Set Valued HJB Equations

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9th Colloquium on BSDEs and Mean Field Systems

Annecy, 6/27-7/1, 2022

HAPPY BIRTHDAY, Jin !

See a special volume honoring Jin
in Numerical Algebra, Control and Optimization

Outline

Some motivations

- Multivariate dynamic risk measures
  - ◇ Feinstein-Rudloff (2015), $\cdots$
- Multivariate superhedging and stochastic target problems
  - ◇ Kabanov (1999), Soner-Touzi (2002), $\cdots$
- Stochastic viability problems
  - ◇ Aubin-Frankowska (2009), Buckdahn-Goreac-Li (2022), $\cdots$
- Time inconsistent problems (e.g. mean variance problems)
  - ◇ Karnam-Ma-Z. (2017)
- (Mean field) games with multiple equilibria
  - ◇ Feinstein-Rudloff-Z. (2022), Iseri-Z. (2021)
- $\cdots\cdots$

Multivariate optimization problem

- $(t, x) \in [0, T] \times \mathbb{R}^d$, $\alpha \in \mathcal{A}_{[t,T]}$,

$$X_s^{t,x,\alpha} = x + \int_t^s b(r, X_r^{t,x,\alpha}, \alpha_r)dr + \int_t^s \sigma(r, X_r^{t,x,\alpha}, \alpha_r)dB_r,$$

$$Y_s^{t,x,\alpha} = g(X_T^{t,x,\alpha}) + \int_s^T f(r, X_r^{t,x,\alpha}, Y_r^{t,x,\alpha}, Z_r^{t,x,\alpha}, \alpha_r)dr - \int_s^T Z_r^{t,x,\alpha} dB_r,$$

where $Y \in \mathbb{R}^m$ for some $m \geq 1$.

- Set value $\mathbb{V}(t, x) := \text{cl}\{Y_t^{t,x,\alpha} : \alpha \in \mathcal{A}_{[t,T]}\} \subset \mathbb{R}^m$.

- Our goal is to characterize $\mathbb{V} : [0, T] \times \mathbb{R}^d \to 2^{\mathbb{R}^m}$ through a set valued HJB equation

The scalar case

- When $m = 1$,

$$\mathbb{V}(t,x) = [\underline{v}(t,x),\ \overline{v}(t,x)]$$

where

$$\underline{v}(t,x) := \inf\Big\{ Y_t^{t,x,\alpha} : \alpha \in \mathcal{A}_{[t,T]} \Big\},$$
$$\overline{v}(t,x) := \sup\Big\{ Y_t^{t,x,\alpha} : \alpha \in \mathcal{A}_{[t,T]} \Big\}.$$

- The set valued HJB for $\mathbb{V}$ will be exactly the standard HJB equations for $\underline{v}$ and $\overline{v}$, but incorporating into one equation.

The moving scalarization

- When $m > 1$, given $\lambda \in \mathbb{R}^m$ with $|\lambda| = 1$,

$$\sup_{\alpha \in \mathcal{A}_{[0,T]}} \lambda \cdot Y_0^{0,x_0,\alpha} = \sup_{y \in \mathbb{V}(0,x_0)} \lambda \cdot y.$$

- Let $\alpha^*_{[0,T]}$ be an optimal control, and denote $X^*_t := X_t^{0,x_0,\alpha^*_{[0,t]}}$.

- Time inconsistency :

$$\alpha^*_{[t,T]} \text{ is typically not optimal for } \sup_{\alpha \in \mathcal{A}_{[t,T]}} \lambda \cdot Y_t^{t,X^*_t,\alpha}$$

- Moving scalarization : finding $\Lambda(t, X_{[0,t]})$ s.t. $\Lambda(0, x_0) = \lambda$ and

$$\alpha^*_{[t,T]} \text{ is optimal for } \sup_{\alpha \in \mathcal{A}_{[t,T]}} \Lambda(t, X^*_{[0,t]}) \cdot Y_t^{t,X^*_t,\alpha}$$

