Volatility is (Mostly) Path-Dependent

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Outline

1. Why Path-Dependent Volatility (PDV)?
2. Is Volatility Path-Dependent? How much? How?
3. The continuous-time empirical Markovian PDV model

This talk is dedicated to the memory of Peter Carr

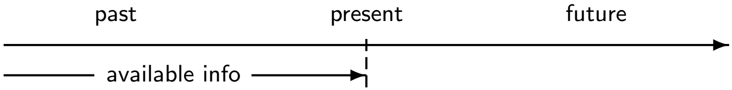
Path-Dependent Volatility

$$\frac{dS_t}{S_t} = \sigma(S_u, u \leq t)\, dW_t$$

- Zero rates, repos, dividends for simplicity
- Volatility drives the dynamics of the asset price S
- Feedback loop from prices to volatility
- Pure feedback model: volatility is an **endogenous** factor
- Main references:
  - Econometrics:
    The whole GARCH literature
  - Derivatives research (macro, pricing models, calibration):
    Hobson-Rogers '98, Guyon '14
  - Econophysics (micro, statistical models):
    Zumbach '09-10, Chicheportiche-Bouchaud '14, Blanc-Donier-Bouchaud '16
  - Recent models with a PDV component:
    Gatheral-Jusselin-Rosenbaum '20, Parent '21

Why Path-Dependent Volatility?

A philosophical argument



- The arrow of time
- Markovian assumption: the future depends on the past only through the present
- Often made just for simplicity and ease of computation, not a fundamental property
- Example: assume that the price of an option depends only on current time t and current asset price S_t: $P(t, S_t)$
- In fact, often, **the present does not capture all information from the past** $\longrightarrow P(t, (S_u, u \leq t))$

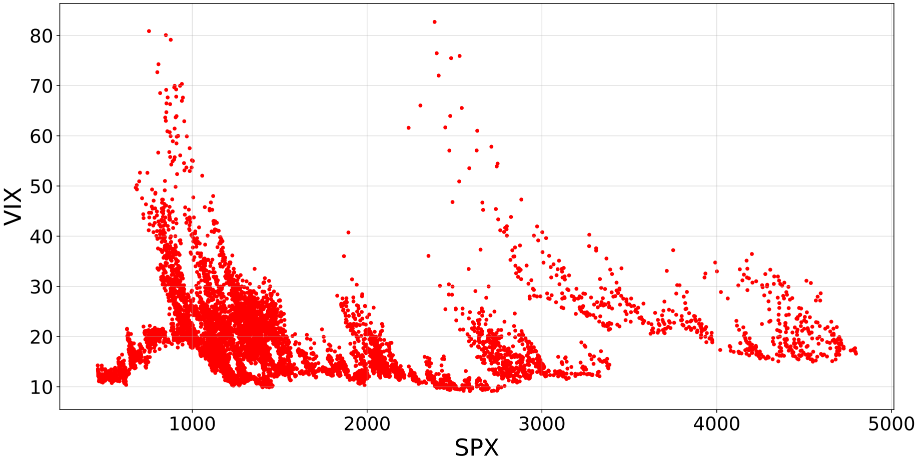
An intuitive argument: a simple quizz

	June 28, 2022		June 28, 2023
SPX	4,000		5,400
VIX			?

An intuitive argument: a simple quizz

	June 28, 2022	May 28, 2023	June 28, 2023
SPX	4,000	6,000	5,400
VIX			?

An intuitive argument: a simple quizz



A financial and scaling argument

- The two basic quantities that possess a natural scale are the **volatility levels** and the **asset returns**
- **A good model should relate these two quantities**: **Path-dependent volatility**
- **LV model** links the **volatility level** to the **asset level**. Does not make much financial sense: well chosen PDV models need not be recalibrated as often as the LV model.
- **SV models** connect the **volatility return** to the **asset price return**. Has limitations:
  - Only very high levels of vol of vol allow fast large movements of volatility
  - Typically a very large mean-reversion is postulated to keep volatility within its natural range.
- **PDV models**, by directly linking **past asset returns** to **volatility levels**, can capture fast large changes in vol more easily and naturally, while maintaining volatility in its natural range. They also provide an explanation for mean-reversion.

A financial and scaling argument

	volatility	depends on	asset
LV	level		level
SV	returns		returns
PDV	level		returns

Path-dependent volatility vs Stochastic volatility

$$\frac{dS_t}{S_t} = \sigma_t\, dW_t, \qquad \sigma_t = f(t, Y_t)$$

$$dY_t = \mu(t, Y_t)\, dt + \nu(t, Y_t) \left(\rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp}\right)$$

$$Y_t = Y_0 + \int_0^t \mu(u, Y_u)\, du + \int_0^t \nu(u, Y_u) \left(\rho \frac{1}{f(u, Y_u)} \frac{dS_u}{S_u} + \sqrt{1-\rho^2}\, dW_u^{\perp}\right)$$

- $\rho = 0$: **SV is strictly path-independent**
  - The asset price is a slave process with absolutely no feedback on volatility:

$$\sigma_t = \varphi(t, (dW_u^{\perp})_{0\le u\le t}) = \psi(t, (W_u^{\perp})_{0\le u\le t})$$

- $\rho \notin \{-1, 0, 1\}$: **SV is partially path-dependent**
  - Partial feedback from asset price to volatility through spot-vol correl(s):

$$\sigma_t = \varphi\left(t, \left(\frac{dS_u}{S_u}\right)_{0\le u\le t}, \left(dW_u^{\perp}\right)_{0\le u\le t}\right) = \psi\left(t, (S_u)_{0\le u\le t}, \left(W_u^{\perp}\right)_{0\le u\le t}\right)$$

- $\rho = \pm 1$: **SV is fully path-dependent**
  - Pure feedback but path-dependence φ, ψ is complicated, implicit:

$$\sigma_t = \varphi\left(t, \left(\frac{dS_u}{S_u}\right)_{0\le u\le t}\right) = \psi(t, (S_u)_{0\le u\le t})$$

Joint calibration of SV models to SPX and VIX smiles

The joint calibration of classical parametric SV models to SPX and VIX smiles leads to

- Very large vol of vol
- Very large mean-reversions (several time scales)
- **Correlations = $\pm 1 \Longrightarrow$ Path-dependent volatility**

See:

- *Inversion of Convex Ordering in the VIX Market* (G. '20)
- *The VIX Future in Bergomi Models: Fast Approximation Formulas and Joint Calibration with S&P 500 Skew* (G. '21)

