight of the past path of y on its current dynamics, while b_0 models the effect of the distribution of wages on their dynamics. A typical example could be a mean reverting term like $b_1(y(t) - \mathbb{E}[y(t)])$ with $b_1 < 0$.

Ex. 2: the delay SDE as an SDE in infinite dimension

Again, the above equation can be rephrased as an SDE for the variable $Y = (Y_0, Y_1)$ in the Hilbert space $H := \mathbb{R} \times L^2(-d, 0; \mathbb{R})$ setting $Y_0(t) = y(t) \in \mathbb{R}$, $Y_1(t) = y(t + \cdot) \in L^2(-d, 0; \mathbb{R})$.
The resulting dynamics is the following Mc Kean - Vlasov SDE in H:

$$dY(t) \;=\; \Big[AY(t) + \overline{b}_0(\mathbb{P}_{Y(t)})\Big]\, dt + \Sigma(Y(t))dZ(t).$$

where A is a suitable first order operator while $\overline{b}_0$ and Σ are zero on the second component, i.e. $\overline{b}_0(\mu) = (b_0(\mu), 0)$ and $\Sigma(y_0, y_1)z = (\sigma y_0 z, 0)$.

Example 3 (MFG): Inter-bank borrowing/lending

In this model (see [Fouque-Zhang '18]) there are N banks. the dynamics for log-monetary reserves for the bank i is

$$dX_i(t) = [\alpha_i(t) - \alpha_i(t-d)]dt + \sigma dW_i(t)$$

where $W_1, \dots, W_N$ are independent Wiener processes. Each bank maximizes, over processes α with values in a compact set $O \subset \mathbb{R}$,

$$J_i(\alpha_i, \alpha_{-i}) = \mathbb{E}\left[\int_0^T f(X(t), \alpha_i(t))dt + g(X(T))\right]$$

where (below $\overline{x} := \frac{1}{N}\sum_{k=1}^N x_k$)

$$f(x, \alpha_i) := -\frac{\alpha_i^2}{2} - \frac{\epsilon}{2}(\overline{x} - x_i)^2; \qquad g(x) = -\frac{c}{2}(\overline{x} - x_i)^2$$

As in Example 2, the problem of each player can be rewritten in a standard way as an optimal control of an SDE in the space $H := \mathbb{R} \times L^2(-d, 0; \mathbb{R})$.

Ex. 3: the infinite dimensional MFG system

In [Fouque-Zhang '18] the authors study the MFG system formally derived as the limit of the above game when $N \to +\infty$.
The state variable in H is $Z = (Z_0, Z_1)$ where $Z_0 \in \mathbb{R}$ is the old state X while $Z_1 \in L^2(-d,0;\mathbb{R})$ is the past of the control α. The state equation in H is

$$dZ(t) = [AZ(t) + B\alpha(t)]dt + GdW(t)$$

with A, B, G suitable operators (A, B unbounded). We call $\mathscr{L}$ the generator of the above SDE when $\alpha \equiv 0$:

$$[\mathscr{L}\phi](z) := \frac{1}{2} TrG^* G\partial_{zz} V + \langle Az, \partial_z V\rangle_H$$

The HJB equation is

$$-\partial_t V - \mathscr{L} V - H_{MAX}(z, \partial_z V, \mu(t)) = 0,$$

where

$$H_{MAX}(z,p,\mu) = \sup_{\alpha \in O} \left\{ \langle B\alpha, p \rangle_H + f_1(z,\alpha,\mu) \right\}$$

with terminal condition $V(T,z) = g_1(\mu(T),z)$. Here f_1 and g_1 are the equivalent of f and g in the MFG setting.

The FPK equation is

$$\partial_t m - \mathscr{L}^* m - div\left(D_p H_{MAX}(z, \partial_z V, m) m\right) = 0; \qquad m(0) = m_0$$

where m_0 is the initial distribution of the state Z.

Outline

Literature on MFC

- Recently, various *papers* have studied mean field control problems in the finite dimensional case. Among them we recall: [Andersson-Djehiche '10], [Buckdahn-Djehiche-Li '11], [Carmona-Delarue-Lachapelle '13], [Carmona-Delarue '15], [Pham-Wei '17], [Lacker '17], [Acciaio-Backhoff-Carmona '19], [Burzoni-Ignazio-Reppen-Soner '20], [Wu-Zhang '20], [Djete-Possamaï-Tan '20].
- We also mention the *books:* [Bensoussan-Freshe-Yam '13] and [Carmona-Delarue '18].

Main results at a glance

- In the first paper we study a (possibly path dependent) McKean-Vlasov control problem under general conditions, proving:
  - the dynamic programming principle;
  - the law invariance property of the value function V;
  - the Ito formula;
  - the fact that V is a viscosity solution of the so-called "Master" HJB equation.
- The second paper studies uniqueness of the solution of the Master HJB equation in a regular finite dimensional case, since it is new also in such setting.
- Further work: regularity of Master HJB and verification theorems.

