Central limit theorems over nonlinear functionals of measures and fluctuations of mean-field interacting particle systems

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Central Limit Theorem

Let $\ell \geq 0$, $\mathcal{P}_\ell(\mathbb{R}^d) = \{\text{prob. meas. } \eta \text{ on } \mathbb{R}^d \text{ s.t. } \int_{\mathbb{R}^d} |x|^{\ell} \eta(dx) < \infty\}$, and $U : \mathcal{P}_\ell(\mathbb{R}^d) \to \mathbb{R}$ be the linear functional defined by

$$U(\mu) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx),$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a measurable s.t. $\sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{1 + |x|^{\ell/2}} < \infty$.

Let $m^N = \frac{1}{N} \sum_{i=1}^N \delta_{\zeta_i}$ where $(\zeta_i)_{i \geq 1}$ are i.i.d. according to $m_0 \in \mathcal{P}_\ell(\mathbb{R}^d)$. We have

$$\sqrt{N}(U(m^N) - U(m_0)) = \sqrt{N} \left( \frac{1}{N} \sum_{i=1}^N \varphi(\zeta_i) - \mathbb{E}[\varphi(\zeta_1)] \right)$$

$\xrightarrow{(d)} N_1(0, \text{Var}(\varphi(\zeta_1)))$ as $N \to \infty$. 

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1. Generalization to nonlinear functionals

2. Fluctuations of interacting particle systems
Linear functional derivative

Notion first introduced by Cardaliaguet, Delarue, Lasry, Lions and used in the literature on mean-field games.

Definition

Let $\ell \geq 0$. A functional $U : \mathcal{P}_\ell(\mathbb{R}^d) \to \mathbb{R}$ admits a linear functional derivative at $\mu \in \mathcal{P}_\ell(\mathbb{R}^d)$ if there exists a measurable function $\mathbb{R}^d \ni y \mapsto \frac{\delta U}{\delta m}(\mu, y)$ such that $\sup_{y \in \mathbb{R}^d} \left| \frac{\delta U}{\delta m}(\mu, y) \right| / (1 + |y|^{\ell}) < \infty$ and

$$\forall \nu \in \mathcal{P}_\ell(\mathbb{R}^d), \quad \lim_{\varepsilon \to 0^+} \frac{U(\mu + \varepsilon(\nu - \mu)) - U(\mu)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(\mu, y)(\nu - \mu)(dy).$$

For $\varphi : \mathbb{R}^d \to \mathbb{R}$ measurable such that $\sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{1 + |x|^{\ell}} < \infty$, the linear functional $U(\mu) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx)$ admits a linear functional derivative at each $\mu \in \mathcal{P}_\ell(\mathbb{R}^d)$ given by $\frac{\delta U}{\delta m}(\mu, y) = \varphi(y)$. 


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$$\forall \nu \in \mathcal{P}_\ell(\mathbb{R}^d), \lim_{\varepsilon \to 0^+} \frac{U(\mu + \varepsilon(\nu - \mu)) - U(\mu)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(\mu, y) (\nu - \mu)(dy).$$

For $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ measurable such that $\sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{1 + |x|^\ell} < \infty$, the linear functional $U(\mu) = \int_{\mathbb{R}^d} \varphi(x) \mu(dx)$ admits a linear functional derivative at each $\mu \in \mathcal{P}_\ell(\mathbb{R}^d)$ given by $\frac{\delta U}{\delta m}(\mu, y) = \varphi(y)$. 
Related integral calculus

**Lemma**

Let $\ell \geq 0$, $m, m' \in \mathcal{P}_\ell(\mathbb{R}^d)$, and suppose that the linear functional derivative of a functional $U : \mathcal{P}_\ell(\mathbb{R}^d) \to \mathbb{R}$ exists on the segment $(m_s := sm' + (1 - s)m)_{s \in [0,1]}$ and that

$$\sup_{(s,y) \in [0,1] \times \mathbb{R}^d} \left| \frac{\delta U}{\delta m}(m_s, y) \right| / (1 + |y|^\ell) < \infty.$$ 

Then

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_s, y)(m' - m)(dy) \, ds.$$
Generalization of the CLT to nonlinear functionals

For $i \in \{1, \cdots, N\}$ and $s \in [0, 1]$, write

$$m_{s}^{N,i} := \frac{N - (i - 1) - s}{N} m_0 + \frac{1}{N} \sum_{j=1}^{i-1} \delta_{\zeta_j} + \frac{s}{N} \delta_{\zeta_i}.$$  

