Dynamic Programming Principle for Stochastic Recursive Optimal Control Problem Driven by a G-Brownian Motion

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Outline

- Basic settings
- Formulation of the control problem
- Dynamic Programming Principle
- Hamilton-Jacobi-Bellman equation

Basic settings: G-expectation space

- $\Omega_T = C_0([0,T];\mathbb{R}^d)$: all $\mathbb{R}^d$-valued continuous functions on $[0,T]$ with $\omega_0 = 0$.
- Canonical process $B_t(\omega) := \omega_t$, for $\omega \in \Omega_T$ and $t \in [0,T]$,

$$\begin{aligned} Lip(\Omega_t) := \quad & \{\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) : \\ & N \geq 1, t_1 < \cdots < t_N \leq t, \varphi \in C_{b.Lip}(\mathbb{R}^{d\times N})\}. \end{aligned}$$

- $G : \mathbb{S}_d \to \mathbb{R}$ is a monotonic and sublinear function iff

$$G(A) = \frac{1}{2} \sup_{\gamma \in \Sigma} \mathrm{tr}[A\gamma] \text{ for } A \in \mathbb{S}_d,$$

  where $\Sigma \subset \mathbb{S}_d^+$ is bounded.

- Peng (2005) constructed the G-expectation $\hat{\mathbb{E}} : Lip(\Omega_T) \to \mathbb{R}$ and the conditional G-expectation $\hat{\mathbb{E}}_t : Lip(\Omega_T) \to Lip(\Omega_t)$.

- For $s_1 \leq s_2 \leq T$ and $\varphi \in C_{b.Lip}(\mathbb{R}^d)$, define

$$\hat{\mathbb{E}}[\varphi(B_{s_2} - B_{s_1})] = u(s_2 - s_1, 0),$$

  where u is the viscosity solution of the following G-heat equation:

$$\partial_t u - G(D_x^2 u) = 0, \ u(0, x) = \varphi(x).$$

- For $X = \varphi_N(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}}) \in Lip(\Omega_T)$, define

$$\hat{\mathbb{E}}_{t_i}[X] = \varphi_i(B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}) \text{ and } \hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[\varphi_1(B_{t_1})],$$

  where

$$\varphi_i(x_1, \ldots, x_i) := \hat{\mathbb{E}}[\varphi_{i+1}(x_1, \ldots, x_i, B_{t_{i+1}} - B_{t_i})].$$

- G-expectation space $(\Omega_T, Lip(\Omega_T), \hat{\mathbb{E}}, (\hat{\mathbb{E}}_t)_{t\in[0,T]})$ is a consistent sublinear expectation space. The canonical process $(B_t)_{t\in[0,T]}$ is called the G-Brownian motion under G-expectation $\hat{\mathbb{E}}$.

Representation theorem of G-expectation

Theorem (Denis-Hu-Peng (2011), Hu-Peng (2009))

There exists a unique weakly compact and convex set of probability measures $\mathcal{P}$ on $(\Omega_T, \mathcal{B}(\Omega_T))$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X] \text{ for all } X \in Lip(\Omega_T),$$

where $\mathcal{B}(\Omega_T) = \sigma(B_s : s \leq T)$.

Characterization of spaces

- $L_G^p(\Omega_t)$ the completion of $Lip(\Omega_t)$ under the norm $(\hat{\mathbb{E}}[|X|^p])^{1/p}$ for $p \geq 1$.
- For $\mathcal{P}$,
$$\mathbb{L}^p(\Omega_t) := \left\{ X \in \mathcal{B}(\Omega_t) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \right\}$$
is a Banach space for $p \geq 1$.
- The capacity associated to $\mathcal{P}$ is defined by
$$c(A) := \sup_{P \in \mathcal{P}} P(A) \text{ for } A \in \mathcal{B}(\Omega_T).$$
A set $A \in \mathcal{B}(\Omega_T)$ is polar if $c(A) = 0$. A property holds "quasi-surely" (q.s. for short) if it holds outside a polar set. We do not distinguish two random variables X and Y if $X = Y$ q.s.

Definition

$X : \Omega_T \to \mathbb{R}$ is called quasi-continuous (q.c.) if for any $\varepsilon > 0$, there exists a closed set F such that $c(F^c) < \varepsilon$ and X is continuous on F. We say that $X : \Omega_T \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function Y with $X = Y$ q.s.