Wasserstein space

Wasserstein space of probability measures on a Polish space H

- We denote by $\mathscr{P}_2(H)$ the set of probability measures on $(H, \mathscr{B}(H))$ with finite second-moment:

$$\int_H |x|^2 \mu(dx) \ < \ +\infty.$$

- We endow $\mathscr{P}_2(H)$ with the 2-Wasserstein metric:

$$\mathscr{W}_2(\mu,\nu) \ := \ \inf\left\{\int_{H\times H} |x-y|^2 \pi(dx,dy) : \pi \in \mathscr{P}_2(H\times H) \right.$$
$$\left. \text{such that } \ \pi(\cdot \times H) = \mu \text{ and } \pi(H\times \cdot) = \nu \right\}^{1/2}.$$

- The space $(\mathscr{P}_2(H), \mathscr{W}_2(H))$ is a Polish space.

Framework

Probabilistic setting

- $(\Omega, \mathscr{F}, \mathbb{P})$ complete probability space.
- $B = (B_t)_{t \geq 0}$ cylindrical Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$ with values in E (Hilbert space).
- $U \colon \Omega \to \mathbb{R}$ is an $\mathscr{F}$-measurable r.v. with uniform distribution on $[0,1]$, independent of $(B_t)_{t \geq 0}$.
- $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ is the $\mathbb{P}$-completion of the filtration generated by $(B_t)_{t \geq 0}$ and U.

Control processes

- $T > 0$ time horizon and O Polish space: the control space.
- The set of admissible control strategies $\mathscr{A}$ is the set of $\mathbb{F}$-progressively measurable processes $\alpha \colon [0, T] \times \Omega \to O$.

McKean-Vlasov control problem

Controlled state process. State space H: Hilbert space. For every $t \in [0, T]$, $\xi \in L^2(\Omega, \mathscr{F}_t, \mathbb{P}; H)$, $\alpha \in \mathscr{A}$,

$$X_s = \xi + \int_t^s [AX_r + b(r, X_r, \mathbb{P}_{X_r}, \alpha_r)] dr + \int_t^s \sigma(r, X_r, \mathbb{P}_{X_r}, \alpha_r) dB_r,$$

for every $s \in [t, T]$. Here $A : D(A) \subseteq H \to H$ is suitable differential operator.

Reward functional. For every $t \in [0, T]$, $\xi \in L^2(\Omega, \mathscr{F}_t, \mathbb{P}; H)$, $\alpha \in \mathscr{A}$,

$$J(t, \xi, \alpha) = \mathbb{E}\left[\int_t^T f\left(s, X_s^{t,\xi,\alpha}, \mathbb{P}_{X_s^{t,\xi,\alpha}}, \alpha_s\right) ds + g\left(X_T^{t,\xi,\alpha}, \mathbb{P}_{X_T^{t,\xi,\alpha}}\right) \right].$$

Lifted value function. For every $t \in [0, T]$, $\xi \in L^2(\Omega, \mathscr{F}_t, \mathbb{P}; H)$,

$$V(t, \xi) = \sup_{\alpha \in \mathscr{A}} J(t, \xi, \alpha).$$

Assumptions

Coefficients

$$b, \sigma, f : [0,T] \times H \times \mathscr{P}_2(H) \times O \to H, \mathscr{L}_2(E,H), \mathbb{R}, \quad g : H \times \mathscr{P}_2(H) \to \mathbb{R},$$

Assumption (A).

- b, σ, f, g are measurable and bounded.
- A generates a C-0-semigroup of pseudo-contractions.
- Hölder/Lipschitz continuity

$$\begin{aligned}
|b(t,x,\mu,a) - b(t',x',\mu',a)|_H &\le K\big(|t-t'|^\gamma + |x-x'| + \mathscr{W}_2(\mu,\mu')\big) \\
|\sigma(t,x,\mu,a) - \sigma(t',x',\mu',a)|_{\mathscr{L}_2} &\le K\big(|t-t'|^\gamma + |x-x'| + \mathscr{W}_2(\mu,\mu')\big) \\
|f(t,x,\mu,a) - f(t',x',\mu',a)| &\le K\big(|t-t'|^\gamma + |x-x'| + \mathscr{W}_2(\mu,\mu')\big) \\
|g(x,\mu) - g(x',\mu')| &\le K\big(|x-x'| + \mathscr{W}_2(\mu,\mu')\big).
\end{aligned}$$

- The Polish space O is *compact*.

Consequences

State equation. Under (A), there exists a unique solution $X^{t,\xi,\alpha}$ to the state equation in the class of continuous processes, $\mathbb{F}$-adapted, satisfying

$$\mathbb{E}\Big[\sup_{t\le s\le T}\big|X_s^{t,\xi,\alpha}\big|^2\Big] \;<\; +\infty.$$

Theorem

Suppose that Assumption (A) holds.

- *V is bounded.*
- *V is jointly continuous.*
- *Lipschitz continuity: $|V(t,\xi)-V(t,\xi')|\le L\sqrt{\mathbb{E}|\xi-\xi'|^2}$.*

Dynamic programming principle for V

Theorem

Under Assumption (A), for every $t, s \in [0, T]$, with $t \leq s$, $\xi \in L^2(\Omega, \mathscr{F}_t, \mathbb{P}; H)$, it holds that

$$V(t,\xi) = \sup_{\alpha \in \mathscr{A}} \left\{ \mathbb{E}\left[\int_t^s f\left(r, X_r^{t,\xi,\alpha}, \mathbb{P}_{X_r^{t,\xi,\alpha}}, \alpha_r\right) dr \right] + V\left(s, X_r^{t,\xi,\alpha}\right) \right\}.$$

Remark. No measurable selection issue as the function V depends on the whole r.v. ξ. In particular, the proof goes along the same lines as in the case of deterministic optimal control.

