American options
in a non-linear incomplete market model

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Based on joint works
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Features of the market

- The risky asset is subject to default (or an exogenous credit event).
- The market is incomplete : not every European contingent claim is replicable.
- The market is non-linear : the dynamics of the wealth process have a non-linear (possibly non-convex) driver.
  The non-linearity of the driver can encode a number of market imperfections : different lending and borrowing interest rates, repo rates, impact of a large investor ...

Goal

Study the superhedging of American options in such non-linear incomplete market models.

The case where the pay-off process of the American option is ***not*** *necessarily* ***right-continuous*** *is beyond this talk.*

We will present :

- a duality result for the seller of the option in terms of a non-linear stochastic problem of control and stopping
- infinitesimal characterization in terms of a non-linear constrained reflected BSDE under the initial probability P.

The proofs are not based on convex duality.

A motivating example : Non-linear incomplete market with default

- Let $T > 0$ be a fixed terminal horizon.
- Let $(\Omega, \mathcal{G}, P)$ be a complete probability space.
- Let W be a one-dimensional Brownian motion.
- ϑ is a random variable which models a default time (or the time of a credit event).
- Let N be the process defined by $N_t := \mathbf{1}_{\vartheta \leq t}$ for all $t \in [0, T]$
- Let $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ be the (augmented) filtration generated by W and N.
- Let M be the compensated martingale defined by

$$M_t := N_t - \int_0^t \lambda_s ds, \quad t \geq 0,$$

where $\lambda_s \geq 0$ is the intensity process.

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- We assume that W is a $\mathbb{G}$-Brownian motion (immersion property).

We consider a market with :

- a risky asset $S = (S_t)_{0 \leq t \leq T}$ with (exogenously) modelled price process

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dM_t) \text{ with } S_0 > 0.$$

  The processes σ, μ, and β are predictable bounded with $\sigma_t > 0$ and $\beta_\vartheta > -1$.

- At least one riskless asset.

- An investor, endowed with an initial wealth $x \in \mathbb{R}$.
- At each time t, the investor chooses the amount φ_t of wealth invested in the risky asset (where $\varphi \in \mathbb{H}^2$).
- The wealth process $V_t^{x,\varphi}$ (or simply V_t) satisfies the following forward dynamics :

$$-dV_t = f(t, V_t, \varphi_t \sigma_t)dt - \varphi_t \sigma_t dW_t - \varphi_t \beta_t dM_t; \quad V_0 = x;$$

where f is a nonlinear (possibly non-convex) Lipschitz driver.

Example of non-linear incomplete market model

Black and Scholes-type market model
with a (possible) default on the risky asset (incompleteness)
and different rates for borrowing R_t and lending r_t (non-linearity)

- risk-free assets (B_t^1) and (B_t^2) with

$$dB_t^1 = B_t^1 r_t dt \qquad dB_t^2 = B_t^2 R_t dt$$

- a risky asset (S_t) with

$$dS_t = S_t \mu_t dt + S_t \sigma_t dW_t + S_{t-} \beta_t dM_t$$

- And the constraints : the riskless asset (B_t^1) can only be bought, and the riskless asset (B_t^2) can only be sold.

The self-financing condition :

$$-dV_t = -\Big((V_t - \varphi_t)^+ r_t - (V_t - \varphi_t)^- R_t + \mu_t \varphi_t\Big) dt - \varphi_t \sigma_t dW_t - \varphi_t \beta_t dM_t$$

We see that the driver is non-linear.

Examples (continued) :

- **linear** driver

$$f(t, V_t, \varphi_t \sigma_t) = -r_t V_t - (\mu_t - r_t)\varphi_t = -r_t V_t - \theta_t \varphi_t \sigma_t,$$

where r_t is the risk-free interest rate, and $\theta_t = (\mu_t - r_t)\sigma_t^{-1}$.