◇ Karnam-Ma-Z. (2017), Feinstein-Rudloff (2021)

Suface evolution equations

- $\mathbb{V}(t) \subset \mathbb{R}^m$: $v(t,y) = f(t,y,n,\partial_y n)$
  - ◇ mean curvature flows, cristal growth, flame propagation, $\cdots$
- Huge literature : Barles-Soner-Souganidis (1993), Giga (2006), $\cdots\cdots\cdots$
- 1st order ODE for $\mathbb{V}(t)$ vs 2nd order PDE for $\mathbb{V}(t,x)$
- Forward problem vs Backward problem
  - ◇ Time change ? Path dependence and information flow ?

Outline

Some notations

- $\mathcal{D}^m$:= set of closed domain in $\mathbb{R}^m$ with smooth boundary
- Consider $\mathbb{V} : [0, T] \times \mathbb{R}^d \to \mathcal{D}^m$
- $\mathbb{V}_b(t, x)$:= boundary of $\mathbb{V}(t, x)$
- $\mathbb{G}_{\mathbb{V}} := \big\{(t, x, y) : (t, x) \in [0, T] \times \mathbb{R}^d, y \in \mathbb{V}_b(t, x)\big\}$, graph of $\mathbb{V}$
- $n(t, x, y)$:= outward unit normal vector
- $\mathcal{T}_{\mathbb{V}}$:= set of $\eta : \mathbb{G}_{\mathbb{V}} \to \mathbb{R}^m$ such that $\eta \cdot n = 0$

The 1st order derivatives

- For simplicity, consider $\mathbb{V} : \mathbb{R} \to \mathcal{D}^m$
- For $y \in \mathbb{V}_b(x)$ and $\varepsilon \approx 0$,

$$I_\varepsilon(x, y) := \text{projection of } y \text{ on } \mathbb{V}_b(x+\varepsilon)$$

Definition. $\partial_x \mathbb{V} : \mathbb{G}_{\mathbb{V}} \to \mathbb{R}^m$

$$I_\varepsilon(x, y) = y + \partial_x \mathbb{V}(x, y)\varepsilon + o(\varepsilon).$$

- **Lemma.** $\partial_x \mathbb{V}$ is parallel to n : $\partial_x \mathbb{V} = [\partial_x \mathbb{V} \cdot n]\ n$
  - ◇ $\partial_x \mathbb{V} \cdot n$ = a scalar derivative defined by Soner-Touzi (2002)

The intrinsic derivative of $f : \mathbb{G}_{\mathbb{V}} \to \mathbb{R}$

- Let $\hat{f}$ be a smooth extension of f to $y \in \mathbb{R}^m$, $\hat{\partial}_x \hat{f}$ and $\hat{\partial}_y \hat{f}$ the standard derivatives.

- For $(x, y) \in \mathbb{G}_{\mathbb{V}}$

$$\partial_y f := \hat{\partial}_y \hat{f} - [\hat{\partial}_y \hat{f} \cdot n] \in \mathcal{T}_{\mathbb{V}};$$

$$\partial_x f := \hat{\partial}_x \hat{f} + \hat{\partial}_y \hat{f} \cdot \partial_x \mathbb{V}.$$

  ◇ $\partial_x f$ is in general neither parallel nor orthogonal to n.

- $\partial_x f, \partial_y f$ are intrinsic : they are independent of the choice of $\hat{f}$.