Joint calibration of SV models to SPX and VIX smiles

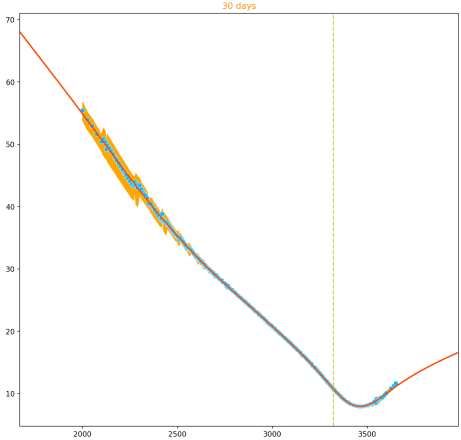


Figure: SPX smile as of January 22, 2020, $T = 30$ days

Joint calibration of SV models to SPX and VIX smiles

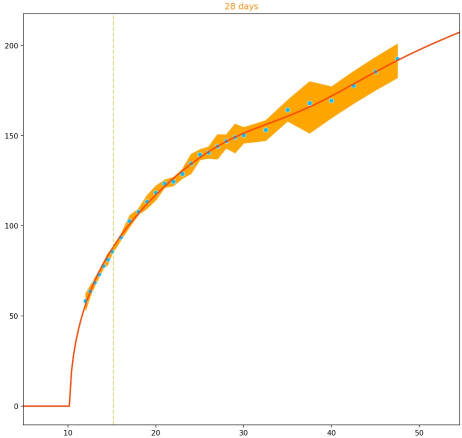


Figure: VIX smile as of January 22, 2020, $T = 28$ days

Joint calibration of SV models to SPX and VIX smiles

- ATM skew:

$$\text{Definition:} \quad \mathcal{S}_T = \left.\frac{d\sigma_{\text{BS}}(K,T)}{\frac{dK}{K}}\right|_{K=F_T}$$

$$\text{SPX, small } T\text{:} \quad \mathcal{S}_T \approx -1.5$$

$$\text{Classical one-factor SV model:} \quad \mathcal{S}_T \underset{T\to 0}{\longrightarrow} \frac{1}{2} \times \text{spot-vol correl} \times \text{vol of vol}$$

- Calibration to short-term ATM SPX skew $\Longrightarrow$

  vol of vol $\geq 3 = 300\% \gg$ short-term ATM VIX implied vol

- $\Longrightarrow$ Use
  - **very large vol of vol**
  - **very large mean-reversion(s)** (so that VIX implied vol $\ll$ vol of vol)
  - -1 **spot-vol correlation(s)**

An information-theoretical/financial economics argument

- Contrary to SV models, PDV models do not require adding extra sources of randomness to generate rich spot-vol dynamics: they explain volatility in a purely **endogenous** way.
- $\Longrightarrow$ Unlike SV models, PDV models are **complete models**: derivatives have a unique, unambiguous price, independent of any preferences or utility functions.
- **All the information exchanged by market participants is recorded in the underlying asset prices**, not just in current prices, but in the history of all past prices.
- Reality is a bit more complex, but we will show that it is actually quite close to this, so it makes sense to **start building a model by extracting all the information that past asset prices contain about volatility**.

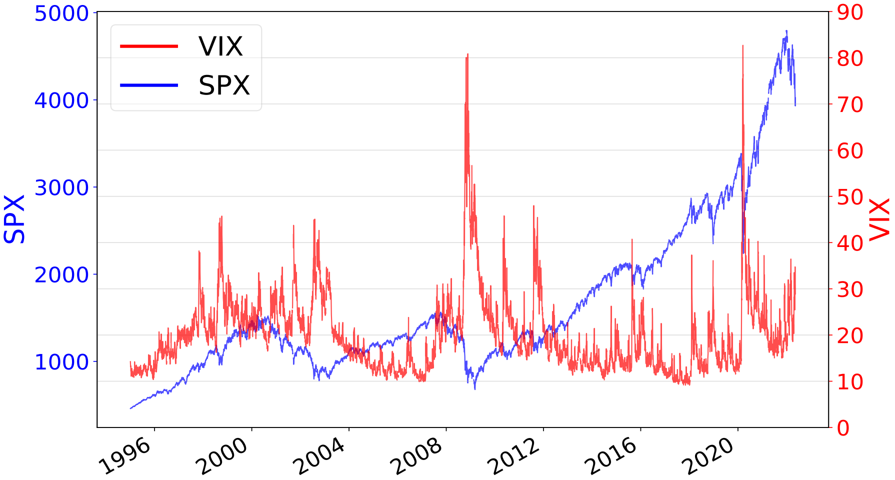
Path-dependent volatility is generic for option pricing

- **All SV models have an equivalent PDV model** in the sense that **all path-dependent options (not only vanilla options)** written on the underlying asset **have the same prices in both models**.
- Brunick and Shreve '13: Given a general Itô process $dS_t = \sigma_t S_t \, dW_t$, there exists a PDV model $d\hat{S}_t = \sigma(t, (\hat{S}_u)_{u \leq t})\hat{S}_t \, d\hat{W}_t$ such that the distributions of the **processes** $(S_t)_{t \geq 0}$ and $(\hat{S}_t)_{t \geq 0}$ are equal; the equivalent PDV is given by

$$\sigma(t, (S_u)_{u \leq t})^2 = \mathbb{E}[\sigma_t^2 | (S_u)_{u \leq t}].$$

- $\Longrightarrow$ The price process $(S_t)_{t \geq 0}$ produced by any SV or stochastic local volatility (SLV) model **can be exactly reproduced by a PDV model**.

Empirical evidence



Empirical evidence

- Much of the GARCH literature
- **Time reversal asymmetry in finance**: Zumbach-Lynch '01, Zumbach '09, Chicheportiche-Bouchaud '14...: "Financial time series are not statistically symmetrical when past and future are interchanged" (BDB '16)
- Leverage effect:
  - Past returns affect (negatively) future realized volatilities, but not the other way round" (BDB '16)
  - $t \to -t$ and $r \to -r$ asymmetry
- ZL '01: time reversal asymmetry even in absence of leverage effect:
  - Weak Zumbach effect: "Past large-scale realized volatilities are more correlated with future small-scale realized volatilities than vice versa" (BDB '16). Most easily captured by PDV models.
  - $t \to -t$ asymmetry, but $r \to -r$ symmetry
- Strong Zumbach effect: "Conditional dynamics of volatility with respect to the past depend not only on past volatility trajectory but also on the historical price path" (GJR '20) $\iff$ **There is some price-path-dependency in the volatility dynamics**

Empirical evidence

Our Machine Learning approach confirms those findings and moreover answer two crucial questions:

1. **How exactly does volatility depend on past price returns (price trends and past squared returns)?**
2. **How much of volatility is path-dependent, i.e., purely endogenous?**

That is, explain volatility as an endogenous factor as best as we can, empirically.

Objectives

(1) Learn path-dependent volatility empirically

- Learn how much of volatility is path-dependent, and how it depends on past asset returns.
- Empirical study: learn implied volatility (VIX) and future Realized Volatility (RV) from SPX path [+ other equity indexes].
- Historical PDV or Empirical PDV or $\mathbb{P}$-PDV.