Examples (continued) :

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  where r_t is the risk-free interest rate, and $\theta_t = (\mu_t - r_t)\sigma_t^{-1}$.
- **different borrowing and lending interest rates** R_t and r_t with $R_t \geq r_t$:
  $f(t, V_t, \varphi_t\sigma_t) = -r_t V_t - \varphi_t(\mu_t - r_t) + (R_t - r_t)(V_t - \varphi_t)^-$.
- **taxes on the profits**
- **literature on counterparty risk**
  $f(t, V_t, \varphi_t\sigma_t) = -r_t(V_t - \varphi_t)^+ + R_t(V_t - \varphi_t)^- - l_t\varphi_t^- + b_t\varphi_t^+ - \varphi_t\mu_t$
- **the effect of a large seller on the default intensity** ; ...

Whatever the form of the driver f, this non-linear market is incomplete.

Let $\eta \in L^2(G_T)$ be the terminal pay-off of a European option.
It might not be possible to find (x, φ) in $\mathbb{R} \times \mathbb{H}^2$ such that

- $V^{x,\varphi}$ satisfies the self-financing condition and
- $V_T^{x,\varphi} = \eta$ (terminal condition)

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- $V_T^{x,\varphi} = \eta$ (terminal condition)

Indeed, the pricing Backward SDE

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \sigma_t^{-1} \beta_t dM_t; \quad V_T = \eta,$$

might not be well-defined.
(Here, we have set as usual $Z_t := \varphi_t \sigma_t$.)

Models of financial markets : Vocabulary

linear complete	linear incomplete
non-linear complete	non-linear incomplete

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Superhedging price of American options

Let $\mathcal{T}$ be the set of all stopping times (a.s. in $[0, T]$).

Let $\xi \in S^2$ be a given RCLL adapted pay-off process.

Seller's superhedging price at time 0

$$u_0 := \inf\{x \in \mathbb{R} : \exists \varphi \in \mathbb{H}^2 \text{ with } V_\tau^{x,\varphi} \geq \xi_\tau, \forall \tau \in \mathcal{T}\}.$$

Dual characterization of the seller's price

Under a suitable integrability assumption on the pay-off process (ξ_t) and a suitable assumption on f ensuring that $\mathscr{E}^f(\cdot)$ is monotone, we have

Theorem (Pricing-hedging duality for the seller)

The superhedging price u_0 for the seller of the American option satisfies

$$u_0 = \sup_{Q \in \mathscr{Q}} \sup_{\tau \in \mathcal{T}} \mathscr{E}^f_{Q,0,\tau}(\xi_\tau).$$

Main objects of the duality formula

Linear case	Non-linear case (f non-linear)
Linear operator $E_R(\cdot)$	Non-linear operator $\mathscr{E}^f_Q$
Martingale (with respect to $E_R(\cdot)$)	Martingale (with respect to $\mathscr{E}^f_Q$)
Set of martingale measures $\mathscr{R}$	Set of measures $\mathscr{Q}$

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Non-linear f-evaluation under Q

Let Q be a probability measure equivalent to P.

By the predictable representation property [1], its density process (ζ_t) satisfies

$$d\zeta_t = \zeta_{t^-}(\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1,$$

where (α_t) and (ν_t) are predictable processes with $\nu_{\vartheta \wedge T} > -1$ a.s.

By Girsanov's theorem,

- $W_t^Q := W_t - \int_0^t \alpha_s ds$ is a Brownian motion under Q, and
- $M_t^Q := M_t - \int_0^t \nu_s \lambda_s ds$ is a martingale under Q.

1. cf. Kusuoka (1999), Jeanblanc and Song (2015)

Let Q be a probability measure equivalent to P.

- Let $\eta \in L^2_Q(\mathcal{G}_T)$
- Consider the pricing BSDE (under Q)

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t^Q - K_t dM_t^Q; \quad X_T = \eta.$$

2. in $\mathcal{S}^2_Q \times \mathbb{H}^2_Q \times \mathbb{H}^2_{Q,\lambda}$

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- Let (X, Z, K) be the unique solution [2] of the BSDE (under Q).

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f-evaluation under Q

For $t \in [0, T]$, we call f-evaluation under Q (at time t), denoted by $\mathcal{E}^f_{Q,t,T}$, the operator defined by :

$$\mathcal{E}^f_{Q,t,T}(\eta) := X_t.$$

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Remark : Note that Shige Peng's g-expectation is a particular case (applications in risk measures).