The derivatives of $\mathbb{V} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^m$

- $\partial_t \mathbb{V}, \partial_{x_i} \mathbb{V} : (t, x, y) \in \mathbb{G}_{\mathbb{V}} \to \mathbb{R}^m$, parallel to n

- Define component wise for each $\partial_{x_i} \mathbb{V}^l : \mathbb{G}_{\mathbb{V}} \to \mathbb{R}$,

$$\partial_{y_k}\partial_{x_i}\mathbb{V}^l, \quad \partial_{x_j}\partial_{x_i}\mathbb{V}^l : \mathbb{G}_{\mathbb{V}} \to \mathbb{R}, \quad k, l = 1, \cdots, m, \quad i, j = 1, \cdots, d.$$

  - ◇ $\partial_{y_k}\partial_{x_i}\mathbb{V} \in \mathcal{T}_{\mathbb{V}}$, but $\partial_{x_j}\partial_{x_i}\mathbb{V}$ is not parallel or orthogonal to n
  - ◇ $\partial_{x_i}\partial_{y_k}\mathbb{V}$ has no meaning
  - ◇ In general $\partial_{x_j}\partial_{x_i}\mathbb{V} \neq \partial_{x_i}\partial_{x_j}\mathbb{V}$

- Lemma. $\partial_{x_j}\partial_{x_i}\mathbb{V} \cdot n = \partial_{x_i}\partial_{x_j}\mathbb{V} \cdot n$

The Itô formula

- Roughly, for $dX_t = b_t dt + \sigma_t dB_t$,

$$d\mathbb{V}(t, X_t) = \Big[\partial_t \mathbb{V} + \partial_x \mathbb{V} \cdot b_t + \frac{1}{2}\mathrm{tr}\left(\partial_{xx}\mathbb{V}\sigma\sigma_t^\top\right)\Big] dt + \partial_x \mathbb{V}\sigma_t dB_t.$$

◇ $\mathrm{tr}\left(\partial_{xx}\mathbb{V}\sigma\sigma^\top\right) := \sum_{i,j=1}^d \partial_{x_j x_i}\mathbb{V}(\sigma\sigma^\top)_{ij} \in \mathbb{R}^m$

Itô formula

Let $\mathbb{V} \in C^{1,2}$ and $dX_t = b_t dt + \sigma_t dB_t$. For $y \in \mathbb{V}_b(0, X_0)$, consider

$$\Gamma_t^y := y + \int_0^t \partial_x \mathbb{V}(s, X_s, \Gamma_s^y)\sigma_s dB_s$$
$$+ \int_0^t \Big[\partial_t \mathbb{V} + \partial_x \mathbb{V} \cdot b_s + \frac{1}{2}\mathrm{tr}\left(\partial_{xx}\mathbb{V}\sigma\sigma_s^\top\right)\Big](s, X_s, \Gamma_s^y) ds.$$

Then $\mathbb{V}_b(t, X_t) = \{\Gamma_t^y : y \in \mathbb{V}_b(0, X_0)\}$, a.s.

The (generalized) Itô formula

Itô formula

Let $\mathbb{V} \in C^{1,2}$ and $dX_t = b_t dt + \sigma_t dB_t$. Fix $\xi, \eta \in \mathcal{T}_{\mathbb{V}}$ smooth. For $y \in \mathbb{V}_b(0, X_0)$, consider

$$\Gamma_t^y := y + \int_0^t \left[\partial_x \mathbb{V}(s, X_s, \Gamma_s^y)\sigma_s + \eta(s, X_s, \Gamma_s^y)\right] dB_s$$
$$+ \int_0^t \Big[\partial_t \mathbb{V} + \partial_x \mathbb{V} \cdot b_s + \frac{1}{2}\mathrm{tr}\left(\partial_{xx}\mathbb{V}\sigma\sigma_s^\top\right)$$
$$+\xi - \mathrm{tr}\left(\eta^\top \partial_x n\sigma + \frac{1}{2}\partial_y n \; \eta\eta^\top\right) n\Big](s, X_s, \Gamma_s^y) ds.$$

Then $\mathbb{V}_b(t, X_t) = \{\Gamma_t^y : y \in \mathbb{V}_b(0, X_0)\}$, a.s.