(2) Build continuous-time Markovian version of empirical PDV model

- Extremely realistic sample paths + SPX and VIX smiles.

(3) Jointly calibrate Model (2) to SPX and VIX smiles

- Modify parameters of historical PDV model to fit market smiles: $\mathbb{P} \neq \mathbb{Q}$.
- Implied PDV or Risk-neutral PDV or $\mathbb{Q}$-PDV.

(4) Add SV to account for the (small) exogenous part: PDSV

- SV component built from the analysis of residuals $\frac{\text{true vol}}{\text{predicted PDV vol}} \approx 1$.

Is Volatility Path-Dependent?

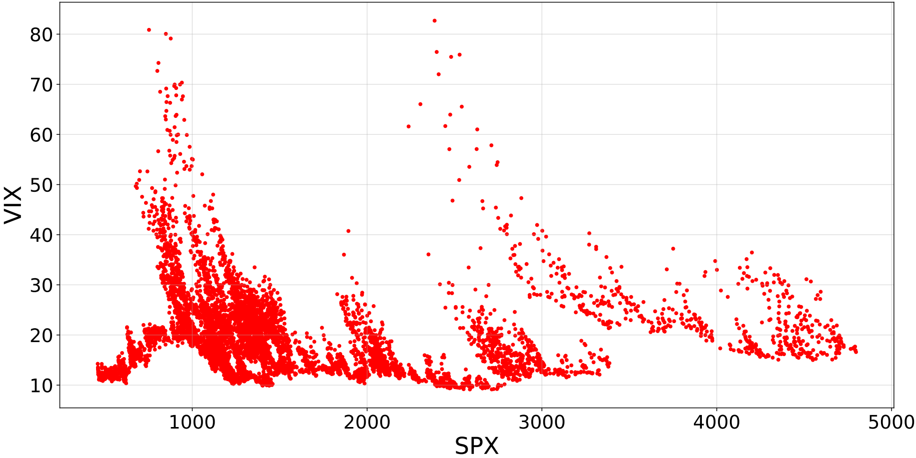
Is volatility path-dependent? A Machine Learning approach

- Objective: **learn from data how much the volatility level depends on past asset returns**.
- Learn Volatility (VIX or RV) from SPX path:

$$\text{Volatility}_t = f(S_u, u \leq t) + \varepsilon$$

- $\longrightarrow$ Historical PDV / Empirical PDV / $\mathbb{P}$-PDV
- Feature engineering: find relevant SPX path features.
- Try various models: various sets of features and parametric forms for f_θ.
- Select the one(s) with the best validation score.
- Check how the models perform on the test set.
- Training set: 2004–18; test set: 2019–22.
- A very challenging test set! Due to the Covid-19 pandemic, the test set includes very different volatility regimes
- **As a result of this analysis, we propose a new, simple PDV model that performs better than existing models**.

Feature engineering



Price path features should be **scale-invariant**

Feature engineering

We focus on **two main types of features**:

[1] Features that capture a **recent trend** in the asset price:

- in order to learn the **leverage effect**: volatility tends to be higher when asset prices fall.
- in order to capture the **strong Zumbach effect**.

[2] Features that capture **recent activity (volatility)** in the asset price (regardless of trend):

- in order to learn **volatility clustering**:
  - periods of large volatility tend to be followed by periods of large volatility.
  - implied volatility tends to be larger when historical volatility is larger.
- in order to capture both the **weak and strong Zumbach effects**.

Trend features

- The most important example of a trend feature is a weighted sum of past daily returns

$$R_{1,t} := \sum_{t_i \leq t} K_1(t - t_i)\, r_{t_i}$$

where

$$r_{t_i} := \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \qquad \text{(scale invariance)}$$

- K_1: convolution kernel that typically decreases towards zero: the impact of a given daily return fades away over time.
- Another example:

$$N_t := \sum_{t_i \leq t} K_N(t - t_i) r_{t_i}^- \quad \text{or more generally } N_t^\varphi := \sum_{t_i \leq t} K_N(t - t_i)\varphi(r_{t_i})$$

with, e.g., $\varphi(r) = r^+$ or r^3 or $(r^-)^2$.
- Another example: spot-to-moving-average ratio

$$U_t := \frac{S_t}{A_t}, \qquad A_t := \sum_{t_i \leq t} K_A(t - t_i)\, S_{t_i}.$$

Activity features (or volatility features)

- The most important example of a volatility feature is a weighted sum of past squared daily returns

$$R_{2,t} := \sum_{t_i \le t} K_2(t - t_i)\, r_{t_i}^2.$$

- K_2-weighted historical volatility:

$$\Sigma_t := \sqrt{R_{2,t}}$$

- Higher even moments of past daily returns may also be considered.

Our model

$$\text{Volatility}_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \Sigma_t, \qquad \beta_0 > 0, \;\; \beta_1 < 0, \;\; \beta_2 \in (0,1)$$

- Volatility_t denotes either some implied volatility (e.g., the VIX) observed at t, or the future realized volatility RV_t (realized over day "$t+1$").
- Leverage effect: $\beta_1 < 0$.
- Volatility clustering, like in GARCH models: $\beta_2 \in (0,1)$.
- Importantly, **both factors $R_{1,t}$ and Σ_t are needed to satisfactorily explain the volatility**.
- We find that **a simple linear model does the job, explaining a very large part of the variability observed in the volatility.**

Kernels

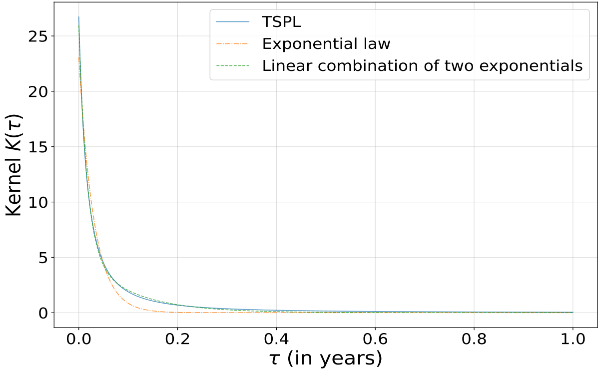


Figure: Typical kernel shapes

Kernels

- The two kernels K_1 and K_2 are distinct.
- Both mix short and long memory
- We consider kernels K_1, K_2 with power-law decay because the data shows that
  1. Very recent daily returns are given much more weight than older daily returns: the weights $K_n(\tau)$ decrease fast for small lags τ.
  2. Nevertheless, volatility has long memory: the weights $K_n(\tau)$ decrease slowly for large τ: persistence of volatility.

The power law aggregates the various time horizons of investors.