Theorem

For $p \geq 1$,

$$\begin{aligned} L_G^p(\Omega_t) = \quad & \{X \in \mathbb{L}^p(\Omega_t) : \lim_{n\to\infty} \sup_{P\in\mathcal{P}} E_P[|X|^p I_{\{|X|>n\}}] = 0, \\ & X \textit{ has a quasi-continuous version}\}. \end{aligned}$$

Spaces of solution

- $M_G^0(0,T) = \{\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) I_{[t_j,t_{j+1})}(t) : \xi_i \in L_{ip}(\Omega_{t_i})\}$.
- $M_G^p(0,T)$: $M_G^0(0,T)$ under $\|\eta\|_{M_G^p} = \{\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds]\}^{1/p}$.
- $H_G^p(0,T)$: $M_G^0(0,T)$ under $\|\eta\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^T |\eta_s|^2 ds)^{p/2}]\}^{1/p}$.

- $S_G^0(0,T) = \{h(t, B_{t_1\wedge t}, \cdots, B_{t_n\wedge t}) : h \in C_{b,Lip}(\mathbb{R}^{n+1})\}$.
- $S_G^p(0,T)$: $S_G^0(0,T)$ under $\|\eta\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t\in[0,T]} |\eta_t|^p]\}^{\frac{1}{p}}$.

Remark G is non-degenerate, i.e., there exists a $\underline{\sigma}^2 > 0$ such that $G(A) - G(B) \geq \underline{\sigma}^2 \mathrm{tr}[A - B]$ for any $A \geq B$.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g_{ij}(s, Y_s, Z_s)d\langle B^i, B^j\rangle_s$$
$$- \int_t^T Z_s dB_s - (K_T - K_t)$$

(H1) $f(\cdot, \cdot, y, z), g_{ij}(\cdot, \cdot, y, z) \in M_G^\beta(0, T)$ for some $\beta > 1$.

(H2) There exists some $L > 0$ such that

$$|f(t, y, z) - f(t, y', z')| + |g_{ij}(t, y, z) - g_{ij}(t, y', z')|$$
$$\leq L(|y - y'| + |z - z'|).$$

Remark Soner-Touzi-Zhang (2012) studied a new type of fully nonlinear BSDE, called 2BSDE, by different formulation and method, and obtained the deep result of the existence and uniqueness theorem for 2BSDE.

Existence and uniqueness theorem

Theorem (Hu-Ji-Peng-Song (2014))

Assume that $\xi \in L_G^\beta(\Omega_T)$ *and* f, g_{ij} *satisfy (H1) and (H2) for some* $\beta > 1$. *Then* G*-BSDE has a unique solution* (Y, Z, K). *Moreover,* $Y \in S_G^\alpha(0,T)$ *and* $Z \in H_G^\alpha(0,T;\mathbb{R}^d)$, $K_T \in L_G^\alpha(\Omega_T)$ *for any* $1 < \alpha < \beta$.

Some useful estimates

Proposition

Assume that $\xi^k \in L_G^\beta(\Omega_T)$ and f^k, g_{ij}^k satisfy (H1) and (H2) for $\beta > 1$, $k = 1, 2$. The solution is denoted by (Y^k, Z^k, K^k). Then there exists a positive constant C depending on α, G, L and T satisfying

$$|\hat{Y}_t|^\alpha \leq C\hat{\mathbb{E}}_t \left[|\hat{\xi}|^\alpha + \left(\int_t^T (|\hat{f}(s)| + |\hat{g}_{ij}(s)|) ds \right)^\alpha \right],$$

where $\hat{Y}_t = Y_t^1 - Y_t^2$, $\hat{\xi} = \xi^1 - \xi^1$, $\hat{f}(s) = f^1(s, Y_s^2) - f^2(s, Y_s^2)$, $\hat{g}_{ij}(s) = g_{ij}^1(s, Y_s^2, Z_s^2) - g_{ij}^2(s, Y_s^2, Z_s^2)$.

Formulation of the control problem

Let $t \in [0,T]$, $\xi \in \cup_{\varepsilon>0} L_G^{2+\varepsilon}(\Omega_t;\mathbb{R}^n)$. Consider the following G-FBSDE:

$$\begin{cases} dX_s^{t,\xi,u} = & b(s,X_s^{t,\xi,u},u_s)ds + h_{ij}(s,X_s^{t,\xi,u},u_s)d\langle B^i,B^j\rangle_s \\ & +\sigma(s,X_s^{t,\xi,u},u_s)dB_s, \\ dY_s^{t,\xi,u} = & -f(s,X_s^{t,\xi,u},Y_s^{t,\xi,u},Z_s^{t,\xi,u},u_s)ds \\ & -g_{ij}(s,X_s^{t,\xi,u},Y_s^{t,\xi,u},Z_s^{t,\xi,u},u_s)d\langle B^i,B^j\rangle_s \\ & +Z_s^{t,\xi,u}dB_s + dK_s^{t,\xi,u}, \\ X_t^{t,\xi,u} = & \xi,\ Y_T^{t,\xi,u} = \Phi(X_T^{t,\xi,u}). \end{cases}$$