Outline

Recall our model

- $(t,x) \in [0,T] \times \mathbb{R}^d$, $\alpha \in \mathcal{A}_{[t,T]}$,

$$X_s^{t,x,\alpha} = x + \int_t^s b(r, X_r^{t,x,\alpha}, \alpha_r)dr + \int_t^s \sigma(r, X_r^{t,x,\alpha}, \alpha_r)dB_r,$$

$$Y_s^{t,x,\alpha} = g(X_T^{t,x,\alpha}) + \int_s^T f(r, X_r^{t,x,\alpha}, Y_r^{t,x,\alpha}, Z_r^{t,x,\alpha}, \alpha_r)dr - \int_s^T Z_r^{t,x,\alpha} dB_r,$$

where $Y \in \mathbb{R}^m$ for some $m \geq 1$.

- Set value $\mathbb{V}(t,x) := \mathrm{cl}\Big\{ Y_t^{t,x,\alpha} : \alpha \in \mathcal{A}_{[t,T]} \Big\}$.

The set valued HJB equation

- The HJB equation : for $(t,x,y) \in \mathbb{G}_{\mathbb{V}}$

$$\sup_{\eta \in \mathcal{T}_{\mathbb{V}}, a \in A} n \cdot \Big[\partial_t \mathbb{V} + \partial_x \mathbb{V} \cdot b + \frac{1}{2}\mathrm{tr}\left(\partial_{xx}\mathbb{V}\sigma\sigma^\top\right) + f(\cdot, \partial_x \mathbb{V}\sigma + \eta, \cdot) - \mathrm{tr}\left(\eta^\top \partial_x n \sigma + \frac{1}{2}\partial_y n\ \eta\eta^\top\right) n\Big](t,x,y,a) = 0; \tag{1}$$

$\mathbb{V}(T,x) = \{g(x)\}.$

  - ◇ n is part of the solution.
  - ◇ Ok to consider non-degenerate $\mathbb{V}(T,x)$.
  - ◇ Recall : $n \cdot \partial_{xx}\mathbb{V} \in \mathbb{R}^{d\times d}$ is symmetric.

The 1-d case : $m = 1$

- In this case we have

$$\mathbb{V}_b(t,x) = \{\overline{y}, \underline{y}\} := \{\overline{v}(t,x), \underline{v}(t,x)\},$$

$$n(t,x,\overline{y}) = 1, \quad n(t,x,\underline{y}) = -1, \quad \eta \equiv 0,$$

$$\partial_t \mathbb{V}(t,x,\overline{y}) = \partial_t \overline{v}(t,x), \quad \partial_t \mathbb{V}(t,x,\underline{y}) = \partial_t \underline{v}(t,x), \quad \cdots$$

- Then (1) becomes the standard HJB equations :

$$\sup_{a\in A}\Big[\partial_t \overline{v} + \partial_x \overline{v}\cdot b + \frac{1}{2}\mathrm{tr}\left(\partial_{xx}\overline{v}\sigma\sigma^\top\right) + f(t,x,\overline{v},\partial_x\overline{v},a)\Big](t,x,a) = 0;$$

$$\sup_{a\in A} -\Big[\partial_t \underline{v} + \partial_x \underline{v}\cdot b + \frac{1}{2}\mathrm{tr}\left(\partial_{xx}\underline{v}\sigma\sigma^\top\right) + f(t,x,\underline{v},\partial_x\underline{v},a)\Big](t,x,a) = 0.$$

The main result

Theorem

(i) If $\mathbb{V}$ is smooth, then it is the unique classical solution of (1).

(ii) Assume further that the Hamiltonian has smooth optimal arguments η^* and $a^* = I(t,x,y)$. Denote further :

$$\xi^* := -\Big[\partial_t \mathbb{V} + \cdots + f(t,x,y,\partial_x \mathbb{V} + \eta^*, I^*)\Big](t,x,y,I^*) \in \mathcal{T}_{\mathbb{V}}.$$