Kernels

- This was checked by running a multivariate lasso regression with variables $R_{1,t}^{(\lambda_j)}$ and $\sqrt{R_{2,t}^{(\mu_k)}}$, where

$$R_{n,t}^{(\lambda)} := \sum_{t_i \le t} K^{(\lambda)}(t - t_i)\, r_{t_i}^n, \qquad K^{(\lambda)}(\tau) := \lambda e^{-\lambda\tau}, \qquad \lambda > 0.$$

- For both $n = 1$ and $n = 2$, lasso selects a multitude of λ's which, combined, form a kernel that looks like a power law, except that for vanishing lags τ the kernels do not seem to blow up (the largest λ's are not selected).
- $\Longrightarrow$ We choose both kernels to be **time-shifted power laws** (TSPL):

$$K(\tau) = K_{\alpha,\delta}(\tau) := Z_{\alpha,\delta}^{-1}(\tau + \delta)^{-\alpha}, \qquad \tau \ge 0, \qquad \alpha > 1, \ \delta > 0,$$

  with only two parameters α, δ.
- The time shift δ (one to a few weeks) guarantees that $K_{\alpha,\delta}(\tau)$ **does not blow up when the lag τ vanishes**.
- If we force δ to be 0, we recover the power-law kernel of rough volatility models. However, our empirical tests always select **positive δ**.

Similar models

- QARCH (Sentana '95):

$$\text{Volatility}_t^2 = \beta_0 + \beta_1 R_{1,t} + \beta_2 R_{2,t}^{\mathrm{Q}}, \qquad R_{2,t}^{\mathrm{Q}} := \sum_{t_i,t_j \le t} K_2^{\mathrm{Q}}(t-t_i, t-t_j)\, r_{t_i} r_{t_j}$$

- Diagonal QARCH model (CB '14, $K_2(\tau) := K_2^{\mathrm{Q}}(\tau,\tau)$):

$$\text{Volatility}_t^2 = \beta_0 + \beta_1 R_{1,t} + \beta_2 R_{2,t} \qquad \text{(M1)}$$

- ZHawkes process (BDB '16):

$$\text{Volatility}_t^2 = \beta_0 + \beta_1 R_{1,t}^2 + \beta_2 R_{2,t} \qquad \text{(M2)}$$

- Discrete-time version of the quadratic rough Heston model (GJR '20, $\theta_0 = 0$):

$$\text{Volatility}_t^2 = \beta_0 + \beta_1 (R_{1,t} - \beta_2)^2 \qquad \text{(M3)}$$

with Mittag-Leffler kernel K_1.

- Discrete-time version of the threshold EWMA Heston model (Parent '21):

$$\text{Volatility}_t^2 = \beta_0 + \beta_1 (R_{1,t} - \beta_2)^2 \mathbf{1}_{\{R_{1,t} \le \beta_2\}} \qquad \text{(M4)}$$

with K_1 an exponential kernel, $K_1(\tau) = \lambda e^{-\lambda\tau}$.

Our model differs in several ways

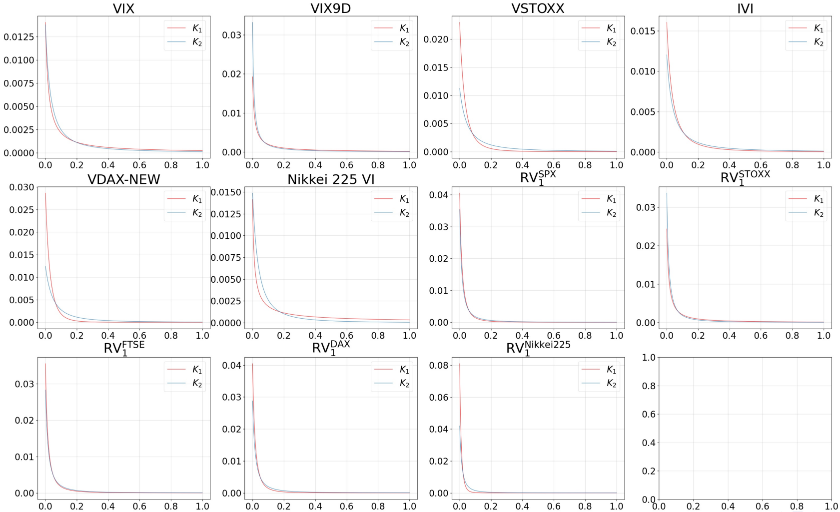
1. All the above models, like almost all ARCH models, model the **square** of the volatility, the variance. Instead, **we directly model the volatility itself**.
2. We use the square root Σ_t of $R_{2,t}$ rather than $R_{2,t}$ itself as one of the linear factors.
3. We use new, explicit parametric forms for the kernels K_1 and K_2, capturing **non-blowing-up power-law-like decays**.
4. Compared with (M3) and (M4), we empirically prove the **importance of including the historical volatility factor Σ_t**.
5. Compared with (M2), we argue that it is **not necessary to include a quadratic factor $R_{1,t}^2$**, as the quadratic-like dependence of the volatility (resp. variance) on $R_{1,t}$ is already captured by the factor Σ_t (resp. $R_{2,t}$).

Results: Implied volatility

	β_0	α_1	δ_1	β_1	α_2	δ_2	β_2
VIX	0.057	1.057	0.020	-23.829	1.597	0.052	0.819
VIX9D	0.045	0.993	0.011	-30.655	1.252	0.011	0.884
VSTOXX	0.032	3.959	0.127	-9.192	1.895	0.089	0.966
IVI	0.022	2.262	0.081	-14.640	1.630	0.063	0.991
VDAX-NEW	0.036	5.540	0.156	-6.149	2.207	0.103	0.922
Nikkei 225 VI	0.055	0.778	0.008	-17.337	2.090	0.077	0.855

Table: Table of optimal parameters for different implied volatility indexes.

Results: Implied volatility



Results: Implied volatility

	Train RMSE	Train r^2	Test RMSE	Test r^2
VIX	0.020	0.946	0.035	0.855
VIX9D	0.023	0.876	0.034	0.914
VSTOXX	0.026	0.929	0.029	0.913
IVI	0.024	0.925	0.030	0.871
VDAX-NEW	0.025	0.934	0.028	0.918
Nikkei 225 VI	0.030	0.890	0.031	0.799

Table: Table of r^2 scores and RMSE for various implied volatility indexes.

Results: Implied volatility

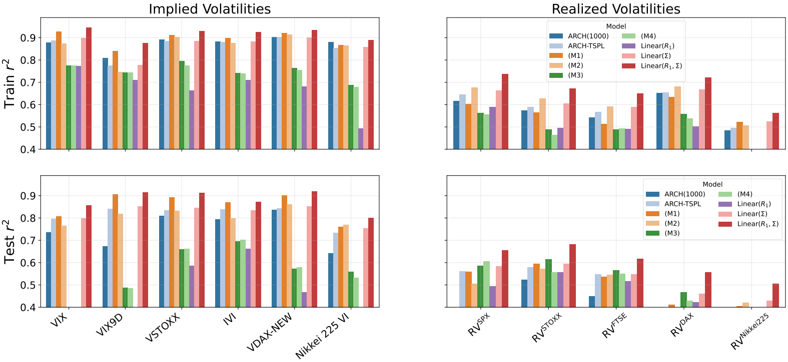


Figure: Comparison of r^2 scores for the different models (M1)-(M4) and our linear models. Top: r^2 score on train set. Bottom: r^2 score on test set. Left: Implied volatilities. Right: Realized volatilities.