Law invariance

Theorem

Under Assumption (A), the map V satisfies the ***law invariance property****: for every $t \in [0,T]$ and $\xi, \xi' \in L^2(\Omega, \mathscr{F}_t, \mathbb{P}; H)$, with $\mathbb{P}_\xi = \mathbb{P}_{\xi'}$, it holds that $V(t,\xi) = V(t,\xi')$.*

- The law invariance was firstly proved in [Cosso-Pham,'18]. However that proof uses a result from [Aliprantis-Border,'06], Corollary 18.23, which is not correct as it is. Hence such a proof does not work.
- Our proof is based on the fact that one can find, for every ξ, ξ' as above, two r.v. U_ξ and $U_{\xi'}$, with uniform distribution on $[0,1]$, such that ξ and U_ξ (and also ξ' and $U_{\xi'}$) are independent.
- We also provide an example where this does not apply.

Value function

By the *law invariance*, we can define the value function

$$v(t,\mu) \;=\; V(t,\xi), \qquad \forall\,(t,\mu) \in [0,T] \times \mathscr{P}_2(H),$$

for any $\xi \in L^2(\Omega, \mathscr{F}_t, \mathbb{P}; H)$.

Corollary

Suppose that Assumption (A) holds.

- *v is bounded.*
- *v is jointly continuous.*
- *Lipschitz continuity:* $|v(t,\mu) - v(t,\mu')| \le L\mathscr{W}_2(\mu,\mu')$.

Corollary

Under Assumption (A), $\forall\, t,s \in [0,T]$, *with* $t \le s$, $\mu \in \mathscr{P}_2(H)$,

$$v(t,\mu) \;=\; \sup_{\alpha\in\mathscr{A}} \left\{ \mathbb{E}\left[\int_t^s f\left(r, X_r^{t,\xi,\alpha}, \mathbb{P}_{X_r^{t,\xi,\alpha}}, \alpha_r\right) dr \right] + v\left(s, \mathbb{P}_{X_s^{t,\xi,\alpha}}\right) \right\},$$

Lifting

Given $u\colon [0,T] \times \mathscr{P}_2(H) \to \mathbb{R}$, we define the lifting of u

$$U\colon [0,T] \times L^2(\Omega, \mathscr{F}, \mathbb{P}; H) \longrightarrow \mathbb{R}$$

as follows

$$U(t,\xi) \ := \ u(t, \mathbb{P}_\xi).$$

Example

Let

$$u(t,\mu) \ = \ h\Big(t, \int_H \varphi_1 \, d\mu, \ldots, \int_H \varphi_n \, d\mu\Big),$$

for some $h\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and $\varphi_i\colon H \to \mathbb{R}$.

Lifting

$$U(t,\xi) \ = \ h\big(t, \mathbb{E}[\varphi_1(\xi)], \ldots, \mathbb{E}[\varphi_n(\xi)]\big).$$

Lions' differential calculus

Definition

u is said to be L-differentiable (or differentiable in the sense of Lions) at $(t,\mu) \in [0,T] \times \mathscr{P}_2(H)$ if there exists $\xi \in L^2(\Omega, \mathscr{F}, \mathbb{P}; H)$ with law μ and U is differentiable (in the sense of Fréchet) at (t,ξ).

Notation for the Fréchet derivative. Let $DU(t,\xi)$ denote the *gradient* of U at (t,ξ), namely the element of $L^2(\Omega, \mathscr{F}, \mathbb{P}; H)$ given by the Riesz representation theorem.

Theorem

Suppose that the lifting U of u admits a continuous Fréchet derivative $DU \colon [0,T] \times L^2(\Omega, \mathscr{F}, \mathbb{P}; H) \to L^2(\Omega, \mathscr{F}, \mathbb{P}; H)$. Then, for any $(t,\mu) \in [0,T] \times \mathscr{P}_2(H)$, there exists a measurable function

$$\partial_\mu u(t,\mu) \colon H \longrightarrow H \qquad \text{such that} \quad DU(t,\xi) = \partial_\mu u(t,\mu)(\xi),$$

for every $\xi \in L^2(\Omega, \mathscr{F}, \mathbb{P}; H)$ with law μ.

Example

Let

$$u(t,\mu) \;=\; h\Big(t, \int_H \varphi_1\, d\mu, \dots, \int_H \varphi_n\, d\mu\Big)$$

and

$$U(t,\xi) \;=\; h\big(t, \mathbb{E}[\varphi_1(\xi)], \dots, \mathbb{E}[\varphi_n(\xi)]\big).$$

Fréchet derivative

$$DU(t,\xi) \;=\; \sum_{i=1}^{n} \frac{\partial h}{\partial y_i}\big(t, \mathbb{E}[\varphi_1(\xi)], \dots, \mathbb{E}[\varphi_n(\xi)]\big) \nabla\varphi_i(\xi).$$

Hence, the L-derivative or measure derivative of u is given by

$$\partial_\mu u(t,\mu)(x) \;=\; \sum_{i=1}^{n} \frac{\partial h}{\partial y_i}\Big(t, \int_H \varphi_1\, d\mu, \dots, \int_H \varphi_n\, d\mu\Big) \nabla\varphi_i(x).$$

Second-order differentiability

$$\partial_\mu u(t,\mu)(x) \quad \longrightarrow \quad \partial_x \partial_\mu u(t,\mu)(x)$$

Definition

$C^{1,2}([0,T]\times\mathscr{P}_2(H))$ is the set of continuous functions $u\colon [0,T]\times\mathscr{P}_2(H)\to\mathbb{R}$ such that:

- the lifting U of u admits a continuous Fréchet derivative $DU\colon [0,T]\times L^2(\Omega,\mathscr{F},\mathbb{P};H)\to L^2(\Omega,\mathscr{F},\mathbb{P};H)$ (this guarantees the existence of $\partial_\mu u$);
- $\partial_\mu u$ is continuous;
- $\partial_t u$ and $\partial_x\partial_\mu u$ exist and are continuous.