Then, for any $y \in \mathbb{V}_b(0,x)$, we have $y = Y_0^{0,x,\alpha^*}$ where

$$\alpha_t^* = I^*(t, X_t^*, \Gamma_t^y);$$

$$X_t^* = x + \int_0^t b(s, X_s^*, I^*(s, X_s^*, \Gamma_s^y))ds + \int_0^t \sigma(\cdot)dB_s;$$

$$\Gamma_t^y = y + \int_0^t \big[\partial_x \mathbb{V}\sigma(\cdot) + \eta^*(s, X_s, \Gamma_s^y)\big]dB_s + \int_0^t [\cdots + \xi^*(\cdot)]ds.$$

The moving scalarization

- Given $\lambda \in \mathbb{R}^m$ with $|\lambda| = 1$, first $\exists!\ y_\lambda \in \mathbb{V}_b(0, x_0)$ such that

$$\sup_{y \in \mathbb{V}(0,x_0)} \lambda \cdot y = \lambda \cdot y_\lambda.$$

In particular, $n(0, x_0, y_\lambda) = \lambda$, the above α^* corresponding to y_λ is optimal, and $X^*_t = X^{0,x_0,\alpha^*}_t$.

- Γ^{y_λ} and hence $n(t, X^*_t, \Gamma^{y_\lambda}_t)$ are $\mathbb{F}^{X^*}$-progressively measurable, hence there exists Λ such that

$$\Lambda(t, X^*_{[0,t]}) = n(t, X^*_t, \Gamma^{y_\lambda}_t).$$

- When $\mathbb{V}(t, X^*_t)$ is convex, $\alpha^*_{[t,T]}$ is optimal for

$$\sup_{\alpha} \Lambda(t, X^*_{[0,t]}) \cdot Y^{t,X^*_t,\alpha}_t.$$

  ◇ The optimality is local if $\mathbb{V}$ is not convex.

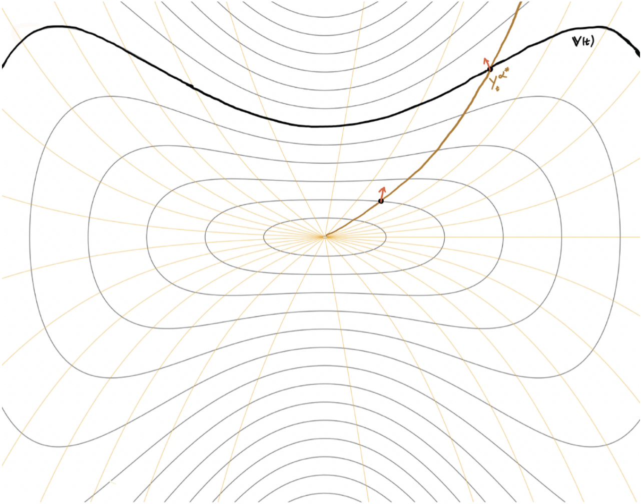
An example

- $d = 1, b = 0, \sigma = 1$
- $m = 2,\ A = \{a \in \mathbb{R}^2 : |a| \leq 1\},\ g = 0$, and

$$f(y,z,a) = \Big(a_1, \quad \frac{2y_1y_2}{\frac{1}{2}+y_1^2}a_1 + (\frac{1}{2}+y_1^2)a_2 - \frac{1}{\frac{1}{2}+y_1^2}\big[y_1z_1^2 - \frac{4y_1^2y_2z_1^2}{\frac{1}{2}+y_1^2} + 2y_1y_2z_1\big]\Big).$$

- Then $\mathbb{V}$ is independent of x and

$$\mathbb{V}_b(t) = \Big\{\Big([T-t]\cos\theta, \quad [T-t]\sin\theta[\frac{1}{2} + (T-t)^2\cos^2(\theta)]\Big) : \theta \in [0, 2\pi]\Big\}.$$


$\mathbb{V}(t)$
$Y_t^{\alpha^*}$

Further research

- Sufficient conditions for the existence of classical solutions
- Weak/viscosity solutions
- Degenerate case (e.g. games)
- Infinitely dimensional case (e.g. mean field games)
- Efficient numerical methods

Thank you very much for your attention !