Results: Implied volatility

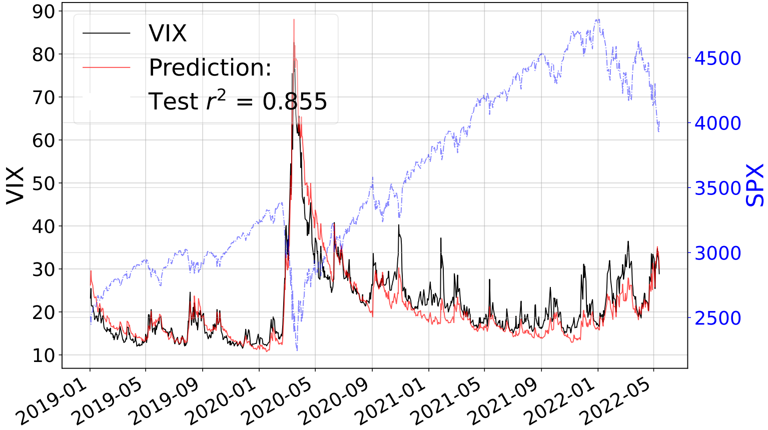


Figure: Time series of VIX and predicted values on test set. The dashed blue line represents the SPX time series.

Results: Implied volatility

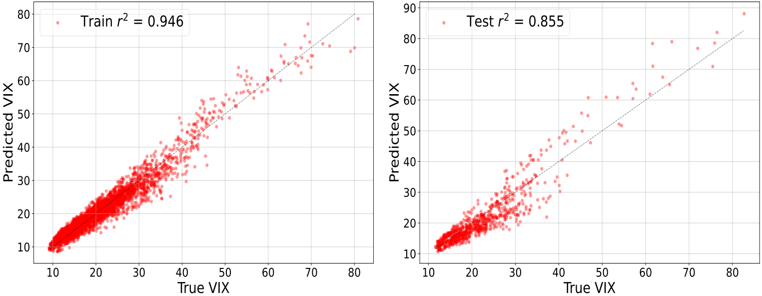


Figure: Predicted VIX vs true VIX on train/test set.

Results: Implied volatility

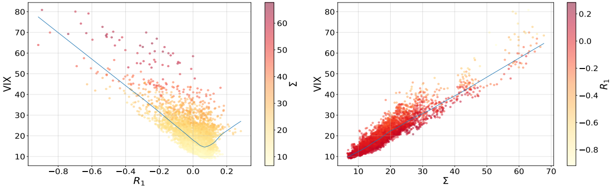


Figure: VIX vs features on the train data set.

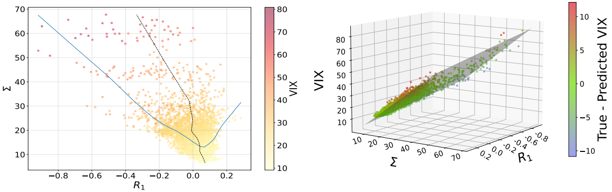


Figure: Σ vs R_1 on the train data set and 3D scatter plot of VIX vs R_1 and Σ

Results: Implied volatility

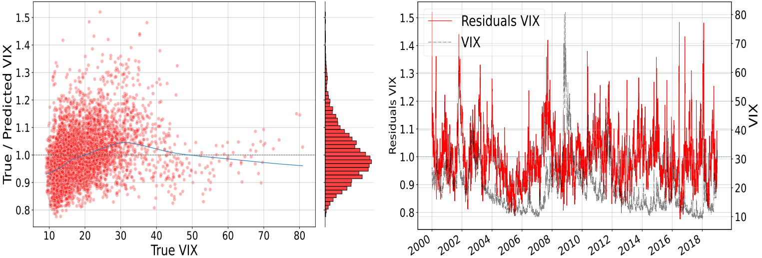


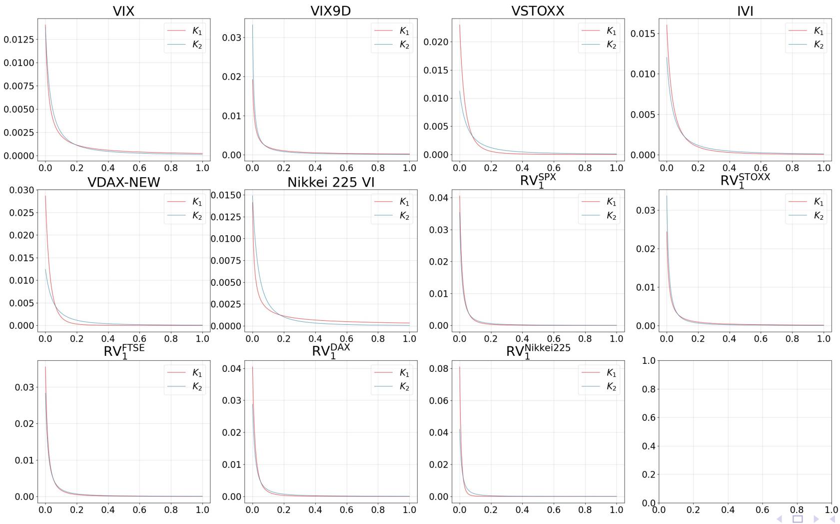
Figure: Residuals plots for VIX predictions.

Results: Realized volatility

	β_0	α_1	δ_1	β_1	α_2	δ_2	β_2
SPX	0.018	2.821	0.044	-10.490	1.860	0.025	0.708
STOXX	0.023	1.306	0.017	-15.567	1.787	0.024	0.697
FTSE	0.017	2.216	0.034	-10.753	1.837	0.031	0.762
DAX	0.001	2.868	0.045	-7.570	1.800	0.029	0.812
NIKKEI	0.032	6.296	0.063	-2.802	2.292	0.030	0.511

Table: Table of optimal parameters for the realized volatility for different indexes.