$C^{1,2}_2([0,T]\times\mathscr{P}_2(H))$ is the subset of $C^{1,2}([0,T]\times\mathscr{P}_2(H))$ of functions $u\colon [0,T]\times\mathscr{P}_2(H)\to\mathbb{R}$ satisfying

$$|\partial_\mu u(t,\mu)(x)| + |\partial_x\partial_\mu u(t,\mu)(x)| \le C\big(1+|x|^2\big).$$

Itô's formula

Theorem

Let $u \in C_2^{1,2}([0,T] \times \mathscr{P}_2(H))$, $t \in [0,T]$, $\xi \in L^2(\Omega, \mathscr{F}_t, \mathbb{P}; H)$. Let also $F : [0,T] \times \Omega \to H$ and $G : [0,T] \times \Omega \to \mathscr{L}_2(E,H)$ be bounded and $\mathbb{F}$-progressively measurable processes. Consider the d-dimensional Itô process

$$X_s \;=\; \xi + \int_t^s F_r\, dr + \int_t^s G_r\, dB_r, \qquad \forall\, s \in [t,T].$$

Then, it holds that

$$\begin{aligned} u(s, \mathbb{P}_{X_s}) \;&=\; u(t, \mathbb{P}_\xi) + \int_t^s \partial_t u(r, \mathbb{P}_{X_r})\, dr \\ &+ \int_t^s \mathbb{E}\Big[\big\langle F_r, \partial_\mu u(r, \mathbb{P}_{X_r})(X_r) \big\rangle\Big] dr \\ &+ \frac{1}{2} \int_t^s \mathbb{E}\Big[\mathrm{tr}\Big(G_r G_r^\intercal \partial_x \partial_\mu u(r, \mathbb{P}_{X_r})(X_r)\Big)\Big] dr, \end{aligned}$$

HJB equation in the Wasserstein space

HJB equation in the Wasserstein space:

$$\begin{cases} -\partial_t u(t,\mu) \\ +F\big(t,\mu,\partial_\mu u(t,\mu)(\cdot),\partial_x\partial_\mu u(t,\mu)(\cdot)\big)=0, & (t,\mu)\in[0,T]\times\mathscr{P}_2(H), \\ u(T,\mu)=\displaystyle\int_H g(x,\mu)\,\mu(dx), & \mu\in\mathscr{P}_2(H), \end{cases}$$

where

$$\begin{aligned} F(t,\mu,p(\cdot),M(\cdot)) \; &= \; -\int_H \sup_{a\in A}\Big\{ f(t,x,\mu,a)+\langle b(t,x,\mu,a),p(x)\rangle \\ &\qquad + \frac{1}{2}\mathrm{tr}\big[(\sigma\sigma^{\intercal})(t,x,\mu,a)M(x)\big]\Big\}\mu(dx), \end{aligned}$$

for all $(t,\mu,p(\cdot),M(\cdot))\in[0,T]\times\mathscr{P}_2(H)\times L^2(H,\mathscr{B}(H),\mu;H)\times$
$\times L^2(H,\mathscr{B}(H),\mu;\mathscr{L}(H))$.

Viscosity solutions

Let

$$u\colon [0,T]\times\mathscr{P}_2(H) \longrightarrow \mathbb{R}$$

be upper semicontinuous.

Viscosity subsolution

- $u(T,\mu) \leq \int_H g(x,\mu)\,\mu(dx)$, for all $\mu \in \mathscr{P}_2(H)$;
- for any $(t,\mu) \in [0,T)\times\mathscr{P}_2(H)$ and $\varphi \in C_2^{1,2}([0,T]\times\mathscr{P}_2(H))$, satisfying

$$(u-\varphi)(t,\mu) = \sup_{(t',\mu')\in[0,T]\times\mathscr{P}_2(H)} (u-\varphi)(t',\mu'),$$

with $(u-\varphi)(t,\mu)=0$, we have

$$-\partial_t\varphi(t,\mu) + F\big(t,\mu,\partial_\mu\varphi(t,\mu)(\cdot),\partial_x\partial_\mu\varphi(t,\mu)(\cdot)\big) \leq 0.$$

Viscosity solutions

Let

$$u\colon [0,T]\times\mathscr{P}_2(H) \longrightarrow \mathbb{R}$$

be lower semicontinuous.