- U is a given compact set of $\mathbb{R}^m$
- $\mathcal{U}[t,T] = M^2(t,T;U)$ the set of all admissible controls u
- $b, h_{ij}, \sigma, f, g_{ij}, \Phi$ are continuous in s and Lipschitz in x, y, z, u

Define the value function

$$V(t,x) := \underset{u \in \mathcal{U}[t,T]}{ess\inf} Y_t^{t,x,u} \text{ for } x \in \mathbb{R}^n.$$

Remark This control problem is a "infsup problem", because

$$Y_t^{t,x,u} = \sup_{P \in \mathcal{P}} E_P\left[\cdot\right].$$

Definition

The essential infimum of $\{Y_t^{t,x,u} \mid u \in \mathcal{U}[t,T]\}$ is a random variable $\zeta \in L_G^2(\Omega_t)$ satisfying:

(i) $\forall u \in \mathcal{U}[t,T]$, $\zeta \leq Y_t^{t,x,u}$ q.s.;

(ii) if η is a random variable satisfying $\eta \leq Y_t^{t,x,u}$ q.s. for any $u \in \mathcal{U}[t,T]$, then $\zeta \geq \eta$ q.s..

Similarly, define the essential infimum of $\{Y_t^{t,\xi,u} \mid u \in \mathcal{U}[t,T]\}$.

Remark The essential infimum may not exist.

Dynamic Programming Principle

Notation:

- $L_{ip}(\Omega_s^t) := \{\varphi(B_{t_1} - B_t, ..., B_{t_n} - B_t) : t_1, ..., t_n \in [t, s]\}$
- $M_G^{0,t}(t,T) := \{\eta_s = \sum_{i=0}^{N-1} \xi_i I_{[t_i, t_{i+1})}(s) : \xi_i \in L_{ip}(\Omega_{t_i}^t)\}$
- $M_G^{2,t}(t,T) := \{$completion of $M_G^{0,t}(t,T)$ under $\| \cdot \|_{M_G^2}\}$
- $\mathcal{U}^t[t,T] := \{u \in M_G^{2,t}(t,T;\mathbb{R}^m)$ with values in $U\}$
- $\mathbb{U}[t,T] := \{u = \sum_{i=1}^{n} I_{A_i} u^i : u^i \in \mathcal{U}^t[t,T], I_{A_i} \in L_G^2(\Omega_t), \Omega = \bigcup_{i=1}^{n} A_i\}$

The value function is well defined

Lemma

Let $u \in \mathcal{U}[t,T]$ be given. Then there exists a sequence $(u^k)_{k\geq 1}$ in $\mathbb{U}[t,T]$ such that

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[\int_t^T |u_s - u_s^k|^2 ds\right] = 0.$$

Based on this lemma, we obtain

Theorem

The value function $V(t,x)$ exists and

$$V(t,x) = \inf_{u\in\mathcal{U}^t[t,T]} Y_t^{t,x,u}.$$

Properties of the value function

Proposition

There exists a constant $C > 0$ such that

$$| V(t,x) - V(t,y) | \leq C \mid x - y \mid \text{ for any } x, y \in \mathbb{R}^n,$$

$$| V(t,x) | \leq C(1 + \mid x \mid) \text{ for any } x \in \mathbb{R}^n.$$

Theorem

For any $\xi \in \cup_{\varepsilon > 0} L_G^{2+\varepsilon}(\Omega_t; \mathbb{R}^n)$,

$$V(t,\xi) = \operatorname*{ess\,inf}_{u \in \mathcal{U}[t,T]} Y_t^{t,\xi,u}.$$

Backward semigroup method (Peng 1997)