Results: Realized volatility

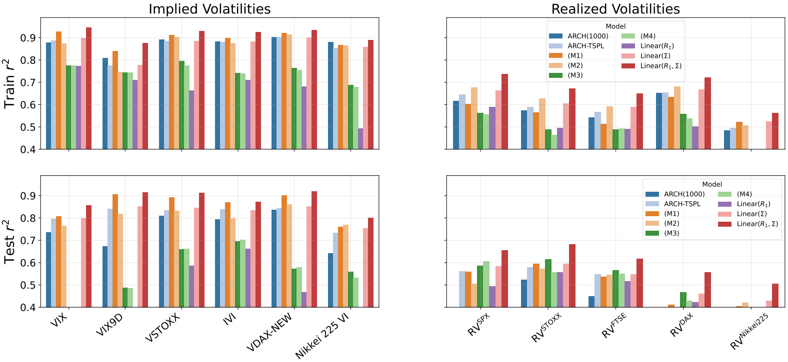


Results: Realized volatility

	Train RMSE	Train r^2	Test RMSE	Test r^2
SPX	0.049	0.738	0.063	0.654
STOXX	0.060	0.672	0.064	0.682
FTSE 100	0.055	0.650	0.066	0.617
DAX	0.057	0.722	0.059	0.557
NIKKEI	0.051	0.563	0.051	0.504

Table: Table of r^2 scores and RMSE for the realized volatility of several indexes

Results: Realized volatility



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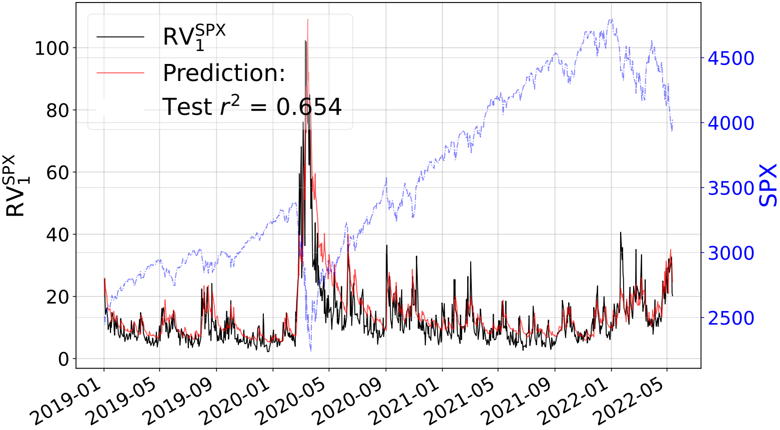
Results: Realized volatility



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Results: Realized volatility

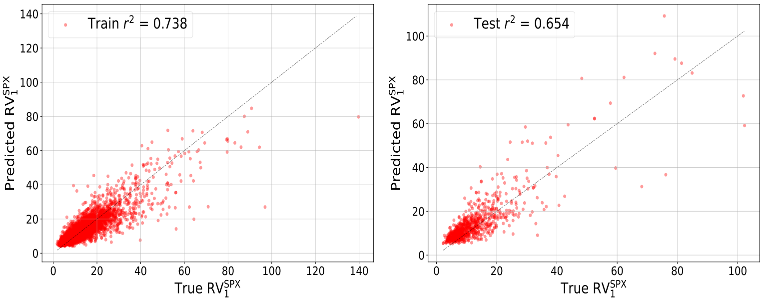

Figure: Predicted VIX vs true VIX on train/test set.

Results: Realized volatility

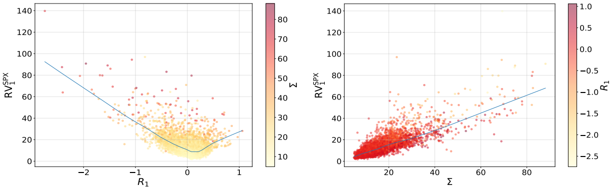

Figure: $\mathrm{RV}^{\mathrm{SPX}}$ vs features on the train data set.

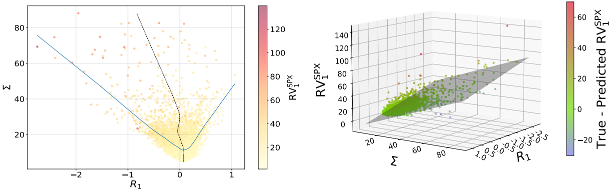

Figure: Σ vs R_1 on the train data set and 3D scatter plot of $\mathrm{RV}^{\mathrm{SPX}}$ vs R_1 and Σ.

Results: Realized volatility

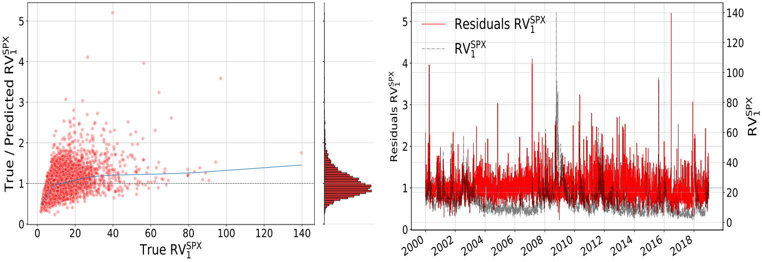

Figure: Residuals plots for RV^{SPX} predictions.

The Continuous-Time Empirical Path-Dependent Volatility Model

The Continuous-Time Empirical Path-Dependent Volatility Model

We now consider the **continuous-time limit** of our empirical PDV model, where we identify Volatility_t as the instantaneous volatility σ_t:

$$\begin{aligned}
\frac{dS_t}{S_t} &= \sigma_t \, dW_t, \\
\sigma_t &= \sigma(R_{1,t}, R_{2,t}) \\
\sigma(R_1, R_2) &= \beta_0 + \beta_1 R_1 + \beta_2 \sqrt{R_2} \\
R_{1,t} &= \int_{-\infty}^{t} K_1(t-u) \frac{dS_u}{S_u} = \int_{-\infty}^{t} K_1(t-u) \sigma_u \, dW_u, \\
R_{2,t} &= \int_{-\infty}^{t} K_2(t-u) \left(\frac{dS_u}{S_u} \right)^2 = \int_{-\infty}^{t} K_2(t-u) \sigma_u^2 \, du.
\end{aligned} \tag{1}$$

The Continuous-Time Empirical Path-Dependent Volatility Model

The dynamics of $R_{1,t}$ and $R_{2,t}$

$$\begin{aligned}
dR_{1,t} &= \left(\int_{-\infty}^{t} K_1'(t-u)\frac{dS_u}{S_u}\right) dt + K_1(0)\frac{dS_t}{S_t} \\
&= \left(\int_{-\infty}^{t} K_1'(t-u)\sigma_u dW_u\right) dt + K_1(0)\sigma_t\, dW_t \\
dR_{2,t} &= \left(\int_{-\infty}^{t} K_2'(t-u)\left(\frac{dS_u}{S_u}\right)^2\right) dt + K_2(0)\left(\frac{dS_t}{S_t}\right)^2 \\
&= \left(K_2(0)\sigma_t^2 + \int_{-\infty}^{t} K_2'(t-u)\sigma_u^2 du\right) dt
\end{aligned}$$

are in general non-Markovian, since for general kernels K_1 and K_2 the integrals in the above drifts are not functions of $(R_{1,t}, R_{2,t})$.