Viscosity supersolution

- $u(T,\mu) \geq \int_H g(x,\mu)\,\mu(dx)$, for all $\mu\in\mathscr{P}_2(H)$;
- for any $(t,\mu)\in[0,T)\times\mathscr{P}_2(H)$ and $\varphi\in C_2^{1,2}([0,T]\times\mathscr{P}_2(H))$, satisfying

$$(u-\varphi)(t,\mu) = \inf_{(t',\mu')\in[0,T]\times\mathscr{P}_2(H)}(u-\varphi)(t',\mu'),$$

with $(u-\varphi)(t,\mu)=0$, we have

$$-\partial_t\varphi(t,\mu)+F\big(t,\mu,\partial_\mu\varphi(t,\mu)(\cdot),\partial_x\partial_\mu\varphi(t,\mu)(\cdot)\big) \geq 0.$$

Viscosity solutions

Let

$$u\colon [0,T]\times\mathscr{P}_2(H) \longrightarrow \mathbb{R}$$

be continuous.

Viscosity solution

- u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Theorem

Suppose that Assumption (A) holds. The value function v is a viscosity solution to equation (HJB).

Proof. DPP + Itô's formula with some technical issues.

Literature

- The theory of second-order Hamilton-Jacobi-Bellman equations in the Wasserstein space is an emerging research topic, whose rigorous investigation is still at an early stage.

- C. Wu, J. Zhang (2020) adopt a different notion of viscosity solution (inspired by the definition of viscosity solution for PPDEs), where the maximum/minimum condition is formulated on compact subsets of the Wasserstein space.

- M. Burzoni, V. Ignazio, A. M. Reppen, H. M. Soner (2020) study a special class of integro-differential Hamilton-Jacobi-Bellman equations of specific type. In particular, b, σ, f, g do not depend on x, moreover the control processes are assumed to be deterministic functions of time.

Comparison theorem and uniqueness

Theorem (Comparison)

Suppose that Assumption (A) holds.
Let $u_1, u_2 \colon [0,T] \times \mathscr{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be continuous and bounded functions.
Suppose that u_1 (resp. u_2) is a viscosity subsolution (resp. supersolution) of equation (HJB).
Then

$$u_1 \;\le\; u_2, \qquad \text{on } [0,T] \times \mathscr{P}_2(\mathbb{R}^d).$$

Corollary (Uniqueness)

Suppose that Assumption (A) holds.
Then, v is the unique bounded and continuous viscosity solution of equation (HJB).

Proof of the comparison theorem

The proof is based on the existence of a candidate solution and consists in showing that

$$u_1 \le v \qquad \text{and} \qquad v \le u_2,$$

with v being the value function.

Proof of $u_1 \le v$

- We proceed by contradiction and assume that there exists $(t_0, \mu_0) \in [0, T) \times \mathscr{P}_2(\mathbb{R}^d)$ such that

$$(u_1 - v)(t_0, \mu_0) > 0. \tag{1}$$

- *Suppose for a moment that* $v \in C_2^{1,2}([0, T] \times \mathscr{P}_2(\mathbb{R}^d))$.
- *Suppose also that there exists* $(\bar{t}, \bar{\mu}) \in [0, T] \times \mathscr{P}_2(\mathbb{R}^d)$ *such that*

$$(u_1 - v)(\bar{t}, \bar{\mu}) = \sup(u_1 - v).$$

- Then, we get a contradiction to (1) using the viscosity subsolution property of u_1 at $(\bar{t}, \bar{\mu})$ with v as test function.

Smoothing of v

Approximation by non-degenerate control problems

- Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ supports an independent d-dimensional Brownian motion W.
- Let $\widehat{\mathscr{A}}$ be the set of processes $\alpha\colon [0,T]\times\Omega \to A$ progressively measurable with respect to the filtration generated by $(B_t)_{t\geq 0}$, $(W_t)_{t\geq 0}$, U.
- Controlled state process: for every $\varepsilon \geq 0$

$$X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}_{X_r}, \alpha_r)\,dr + \int_t^s \sigma(r, X_r, \mathbb{P}_{X_r}, \alpha_r)\,dB_r + \varepsilon(W_s - W_t)$$

- Reward functional:

$$J_\varepsilon(t,\xi,\alpha) = \mathbb{E}\left[\int_t^T f\left(s, X_s^{\varepsilon,t,\xi,\alpha}, \mathbb{P}_{X_s^{\varepsilon,t,\xi,\alpha}}, \alpha_s\right)ds + g\left(X_T^{\varepsilon,t,\xi,\alpha}, \mathbb{P}_{X_T^{\varepsilon,t,\xi,\alpha}}\right)\right]$$

- Value function: $v_\varepsilon(t,\mu) = \sup_{\alpha\in\widehat{\mathscr{A}}} J_\varepsilon(t,\xi,\alpha)$.

Approximation by non-degenerate control problems

The case $\varepsilon = 0$

Notice that it is not a priori clear if

$$v_0 \;\equiv\; v.$$

However, under Assumption (A), both v_0 and v solve the same HJB equation. Therefore, the equality $v_0 \equiv v$ follows from the *comparison theorem*.

Approximation

It holds that

$$|v_\varepsilon(t,\mu) - v_0(t,\mu)| \;\leq\; C_{K,T}\,\varepsilon$$

for some constant $C_{K,T}$, depending only on K and T.