Using $m_{0}^{N,i} = m_{1}^{N,i-1}$, one has the telescoping sum

$$U(m^{N}) - U(m_{0}) = U(m_{1}^{N,N}) - U(m_{0}^{N,1}) = \sum_{i=1}^{N} \left( U(m_{1}^{N,i}) - U(m_{0}^{N,i}) \right)$$  

$$= \sum_{i=1}^{N} \int_{s=0}^{1} \int_{\mathbb{R}^{d}} \frac{\delta U}{\delta m_{s}^{N,i}}(m_{s}^{N,i}, x) \frac{\delta_{\zeta_i} - m_0}{N} (dx) ds.$$  

Note that $m_{0}^{N,i} = m_{1}^{N,i-1} = \frac{N-(i-1)}{N} m_0 + \frac{1}{N} \sum_{j=1}^{i-1} \delta_{\zeta_j}$ is indep. from $\zeta_i$. 
Generalization of the CLT to nonlinear functionals $U$

\[
\sqrt{N}(U(m^N) - U(m_0)) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_0^{N,i}, x)(\delta \zeta - m_0) \cdot (dx) + R_N
\]

\[
R_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{s=0}^{1} \int_{\mathbb{R}^d} \left( \frac{\delta U}{\delta m}(m_s^{N,i}, x) - \frac{\delta U}{\delta m}(m_0^{N,i}, x) \right)(\delta \zeta - m_0) \cdot (dx)ds
\]

Find assumptions on $U$ ensuring that

- by the Central Limit Theorem for martingales and using

\[
\sup_{1 \leq i \leq N, s \in [0,1]} W_{\ell}(m_s^{N,i}, m_0) \to 0,
\]

the first term converges in law to $\mathcal{N}_1(0, \text{Var}(\frac{\delta U}{\delta m}(m_0, \zeta_1)))$,

- the remainder $R_N$ vanishes as $N \to \infty$. 

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\[ \sqrt{N}(U(m^N) - U(m_0)) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \delta U \left( m_0^N, x \right) (\delta \zeta_i - m_0)(dx) + R_N \]

\[ R_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_{s=0}^{1} \int_{\mathbb{R}^d} \left( \frac{\delta U}{\delta m} \left( m_s^N, x \right) - \frac{\delta U}{\delta m} \left( m_0^N, x \right) \right) (\delta \zeta_i - m_0)(dx)ds \]

Find assumptions on $U$ ensuring that

- by the Central Limit Theorem for martingales and using

\[ \sup_{1 \leq i \leq N, s \in [0, 1]} W_{\ell} \left( m_s^N, m_0 \right) \xrightarrow{N \to \infty} 0, \]

the first term converges in law to $\mathcal{N}_1(0, \text{Var} \left( \frac{\delta U}{\delta m} \left( m_0, \zeta_1 \right) \right))$,

- the remainder $R_N$ vanishes as $N \to \infty$. 
Remarks

- For linear functionals $U(\mu) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx)$, $\frac{\delta U}{\delta m}(\mu, x) = \varphi(x)$ and $\text{Var}(\frac{\delta U}{\delta m}(m_0, \zeta_1)) = \text{Var}(\varphi(\zeta_1))$.

- To prove the CLT, our decomposition requires less regularity on $U$ than the one previously considered by Delarue, Lacker, Ramanan EJP 19 (see also Szpruch and Tse AAP 21) to prove that $\sup_{N \in \mathbb{N}} N^2 \mathbb{E}[(U(m^N) - U(m_0))^4] < \infty$:

\[
U(m^N) - U(m_0) = \frac{1}{N} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_0, x)(\delta \zeta_i - m_0)(dx) + \frac{1}{N} \sum_{i=1}^{N} \int_{s=0}^{1} \int_{\mathbb{R}^d} \left( \frac{\delta U}{\delta m}(m^N_{s,i}, x) - \frac{\delta U}{\delta m}(m_0, x) \right)(\delta \zeta_i - m_0)(dx)ds
\]

\[
\text{TV}(m^N_{s,i}, m_0) \leq \frac{i+s-1}{N}
\]
Remarks

- For linear functionals $U(\mu) = \int_{\mathbb{R}^d} \varphi(x) \mu(dx)$, $\frac{\delta U}{\delta m}(\mu, x) = \varphi(x)$ and $\text{Var}(\frac{\delta U}{\delta m}(m_0, \zeta_1)) = \text{Var}(\varphi(\zeta_1))$.