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Martingale (with respect to $E_R(\cdot)$)	Martingale (with respect to $\mathscr{E}_Q^f$)
Set of martingale measures $\mathscr{R}$	Set of measures $\mathscr{Q}$

Definition (Martingale with respect to $\mathscr{E}^f_Q$)

Let $(Y_t) \in \mathbb{S}^2_Q$.
The process (Y_t) is called a (strong) $\mathscr{E}^f_Q$-martingale, denoted also (f, Q)-martingale, if

$$\mathscr{E}^f_{Q,\sigma,\tau}(Y_\tau) = Y_\sigma \text{ a.s. for all stopping times } \sigma, \tau \text{ such that } \sigma \leq \tau.$$

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The notion of strong $\mathscr{E}^f_Q$-supermartingale is defined in a similar manner.

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Definition (The set $\mathscr{Q}$)

An equivalent probability measure Q is in $\mathscr{Q}$ if :
for all $x \in \mathbb{R}$, for all $\varphi \in \mathbb{H}^2 \cap \mathbb{H}^2_Q$, the wealth process $V^{x,\varphi}$ is a (strong) $\mathscr{E}^f_Q$-martingale.

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Remarks :

- The set $\mathscr{Q}$ does not depend on the driver f.
- There is a one-to-one correspondence between the set $\mathscr{Q}$ and the set $\mathscr{R}$.

Under a suitable integrability assumption on (ξ_t), and an additional assumption on f to ensure the monotonicity of $\mathscr{E}^f(\cdot)$, we have

Theorem (Pricing-hedging duality for the seller)

$$u_0 = \sup_{Q\in\mathscr{Q}} \sup_{\tau\in\mathcal{T}} \mathscr{E}^f_{Q,0,\tau}(\xi_\tau).$$

Intermediary results in the proof of the duality for the seller

The dual value problem

$\longrightarrow$ Passes through the study of the dual value problem.

Let $X(S)$ be the value at a given stopping time time S of the dual problem, that is,

$$X(S) := \underset{Q \in \mathscr{Q}}{\text{ess sup}}\ \underset{\tau \in \mathcal{T}_S}{\text{ess sup}}\ \mathscr{E}^f_{Q,S,\tau}(\xi_\tau) \text{ a.s.}$$

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This is a non-linear mixed control/stopping problem.

Main steps

1. Aggregation step
2. Minimality characterization
3. Non-linear optional decomposition
4. Pricing-hedging duality

Study the dual problem

$$X(S) := \underset{Q \in \mathcal{Q}}{ess \sup}\ \underset{\tau \in \mathcal{T}_S}{ess \sup}\ \mathcal{E}^f_{Q,S,\tau}(\xi_\tau) = X_S \quad \text{a.s.}$$

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1. Aggregation step
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Theorem (Minimality characterization)

- The dual value process (X_t) is the smallest non-linear $\mathcal{E}^f_Q$-supermartingale for all $Q \in \mathcal{Q}$, dominating the pay-off process (ξ_t).

$$X(S) := ess \sup_{Q \in \mathcal{Q}} ess \sup_{\tau \in \mathcal{T}_S} \mathcal{E}^f_{Q,S,\tau}(\xi_\tau) = X_S \quad \text{a.s.}$$

1. Aggregation step
2. Minimality characterization
3. Non-linear optional decomposition
   $\rightarrow$ Structure of the processes which are non-linear supermartingales (simultaneously) in all the auxiliary models
4. Pricing-hedging duality

Theorem (Non-linear optional decomposition)

Let (χ_t) be an RCLL non-linear optional $\mathscr{E}^f_Q$-supermartingale for all $Q \in \mathscr{Q}$.

Then, there exist :

- a unique adapted process $Z \in \mathbb{H}^2$,
- a unique nondecreasing optional RCLL process h with $h_0 = 0$,

such that

$$-d\chi_t = \underbrace{f(t,\chi_t,Z_t)dt}_{\text{non-linear part}} - \underbrace{Z_t\sigma_t^{-1}(\sigma_t dW_t + \beta_t dM_t)}_{\text{martingale part}} + dh_t.$$

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Moreover, the converse statement holds.