Smoothing of v

Cooperative n-player game

- For every $n \in \mathbb{N}$, suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ supports independent Brownian motions

$$B^1, \ldots, B^n, W^1, \ldots, W^n$$

- Let $\widehat{\mathscr{A}}^n$ be the set of processes $\alpha\colon [0,T] \times \Omega \to A$ progressively measurable with respect to the filtration generated by $(B^1_t)_t, \ldots, (B^n_t)_t$, $(W^1_t)_t, \ldots, (W^n_t)_t$, U.

System of controlled state processes: for every $\varepsilon \geq 0$, $n \in \mathbb{N}$

$$X^i_s = \xi^i + \int_t^s b(r, X^i_r, \widehat{\mu}^n_r, \alpha^i_r) dr + \int_t^s \sigma(r, X^i_r, \widehat{\mu}^n_r, \alpha^i_r) dB^i_r + \varepsilon (W^i_s - W^i_t),$$

for $i = 1, \ldots, n$, with $\widehat{\mu}^n_r = \frac{1}{n} \sum_{j=1}^n \delta_{X^j_r}$.

Cooperative n-player game

- Reward functional:

$$J_{\varepsilon,n}(t,\overline{\xi},\overline{\alpha}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\bigg[\int_t^T f\big(s,X_s^{i,\varepsilon,t,\overline{\xi},\overline{\alpha}},\widehat{\mu}_s^{n,\varepsilon,t,\overline{\xi},\overline{\alpha}},\alpha_s^i\big)\,ds + g\big(X_T^{i,\varepsilon,t,\overline{\xi},\overline{\alpha}},\widehat{\mu}_T^{n,\varepsilon,t,\overline{\xi},\overline{\alpha}}\big)\bigg],$$

with $\overline{\xi}=(\xi^1,\dots,\xi^n)$ and $\overline{\alpha}=(\alpha^1,\dots,\alpha^n)$.

- Value function:

$$\widetilde{v}_{\varepsilon,n}(t,\overline{\mu}) = \sup_{\overline{\alpha}\in\widehat{\mathscr{A}}^n} J_{\varepsilon,n}(t,\overline{\xi},\overline{\alpha}),$$

for every $(t,\overline{\mu})\in[0,T]\times\mathscr{P}_2(\mathbb{R}^{dn})$, with $\overline{\xi}:\Omega\to\mathbb{R}^{dn}$ such that $\mathbb{P}_{\overline{\xi}}=\overline{\mu}$.

- Function $v_{\varepsilon,n}$:

$$v_{\varepsilon,n}(t,\mu) = \widetilde{v}_{\varepsilon,n}(t,\mu\otimes\cdots\otimes\mu).$$

Propagation of chaos result

Theorem

Suppose that Assumption (A) holds. Let $\varepsilon > 0$ and $(t,\mu) \in \mathscr{P}_2(\mathbb{R}^d)$. If there exists $q > 2$ such that $\mu \in \mathscr{P}_q(\mathbb{R}^d)$, then

$$\lim_{n\to+\infty} v_{\varepsilon,n}(t,\mu) \;=\; v_\varepsilon(t,\mu).$$

Proof. See Theorem 3.3 in Djete (2021). □

Smooth finite-dimensional approximations

Theorem

Suppose that Assumption (A) holds. For every $\varepsilon > 0$, $n \in \mathbb{N}$:

- $v_{\varepsilon,n} \in C_2^{1,2}([0,T] \times \mathscr{P}_2(\mathbb{R}^d))$ *and there exists* $\overline{v}_{\varepsilon,n} \in C^{1,2}([0,T] \times \mathbb{R}^{dn})$ *such that*

$$v_{\varepsilon,n}(t,\mu) \;=\; \int_{\mathbb{R}^{dn}} \overline{v}_{\varepsilon,n}(t,x_1,\ldots,x_n)\,\mu(dx_1)\cdots\mu(dx_n).$$

- $v_{\varepsilon,n}$ *solves the following equation on* $[0,T] \times \mathscr{P}_2(\mathbb{R}^d)$*:*

$$\begin{cases} \partial_t v_{\varepsilon,n}(t,\mu) + \int_{\mathbb{R}^{dn}} \sum_{i=1}^n \sup_{a_i \in A} \Big\{ \langle b(t,x_i,\widehat{\mu}^{n,\overline{x}},a_i), \partial_{x_i}\overline{v}_{\varepsilon,n}(t,\overline{x})\rangle \\ + \frac{1}{2}\mathrm{tr}\big[\big((\sigma\sigma^\intercal)(t,x_i,\widehat{\mu}^{n,\overline{x}},a_i) + \varepsilon^2\big)\partial^2_{x_i x_i}\overline{v}_{\varepsilon,n}(t,\overline{x})\big] \\ + \frac{1}{n} f(t,x_i,\widehat{\mu}^{n,\overline{x}},a_i)\Big\}\mu(dx_1)\otimes\cdots\otimes\mu(dx_n) = 0, \\ v_{\varepsilon,n}(T,\mu) = \frac{1}{n}\sum_{i=1}^n \int_{\mathbb{R}^{dn}} g(x_i,\widehat{\mu}^{n,\overline{x}})\,\mu(dx_1)\otimes\cdots\otimes\mu(dx_n), \end{cases}$$

Smooth variational principle

Replace

- *There exists* $(\bar{t}, \bar{\mu}) \in [0, T] \times \mathscr{P}_2(\mathbb{R}^d)$ *such that*

$$(u_1 - v)(\bar{t}, \bar{\mu}) \;=\; \sup(u_1 - v).$$

with

- *There exists* $(\bar{t}, \bar{\mu}) \in [0, T] \times \mathscr{P}_2(\mathbb{R}^d)$ *such that*

$$(u_1 - (v + \delta\varphi))(\bar{t}, \bar{\mu}) \;=\; \sup(u_1 - (v + \delta\varphi)),$$

for some ***smooth*** *perturbation* φ.