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\]

\[
+ \frac{1}{N} \sum_{i=1}^{N} \int_{s=0}^{1} \int_{\mathbb{R}^d} \left( \frac{\delta U}{\delta m}(m_s^{N,i}, x) - \frac{\delta U}{\delta m}(m_0, x) \right) (\delta \zeta_i - m_0)(dx) ds
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$\text{TV}(m_s^{N,i}, m_0) \leq \frac{i+s-1}{N}$
Theorem

Let \( \ell \geq 0 \), \( m_0 \in \mathcal{P}_\ell(\mathbb{R}^d) \) and \( m^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\zeta_i} \), with \( (\zeta_i)_{i \geq 1} \) i.i.d. \( \sim m_0 \). Suppose that there exists \( r > 0 \) such that

- \( U \) admits a linear functional derivative on the ball \( B_{W_\ell}(m_0, r) \),
- \( \exists C < \infty, \forall (\mu, x) \in B_{W_\ell}(m_0, r) \times \mathbb{R}^d, \left| \frac{\delta U}{\delta m}(\mu, x) \right| \leq C \left( 1 + |x|^{\ell/2} \right) \)
- \( \exists \alpha \in (1/2, 1], \exists C < \infty, \forall \mu_1, \mu_2 \in B_{W_\ell}(m_0, r), \forall x \in \mathbb{R}^d, \left| \frac{\delta U}{\delta m}(\mu_2, x) - \frac{\delta U}{\delta m}(\mu_1, x) \right| \leq C(1 + |x|^{\ell})TV^\alpha(\mu_2, \mu_1), \)
- \( \sup_{x \in \mathbb{R}^d} \left| \frac{\delta U}{\delta m}(\mu, x) - \frac{\delta U}{\delta m}(m_0, x) \right| \) converges to 0 when \( W_\ell(\mu, m_0) \rightarrow 0 \).

Then the following convergence in distribution holds:

\[
\sqrt{N} \left( U(m^N) - U(m_0) \right) \xrightarrow{d} \mathcal{N} \left( 0, \text{Var} \left( \frac{\delta U}{\delta m}(m_0, \zeta_1) \right) \right).
\]
Wasserstein distances

- For $\ell > 0$, we endow $P_\ell(\mathbb{R}^d)$ with the Wasserstein distance:

  $$W_\ell(\mu, \nu) = \left( \inf_{\pi \in P(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^\ell \pi(dx, dy) \right)^{1/\ell \vee 1}$$

  where $P(\mu, \nu) = \{ \pi \in P_\ell(\mathbb{R}^{2d}) : \forall A \in \mathcal{B}(\mathbb{R}^d), \pi(A \times \mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d \times A) = \nu(A) \}$

- We endow $P_0(\mathbb{R}^d)$ with

  $$W_0(\mu, \nu) = \inf_{\pi \in P(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} (|y - x| \wedge 1) \pi(dx, dy),$$

  which metricizes the weak convergence topology.

- When $m_0$ is discrete, for $D_\ell(\mu_2, \mu_1) := \int_{\mathbb{R}^d} (1 + |y|^\ell) |\mu_2 - \mu_1|(dy)$,

  $$\sup_{1 \leq i \leq N, s \in [0,1]} D_\ell(m^{N,i}_s, m_0) \xrightarrow{N \to \infty} 0,$$

  and we may replace $W_\ell$ by the stronger metric $D_\ell$ in the hypotheses.
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$$\sup_{1 \leq i \leq N, s \in [0,1]} D_\ell(m_s^{N,i}, m_0) \to 0 \text{ as } N \to \infty,$$

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Related results in the statistical literature

- **Boos Serfling Ann. Stat 80** dimension \( d = 1 \), existence of a Gateaux differential of \( U \) at \( m_0 \) linear in the measure such that

\[
U(m^N) - U(m_0) - \frac{1}{N} \sum_{i=1}^{N} dU(m_0, \delta_{\xi_i} - m_0) = o \left( \left\| (m^N - m_0)((-\infty, \cdot]) \right\|_\infty \right)
\]

(almost amounts to Fréchet diff. at \( m_0 \) w.r.t. Kolmogorov dist.).

- **Dudley 90** : "Gateaux derivative considered too weak"
  Existence of a class \( \mathcal{F} \) of measurable functions s.t.
  - Fréchet differentiability at \( m_0 \) with respect to
    \[
    \left\| \mu - m_0 \right\| = \sup_{f \in \mathcal{F}} \left| \int f(x)(\mu - m_0)(dx) \right|
    \]
  - a CLT for empirical measures holds with respect to uniform convergence over \( \mathcal{F} \).

→ balance needed.