A (too) simple Markovian approximation: the 2-Factor PDV model

- The simplest kernels yielding a Markovian model are the (normalized) exponential kernels $K_1(\tau) := \lambda_1 e^{-\lambda_1 \tau}$ and $K_2(\tau) := \lambda_2 e^{-\lambda_2 \tau}$, $\lambda_1, \lambda_2 > 0$.
- $K_1' = -\lambda_1 K_1$ and $K_2' = -\lambda_2 K_2$ so both $(R_{1,t}, R_{2,t})$ and $(S_t, R_{1,t}, R_{2,t})$ have Markovian dynamics:

$$\frac{dS_t}{S_t} = \sigma(R_{1,t}, R_{2,t})\, dW_t, \qquad \sigma(R_1, R_2) = \beta_0 + \beta_1 R_1 + \beta_2 \sqrt{R_2},$$

$$dR_{1,t} = \lambda_1 \left(\frac{dS_t}{S_t} - R_{1,t}\, dt \right) = \lambda_1 \big(\sigma(R_{1,t}, R_{2,t})\, dW_t - R_{1,t}\, dt \big),$$

$$dR_{2,t} = \lambda_2 \left(\left(\frac{dS_t}{S_t} \right)^2 - R_{2,t}\, dt \right) = \lambda_2 \left(\sigma(R_{1,t}, R_{2,t})^2 - R_{2,t} \right) dt.$$

- We call this model the **2-Factor PDV model** (2FPDV model).

The 2-Factor PDV model

- Choosing K_1 and K_2 to be single exponential kernels fails to capture the mix of short and long memory in both R_1 and R_2 observed in the data.
- We will capture this mix of short and long memory in a Markovian way by choosing K_1 and K_2 to be **linear combinations** of exponential kernels.
- Dynamics of the volatility $\sigma_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \sqrt{R_{2,t}}$ reads

$$d\sigma_t = \left(-\beta_1 \lambda_1 R_{1,t} + \frac{\beta_2 \lambda_2}{2} \frac{\sigma_t^2 - R_{2,t}}{\sqrt{R_{2,t}}} \right) dt + \beta_1 \lambda_1 \sigma_t \, dW_t. \qquad (2)$$

- **Constant instantaneous vol of instantaneous vol** but rich drift
- **Volatility clustering via mean-reversion** + **explanation for mean-reversion**
- Price-path-dependence of volatility dynamics: **strong Zumbach effect**
- Nonnegativity of volatility guaranteed if $\lambda_2 < 2\lambda_1$

A better Markovian approximation: the 4-Factor PDV model

- Approximate TSPL kernel $\tau \mapsto Z_{\alpha,\delta}^{-1}(\tau+\delta)^{-\alpha}$ by a linear combination of two exponential kernels, $\tau \mapsto (1-\theta)\lambda_0 e^{-\lambda_0 \tau} + \theta\lambda_1 e^{-\lambda_1 \tau}$ with $\theta \in [0,1]$ and $\lambda_0 > \lambda_1 > 0$.
- **Short memory: large** λ_0.
- **Long memory: small** λ_1.
- θ is a mixing factor.

TSPL vs linear combination of two exponentials

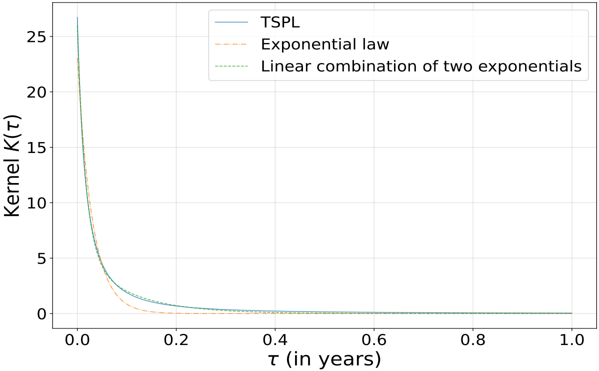

Figure: TSPL kernel K_1 and its approximations by an exponential and by a linear combination of two exponentials.

The 4-Factor PDV model

Introduce parameters $\theta_1, \lambda_{1,0}, \lambda_{1,1}$ and $\theta_2, \lambda_{2,0}, \lambda_{2,1}$ for the approximation of the TSPL kernels K_1 and K_2. For $n \in \{1,2\}$ and $j \in \{0,1\}$, denote

$$R_{n,j,t} := \int_{-\infty}^{t} \lambda_{n,j} e^{-\lambda_{n,j}(t-u)} \left(\frac{dS_u}{S_u}\right)^n.$$

$$\begin{aligned}
\frac{dS_t}{S_t} &= \sigma_t \, dW_t \\
\sigma_t &= \sigma(R_{1,t}, R_{2,t}) \\
\sigma(R_1, R_2) &= \beta_0 + \beta_1 R_1 + \beta_2 \sqrt{R_2} \\
R_{1,t} &= (1-\theta_1) R_{1,0,t} + \theta_1 R_{1,1,t} \\
R_{2,t} &= (1-\theta_2) R_{2,0,t} + \theta_2 R_{2,1,t} \\
dR_{1,j,t} &= \lambda_{1,j} \left(\frac{dS_t}{S_t} - R_{1,j,t}\, dt\right) = \lambda_{1,j} \left(\sigma(R_{1,t}, R_{2,t})\, dW_t - R_{1,j,t}\, dt\right), \\
dR_{2,j,t} &= \lambda_{2,j} \left(\left(\frac{dS_t}{S_t}\right)^2 - R_{2,j,t}\, dt\right) = \lambda_{2,j} \left(\sigma(R_{1,t}, R_{2,t})^2 - R_{2,j,t}\right) dt.
\end{aligned}$$

The 4-Factor PDV model

The dynamics of the instantaneous volatility reads

$$d\sigma_t = \beta_1 \left((1-\theta_1)\lambda_{1,0} + \theta_1\lambda_{1,1}\right)\sigma_t\, dW_t + \Big\{ -\beta_1 \left((1-\theta_1)\lambda_{1,0} R_{1,0,t} + \theta_1\lambda_{1,1} R_{1,1,t}\right) + \frac{\beta_2}{2} \frac{\left((1-\theta_2)\lambda_{2,0} + \theta_2\lambda_{2,1}\right)\sigma_t^2 - \left((1-\theta_2)\lambda_{2,0} R_{2,0,t} + \theta_2\lambda_{2,1} R_{2,1,t}\right)}{\sqrt{R_{2,t}}} \Big\}\, dt \qquad (3)$$

and satisfies similar qualitative properties as dynamics (2):

- The drift of σ_t produces **volatility clustering via a clear trend of mean reversion of volatility**.
- The **lognormal volatility of σ_t is constant**.
- The **dynamics of (σ_t) are price-path-dependent**: the drift of σ_t cannot be written as a function of just the past values $(\sigma_u)_{u\leq t}$ of the volatility; it depends on the past asset returns through $R_{1,0,t}$ and $R_{1,1,t}$.