Smooth variational principle

Borwein-Preiss generalization of Ekeland's principle works with a ***gauge-type function***, namely a map

$$\Psi\colon ([0,T]\times\mathscr{P}_2(\mathbb{R}^d))^2 \to [0,+\infty)$$

satisfying:

- Ψ is continuous on $([0,T]\times\mathscr{P}_2(\mathbb{R}^d))^2$.
- $\Psi((t,\mu),(t,\mu))=0$, for every $(t,\mu)\in[0,T]\times\mathscr{P}_2(\mathbb{R}^d)$.
- For every $\varepsilon>0$ there exists $\eta>0$ such that, for all $(t',\mu'),(t'',\mu'')\in[0,T]\times\mathscr{P}_2(\mathbb{R}^d)$, the inequality $\Psi((t',\mu'),(t'',\mu''))\le\eta$ implies $|t'-t''|+\mathscr{W}_2(\mu',\mu'')<\varepsilon$.

An example of gauge-type function is the distance itself:

$$((t,\mu),(s,\nu)) \longrightarrow |t-s|+\mathscr{W}_2(\mu,\nu),$$

which is however not smooth enough.

Smooth variational principle

Borwein-Preiss generalization of Ekeland's principle works with a ***gauge-type function***, namely a map

$$\Psi : ([0,T] \times \mathscr{P}_2(\mathbb{R}^d))^2 \to [0,+\infty)$$

satisfying:

- Ψ is continuous on $([0,T] \times \mathscr{P}_2(\mathbb{R}^d))^2$.
- $\Psi((t,\mu),(t,\mu)) = 0$, for every $(t,\mu) \in [0,T] \times \mathscr{P}_2(\mathbb{R}^d)$.
- For every $\varepsilon > 0$ there exists $\eta > 0$ such that, for all $(t',\mu'),(t'',\mu'') \in [0,T] \times \mathscr{P}_2(\mathbb{R}^d)$, the inequality $\Psi((t',\mu'),(t'',\mu'')) \leq \eta$ implies $|t'-t''| + \mathcal{W}_2(\mu',\mu'') < \varepsilon$.

Even if we take the square

$$((t,\mu),(s,\nu)) \longrightarrow |t-s|^2 + \mathcal{W}_2(\mu,\nu)^2$$

it is still not smooth enough.

Sharp upper bound: the case $d = 1$

In the case $d = 1$, ad hoc gauge-type functions may be constructed in easier ways, as for instance relying on the following sharp upper bound:

$$\mathcal{W}_2(\mu,\nu)^2 \ \leq \ 4\int_{-\infty}^{+\infty} |x|\left|F_\mu(x) - F_\nu(x)\right| dx, \qquad \forall\, \mu,\nu \in \mathscr{P}_2(\mathbb{R}),$$

where F_μ and F_ν are the CDF's of μ and ν, respectively.

Notice that

$$\big((t,\mu),(s,\nu)\big) \ \longrightarrow \ |t-s|^2 + \int_{-\infty}^{+\infty} |x|\left|F_\mu(x) - F_\nu(x)\right| dx$$

is a gauge-type function.

Sharp upper bound: the general case

Lemma

For every integer $\ell \geq 0$, let $\mathscr{P}_\ell$ denote the partition of $(-1,1]^d$ into $2^{d\ell}$ translations of $(-2^{-\ell},2^{-\ell}]^d$. Moreover, let $B_0 := (-1,1]^d$ and, for every integer $n \geq 1$, $B_n := (-2^n,2^n]^d \setminus (-2^{n-1},2^{n-1}]^d$. Then, for every $\mu,\nu \in \mathscr{P}_2(\mathbb{R}^d)$, the following inequality holds:

$$\begin{aligned}&\big(W_2(\mu,\nu)\big)^2\\ &\leq c_d \sum_{n\geq 0} 2^{2n} \sum_{\ell\geq 0} 2^{-2\ell} \sum_{B\in\mathscr{P}_\ell} \big|\mu\big((2^n B)\cap B_n\big) - \nu\big((2^n B)\cap B_n\big)\big|,\end{aligned}$$

where $2^n B := \{2^n x \in \mathbb{R}^d : x \in B\}$ and $c_d > 0$ is a constant depending only on d.

Proof. See Fournier and Guillin (2015) or the book (first volume) by Carmona and Delarue. □

Smooth gauge-type function

Notice that

$$
\begin{aligned}
((t,\mu),(s,\nu)) \longrightarrow\ & |t-s|^2 \\
& + c_d \sum_{n\geq 0} 2^{2n} \sum_{\ell\geq 0} 2^{-2\ell} \sum_{B\in\mathscr{P}_\ell} \Big|\mu\big((2^n B)\cap B_n\big) - \nu\big((2^n B)\cap B_n\big)\Big|
\end{aligned}
$$

is a gauge-type function, which is however not smooth. We smooth it proceeding as follows.