Integral calculus related to the (Gateaux) linear functional derivative
→ versatile tool permitting to go beyond the i.i.d. case (Flenghi J. 22)
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Interacting particle system

\[
\begin{cases}
Y_{t}^{i,N} = \zeta_i + \int_{0}^{t} b(Y_{s}^{i,N}, \mu_{s}^{N}) \, ds + \int_{0}^{t} \sigma(Y_{s}^{i,N}, \mu_{s}^{N}) \, dW_{s}^{i}, & 1 \leq i \leq N, \quad t \geq 0, \\
\mu_{s}^{N} := \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_{s}^{i,N}},
\end{cases}
\]

where
- \( b : \mathbb{R}^{d} \times \mathcal{P}_2(\mathbb{R}^{d}) \rightarrow \mathbb{R}^{d}, \sigma : \mathbb{R}^{d} \times \mathcal{P}_2(\mathbb{R}^{d}) \rightarrow \mathbb{R}^{d \times d'} \) Lipschitz,
- \((W_{i}, \zeta_{i})_{i \geq 1}\ i.i.d.\ with\ W_{i}\ a\ d'-dimensional\ Brownian\ motion\ independent\ from\ the\ \mathbb{R}^{d}-valued\ initial\ random\ vector\ \zeta_{i} \sim m_0.\)

Mean-field limit as \( N \rightarrow \infty \): SDE nonlinear in the sense of McKean

\[
\begin{cases}
X_{t} = \zeta + \int_{0}^{t} b(X_{s}, \mu_{s}^{\infty}) \, ds + \int_{0}^{t} \sigma(X_{s}, \mu_{s}^{\infty}) \, dW_{s}, & t \geq 0, \\
\mu_{s}^{\infty} := \text{Law}(X_{s}),
\end{cases}
\]

with \( W \ d'-dimensional\ Brownian\ motion \perp \zeta \sim m_0.\)

Question: for \( \Phi : \mathcal{P}_2(\mathbb{R}^{d}) \rightarrow \mathbb{R} \) a nonlinear functional, limiting behaviour of the fluctuations process

\[
(\sqrt{N}[\Phi(\mu_{t}^{N}) - \Phi(\mu_{t}^{\infty})])_{t \geq 0}?
\]
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Question : for \(\Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) a nonlinear functional, limiting behaviour of the fluctuations process

\[ (\sqrt{N} [\Phi(\mu^{N}_t) - \Phi(\mu^{\infty}_t)])_{t > 0}? \]
Regularity class

**Definition**

A function $f : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ belongs to class $\mathcal{M}_k(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, if the derivatives $D^{(j,n,\beta)} f(x, \mu, y_1, \ldots, y_n)$ exist for every multi-index $(j, n, \beta)$ such that $|(j, n, \beta)| \leq k$ and satisfy

$$\forall x, y_1, \ldots, y_n \in \mathbb{R}^d, \forall \mu \in \mathcal{P}_2(\mathbb{R}^d), \quad |D^{(j,n,\beta)} f(x, \mu, y_1, \ldots, y_n)| \leq C$$

and

$$\forall x', y'_1, \ldots, y'_n \in \mathbb{R}^d, \forall \mu' \in \mathcal{P}_2(\mathbb{R}^d),$$

$$\left|D^{(j,n,\beta)} f(x, \mu, y_1, \ldots, y_n) - D^{(j,n,\beta)} f(x', \mu', y'_1, \ldots, y'_n)\right| \leq C \left(|x - x'| + \sum_{i=1}^n |y_i - y'_i| + W_2(\mu, \mu')\right)$$

The $n$ derivatives w.r.t. $\mu$ are taken in the Lions sense

$$D^{(0,1,0)} f(x, \mu) = \partial_{y_1} \frac{\delta U}{\delta m}(x, \mu, y_1).$$
Master equation

Let $\mathcal{V} : \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\mathcal{V}(t, \mathcal{L}(\theta)) = \Phi(\mathcal{L}(X^\theta_t))$$

where, for $\theta$ a square integrable $\mathbb{R}^d$-valued initial random vector $\perp \mathcal{W}$,

$$X^\theta_t = \theta + \int_0^t b(X^\theta_s, \mathcal{L}(X^\theta_s)) \, ds + \int_0^t \sigma(X^\theta_s, \mathcal{L}(X^\theta_s)) \, d\mathcal{W}_s,$$

$t \geq 0$.