The 4-Factor PDV model: drift of the volatility

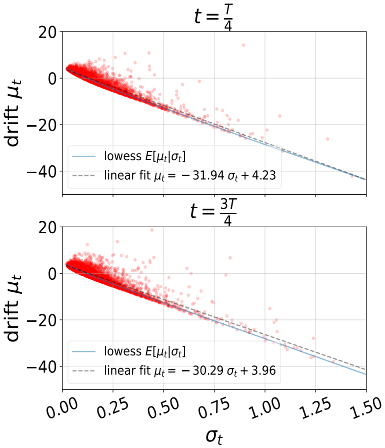

Figure: Drift of σ_t vs σ_t for different maturities and for $N = 10$k paths, $T = 1$ year.

The 4-Factor PDV model: sample paths

β_0	β_1	$\lambda_{1,0}$	$\lambda_{1,1}$	θ_1	β_2	$\lambda_{2,0}$	$\lambda_{2,1}$	θ_2
0.04	-0.105	62	10	0.21	0.6	40	3	0.42

Table: Parameters for the simulation of the 4-Factor PDV Model

The 4-Factor PDV model: sample paths

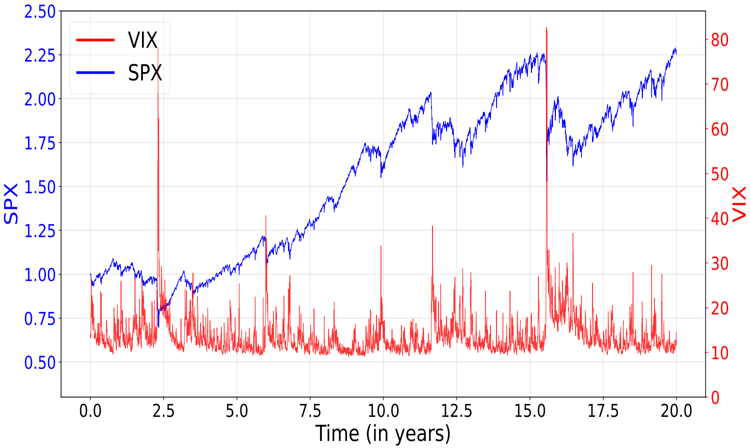

Figure: SPX and VIX timeseries on a typical path of 20 years.

The 4-Factor PDV model: scatter plots

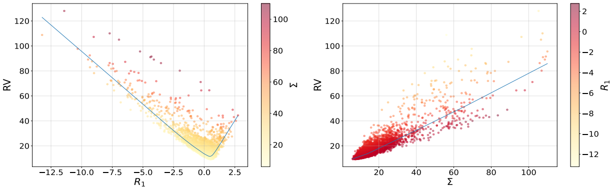

Figure: VIX vs features on 5 simulated paths of 20 years.

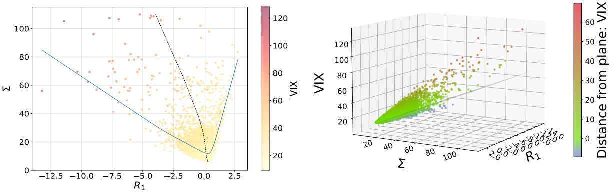

Figure: Σ vs R_1 and 3D scatter plot of VIX vs R_1 and Σ.

The 4-Factor PDV model: scatter plots

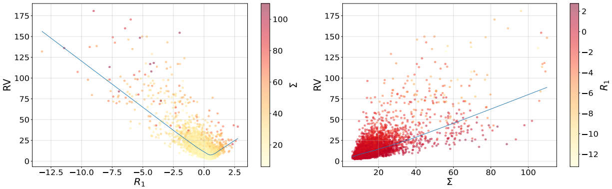

Figure: RV vs features on 5 simulated paths of 20 years.

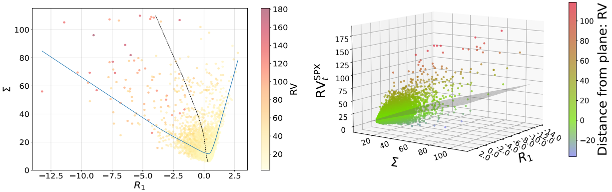

Figure: Σ vs R_1 and 3D scatter plot of RV vs R_1 and Σ.

The 4-Factor PDV model: very realistic smiles

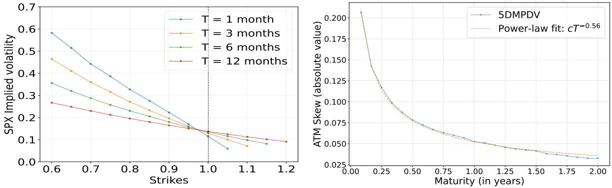

Figure: Model SPX smiles and term-structure of ATM skew.

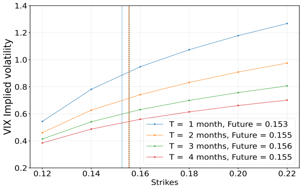

Figure: Model VIX smiles.

Conclusion

- **Volatility is (mostly) path-dependent, endogenous.**
- Volatility is very well explained by recent past asset returns only: train $r^2 \approx \mathbf{0.9}$, test $r^2 \approx \mathbf{0.9}$ on implied volatility data; train $r^2 \approx \mathbf{0.7}$, test $r^2 \approx \mathbf{0.7}$ on (noisy) realized volatility data.
- We have found a simple path-dependent volatility model that accurately explains the current VIX or RV value by recent SPX returns.
- We directly model the volatility level (not the vol changes).
- By design, dependence on trend features (MA of past returns) $\Longrightarrow$ leverage effect + strong Zumbach effect...
- ...but it is not enough: volatility features (MA of past squared returns = historical volatility) are needed too; they capture volatility clustering + weak Zumbach effect.
- Multi-scale trading memory: different time scales of path-dependence are needed $\Longleftrightarrow$ various time horizons of investors/traders
- Using EWMA yields easy-to-simulate Markovian models...
- ...but still generates spurious roughness caused by noisy estimation of RV.

Conclusion

- Volatility is not purely path-dependent: some of it depends on news, new information.
- The (small) exogenous part can then be incorporated using another source of randomness, e.g.,

$$\frac{dS_t}{S_t} = a_t\,\sigma(S_u, u \leq t)\,dW_t$$

  where a_t is some stochastic volatility, for instance: **PDSV**
- The ratio residuals $\frac{\mathrm{VIX}_t}{f(S_u, u \leq t)}$ help define relevant stochastic dynamics for (a_t).

We believe this is the right way of modeling volatility:
(1) Model the purely endogenous part of volatility as best as we can.
(2) Then add the exogenous part, if needed.

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