For $\eta \in \cup_{\varepsilon>0} L_G^{2+\varepsilon}(\Omega_{t+\delta})$, define

$$\mathbb{G}_{t,t+\delta}^{t,x,u}[\eta] := \tilde{Y}_t^{t,x,u},$$

where

$$\begin{cases} dX_s^{t,x,u} = & b(s, X_s^{t,x,u}, u_s)ds + h_{ij}(s, X_s^{t,x,u}, u_s)d\langle B^i, B^j\rangle_s \\ & +\sigma(s, X_s^{t,x,u}, u_s)dB_s, \\ d\tilde{Y}_s^{t,x,u} = & -f(s, X_s^{t,x,u}, \tilde{Y}_s^{t,x,u}, \tilde{Z}_s^{t,x,u}, u_s)ds \\ & -g_{ij}(s, X_s^{t,x,u}, \tilde{Y}_s^{t,x,u}, \tilde{Z}_s^{t,x,u}, u_s)d\langle B^i, B^j\rangle_s \\ & +\tilde{Z}_s^{t,x,u}dB_s + d\tilde{K}_s^{t,x,u}, \\ X_t^{t,x,u} = & x, \ \tilde{Y}_{t+\delta}^{t,x,u} = \eta, \ s \in [t, t+\delta]. \end{cases}$$

Dynamic Programming Principle

Theorem

For any $t < t + \delta \leq T$, $x \in \mathbb{R}^n$, we have

$$\begin{aligned} V(t,x) &= \operatorname*{ess\,inf}_{u(\cdot)\in\mathcal{U}[t,t+\delta]} \mathbb{G}^{t,x,u}_{t,t+\delta}[V(t+\delta, X^{t,x,u}_{t+\delta})] \\ &= \inf_{u(\cdot)\in\mathcal{U}^t[t,t+\delta]} \mathbb{G}^{t,x,u}_{t,t+\delta}[V(t+\delta, X^{t,x,u}_{t+\delta})]. \end{aligned}$$

Key point of proof

- For each give $u(\cdot) \in \mathcal{U}[t, t+\delta]$, the idea is to find $\tilde{u}(\cdot) \in \mathcal{U}[t+\delta, T]$such that

$$Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x,u}, \tilde{u}} \approx V(t+\delta, X_{t+\delta}^{t,x,u}).$$

- Usual method: $A_i = [x_i, x_{i+1})$, $i \leq k$, is a partition of $\mathbb{R}^n$, choose

$$\tilde{u}(\cdot) = \sum_{i=1}^{k} I_{\{X_{t+\delta}^{t,x,u} \in A_i\}} \bar{u}_i(\cdot),$$

where

$$Y_{t+\delta}^{t+\delta, x_i, \bar{u}_i} \approx V(t+\delta, x_i).$$

- Difficulty: $I_{\{X_{t+\delta}^{t,x,u} \in A_i\}}$ may not in L_G^1, then $\tilde{u}$ may not in $\mathcal{U}[t+\delta, T]$. For example:

$$I_{\{\langle B \rangle_\delta \in A_i\}} \notin L_G^1.$$

Our Method

- Implied partition method: find Y such that $I_{\{X_{t+\delta}^{t,x,u}+Y\in A_i\}}$, $I_{\{Y\in A_j\}} \in L_G^1$, using

$$\tilde{u}(\cdot) = \sum_{i,j=1}^{k} I_{\{X_s^{t,x,u}+Y\in A_i\}\cap\{Y\in A_j\}}\bar{u}_{ij}(\cdot),$$

where

$$Y_{t+\delta}^{t+\delta,x_i-x_j,\bar{u}_{ij}} \approx V(t+\delta, x_i - x_j).$$

- Recently, Hu-Ji-Li (2022) obtained that, for any $\xi \in L_G^2(\Omega_{t+\delta})$, there exists a sequence $\xi_k = \sum_{i=1}^{N_k} x_i^k I_{A_i^k}$, $k \geq 1$, such that

$$\lim_{k\to\infty} \hat{\mathbb{E}}\left[|\xi - \xi_k|^2\right] = 0,$$

where $x_i^k \in \mathbb{R}$, $I_{A_i^k} \in L_G^2(\Omega_{t+\delta})$.

Hamilton-Jacobi-Bellman equation

Theorem

V is the unique viscosity solution of the following HJB equation:

$$\begin{cases} \partial_t V + \inf_{u \in U} H(t, x, V, \partial_x V, \partial^2_{xx} V, u) = 0 \\ V(T, x) = \Phi(x), \qquad x \in \mathbb{R}^n \end{cases}$$

where

$$\begin{aligned} H(t, x, v, p, A, u) = &\quad G(F) + \langle p, b(t, x, u) \rangle + f(t, x, v, \sigma^T(t, x, u)p, u), \\ F_{ij}(t, x, v, p, A, u) = &\quad (\sigma^T(t, x, u) A \sigma(t, x, u))_{ij} + 2\langle p, h_{ij}(t, x, u) \rangle \\ &\quad + 2 g_{ij}(t, x, v, \sigma^T(t, x, u)p, u). \end{aligned}$$

Thank you!