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Literature : (linear case) *El Karoui and Quenez (1995) ; Kramkov (1996) ; Föllmer and Kabanov (1998)*.
(non-linear case) Bouchard, Possamaï, and Tan (2016) ; Possamaï,Tan, Zhou (2018) ; M.G., Quenez, Sulem (2020)

$$X(S) := ess \sup_{Q \in \mathcal{Q}} ess \sup_{\tau \in \mathcal{T}_S} \mathcal{E}^f_{Q,S,\tau}(\xi_\tau) = X_S \quad \text{a.s.}$$

1. Aggregation step
2. Minimality characterization
3. Non-linear optional decomposition
4. Pricing-hedging duality and a first infinitesimal characterization

1st infinitesimal characterization

Supersolution of a Reflected BSDE with optional non-decreasing process

A process $Y \in S^2$ is said to be a supersolution of the optional Reflected BSDE with driver f and obstacle (ξ_t) if

- there exists $Z \in \mathbb{H}^2$ and
- there exists a nondecreasing optional RCLL process h, with $h_0 = 0$ and $E[(h_T)^2] < \infty$

such that

$$-dY_t = f(t, Y_t, Z_t)dt - \sigma_s^{-1} Z_s(\sigma_s dW_s + \beta_s dM_s) + dh_t; \text{a.s.}$$
$$Y_T = \xi_T \;\text{ and }\; Y_t \geq \xi_t \text{ for all } t \text{ a.s.}$$

Theorem (1st infinitesimal characterization)

The dual value process (X_t) is the minimal supersolution of the Reflected BSDE with optional non-decreasing process.

2nd infinitesimal characterization

The 2nd infinitesimal characterization is due to the following result :

A process (χ_t) is an f, Q-supermartingale for all $Q \in \mathscr{Q}$

iff

(χ_t) admits a non-linear optional decomposition

iff

(χ_t) admits a non-linear predictable decomposition with constraints.

Reflected BSDE with constraints

- It is possible to characterize the seller's superhedging price as the minimal supersolution of a non-linear Reflected BSDE with constraints.
- This provides a connection to earlier literature on BSDE with jump constraints (Ma, Zhang, ...)

Supersolution of a constrained Reflected BSDE

Let $\xi \in S^2$. A process $Y \in S^2$ is said to be a *supersolution* of the *constrained reflected BSDE* with driver f and obstacle ξ if there exists a predictable process $(Z, K, A) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2$ such that

$$
\begin{aligned}
& -dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t + dA_t; \\
& Y_T = \xi_T \text{ a.s.} \quad \text{and} \, Y_t \geq \xi_t \text{ for all } t \in [0, T] \text{ a.s.}; \\
& A_\cdot + \int_0^\cdot (K_s - \beta_s \sigma_s^{-1} Z_s)\lambda_s ds \text{ is non-decreasing, and,} \\
& (K_t - \beta_t \sigma_t^{-1} Z_t)\lambda_t \leq 0, \; t \in [0, T], \; dP \otimes dt\text{-a.e.}
\end{aligned}
$$

Theorem (2nd infinitesimal characterization)

The seller's price u_0 is equal to the minimal supersolution Y_0 at time 0 of the constrained Reflected BSDE.

Duality for the buyer

Buyer's superhedging price at time 0

$$\tilde{u}_0 := \sup\{z \in \mathbb{R} : \ \exists(\tau, \varphi) \in \mathcal{T} \times \mathbb{H}^2 \text{ with } \xi_\tau \geq -V_\tau^{-z,\varphi}\}.$$

Buyer's superhedging price at time 0

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Theorem (Pricing-hedging duality for the buyer)

If (ξ_t) is l.u.s.c., the superhedging price $\tilde{u}_0$ for the buyer of the American option satisfies

$$\tilde{u}_0 = \inf_{Q \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} -\mathcal{E}^f_{Q,0,\tau}(-\xi_\tau) = \sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{Q}} -\mathcal{E}^f_{Q,0,\tau}(-\xi_\tau).$$

G.M., Quenez M.-C., and A. Sulem : European options in a non-linear incomplete market model with default, SIAM Journal on Financial Mathematics, 2020.

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