- We replace $|\mu((2^n B)\cap B_n) - \nu((2^n B)\cap B_n)|$ by
$$
\Big(|\mu((2^n B)\cap B_n) - \nu((2^n B)\cap B_n)|^2 + \delta_{n,\ell}^2\Big)^{1/2} - \delta_{n,\ell}
$$
with $\delta_{n,\ell} = 2^{-(4n+2d\ell)}$.
- We replace $\mu((2^n B)\cap B_n)$ and $\nu((2^n B)\cap B_n)$ respectively by
$$
(\mu * \mathcal{N}_1)((2^n B)\cap B_n) \qquad \text{and} \qquad (\nu * \mathcal{N}_1)((2^n B)\cap B_n).
$$

Smooth gauge-type function

Theorem

Define the map $\rho_2\colon ([0,T]\times\mathscr{P}_2(\mathbb{R}^d))^2\to\mathbb{R}$ *as*

$$\begin{aligned}\rho_2\big((t,\mu),(s,\nu)\big) &= |t-s|^2+\\ &\quad + c_d\sum_{n\geq 0}2^{2n}\sum_{\ell\geq 0}2^{-2\ell}\sum_{B\in\mathscr{P}_\ell}\Big\{\Big(\big|(\mu*\mathscr{N}_1)\big((2^nB)\cap B_n\big)\\ &\quad -(\nu*\mathscr{N}_1)\big((2^nB)\cap B_n\big)\big|^2+\delta_{n,\ell}^2\Big)^{1/2}-\delta_{n,\ell}\Big\},\end{aligned}$$

with $\delta_{n,\ell}:=2^{-(4n+2d\ell)}$*. Then, the following holds.*

- ρ_2 *is a gauge-type function.*
- *for every fixed* $(t_0,\mu_0)\in[0,T]\times\mathscr{P}_2(\mathbb{R}^d)$*, the map* $(t,\mu)\mapsto\rho_2\big((t,\mu),(t_0,\mu_0)\big)$ *is in* $C_2^{1,2}([0,T]\times\mathscr{P}_2(\mathbb{R}^d))$*.*

Outline

1. Motivation and examples

2. Our results on Mean Field Control
   - Our Mean Field Control setting and the law invariance
   - Differential calculus on Wasserstein space and HJB equation
   - Comparison theorem and uniqueness

3. Ongoing results on Mean Field Games

A basic MFG system

H is a separable Hilbert space. The state variable in H is Z. The state equation for the players in H is linear with additive noise:

$$dZ(t) = [AZ(t) + B\alpha(t)]dt + GdW(t)$$

with A, B, G suitable operators (A, B possibly unbounded). We call $\mathscr{L}$ the generator of the above SDE when $\alpha \equiv 0$:

$$[\mathscr{L}\phi](z) := \frac{1}{2} TrG^* G\partial_{zz} V + \langle Az, \partial_z V\rangle_H$$

This an infinite dimensional Ornstein-Uhlenbeck operator, studied e.g. in various papers of Da-Prato, Lunardi, Zabczyk, etc.
We call $P_t^{\mathscr{L}} = e^{t\mathscr{L}}$ the semigroup generated by this operator, which is the transition semigroup associated to Z when $\alpha \equiv 0$.

The agents maximize

$$J(\alpha) = \mathbb{E}\left[\int_0^T f_0(Z(t),\alpha(t),m(t))dt + g_0(Z(T),m(T))\right]$$

whose supremum over the set $\mathscr{A}$ of all admissible strategies is the value function V.

The HJB equation is

$$-\partial_t V(t,z) - \mathscr{L} V(t,z) - H_{MAX}(z,\partial_z V(t,z),m(t)) = 0,$$

where

$$H_{MAX}(z,p,m) = \sup_{\alpha\in O}\left\{\langle B\alpha,p\rangle_H + f_0(z,\alpha,m)\right\}$$

with terminal condition $V(T,z) = g_0(z,m(T))$.

The FPK equation is

$$\partial_t m - \mathscr{L}^* m - div\left(D_p H_{MAX}(z,\partial_z V,m)m\right) = 0; \qquad m(0) = m_0$$

where m_0 is the initial distribution of the state Z.

First result and method

We obtain a first existence result for the above MFG (writing is ongoing) system under the following assumption

- f_0, g_0 Lipschitz continuous and bounded
- A generates a C_0-semigroup and B is a bounded operator
- A, G satisfy the so-called null controllability condition which guarantees a strong smoothing property of the semigroup $P_t^{\mathscr{L}}$.

Method:

- Write the HJB equation in integral (mild) form using the semigroup $P_t^{\mathscr{L}}$.
- Find existence and uniqueness of HJB in such form for any given $m(\cdot)$ in a compact set of measures
- Show existence of the fixed point using suitable compactness properties in the space of measures.

Difficulties and further work

Problems:

- Compactness in the space of measures is more involved.
- Missing regularity theorems for FPK equations in infinite dimension.

Further work

- Extend monotonicity arguments of Lasry-Lions to study uniqueness.
- Linear quadratic case using the current theory of Riccati equations in inifinite dimension (see e.g. [Da Prato-Ichikawa 86] or [Tessitore, 90])
- Study the Master Equation in such cases
- Extend our results to the case of delay state equation like the ones of [Fouque-Zhang 18], using the tools introduced in [G.-Masiero '17 and '22]

Thank you!