Buckdahn, Li, Peng, Rainer AP 17 \rightarrow if $\Phi \in \mathcal{M}_2(\mathcal{P}_2(\mathbb{R}^d))$, $b, \sigma \in \mathcal{M}_2(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, then $\mathcal{V}$ satisfies the master equation:

$$\begin{cases}
\partial_s \mathcal{V}(s, \mu) = \int_{\mathbb{R}^d} \left[ \partial_\mu \mathcal{V}(s, \mu)(x) \cdot b(x, \mu) + \frac{1}{2} \text{Tr}(\partial_x \partial_\mu \mathcal{V}(s, \mu)(x)a(x, \mu)) \right] \mu(dx) \\
\mathcal{V}(0, \mu) = \Phi(\mu),
\end{cases}$$

where $a(x, \mu) \coloneqq \sigma(x, \mu)\sigma(x, \mu)^*$.
Master equation

Let $\mathcal{V}: \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ defined by

$$\mathcal{V}(t, \mathcal{L}(\theta)) = \Phi(\mathcal{L}(X^\theta_t))$$

where, for $\theta$ a square integrable $\mathbb{R}^d$-valued initial random vector $\perp \mathcal{W}$,

$$X^\theta_t = \theta + \int_0^t b(X^\theta_s, \mathcal{L}(X^\theta_s)) \, ds + \int_0^t \sigma(X^\theta_s, \mathcal{L}(X^\theta_s)) \, d\mathcal{W}_s, \quad t \geq 0.$$ 

Buckdahn, Li, Peng, Rainer AP 17 → if $\Phi \in \mathcal{M}_2(\mathcal{P}_2(\mathbb{R}^d))$, $b, \sigma \in \mathcal{M}_2(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, then $\mathcal{V}$ satisfies the master equation:

$$\begin{cases}
\partial_s \mathcal{V}(s, \mu) = \int_{\mathbb{R}^d} \left[ \partial_\mu \mathcal{V}(s, \mu)(x) \cdot b(x, \mu) + \frac{1}{2} \text{Tr}(\partial_x \partial_\mu \mathcal{V}(s, \mu)(x) a(x, \mu)) \right] \mu(dx) \\
\mathcal{V}(0, \mu) = \Phi(\mu),
\end{cases}$$

where $a(x, \mu) := \sigma(x, \mu)\sigma(x, \mu)^*$. 

\[ \sqrt{N} \left[ \Phi(\mu^N_t) - \Phi(\mu^\infty_t) \right] = \sqrt{N} (\mathcal{V}(t, m^N) - \mathcal{V}(t, m_0)) \]

\[ + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_0^t \partial_\mu \mathcal{V}^* (t - s, \mu^N_s) (Y_{s,i}^N) \sigma(Y_{s,i}^N, \mu^N_s) dW_i^s \equiv I_t^N \]

\[ + \int_0^t \frac{1}{2} \sum_{i=1}^{N} \left[ \frac{1}{N^{3/2}} \operatorname{Tr} \left( a(Y_{s,i}^N, \mu^N_s) \partial_\mu^2 \mathcal{V} (t - s, \mu^N_s) (Y_{s,i}^N, Y_{s,i}^N) \right) \right] ds. \]

\[ \langle I_t^N \rangle_t = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \partial_\mu \mathcal{V}^* (t - s, \mu^N_s) (Y_{s,i}^N) a(Y_{s,i}^N, \mu^N_s) \partial_\mu \mathcal{V} (t - s, \mu^N_s) (Y_{s,i}^N) ds \]

\[ \overset{N\to\infty}{\longrightarrow} \int_0^t \int_{\mathbb{R}^d} \partial_\mu \mathcal{V}^* (t - s, \mu^\infty_s) (y) a(y, \mu^\infty_s) \mathcal{V} (t - s, \mu^\infty_s) (y) \mu^\infty_s (dy) ds. \]
Generalization to nonlinear functionals

Fluctuations of interacting particle systems

Central limit theorems over nonlinear functions of measures

\[ \sqrt{N} \left[ \Phi(\mu_t^N) - \Phi(\mu_t^\infty) \right] = \sqrt{N} \left( \mathcal{V}(t, m^N) - \mathcal{V}(t, m_0) \right) \]

fluctuations of the initial emp. meas.

\[ + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \int_0^t \partial_\mu \mathcal{V}^*(t - s, \mu_s^N)(Y_s^{i,N})\sigma(Y_s^{i,N}, \mu_s^N) dW^i_s \quad := I^N_t \]

\[ + \int_0^t \frac{1}{2} \sum_{i=1}^{N} O(N^{-1/2}) \right \} \right] ds. \]

\[ \left\langle I^N_t \right\rangle = \frac{1}{N} \sum_{i=1}^{N} \int_0^t \partial_\mu \mathcal{V}^*(t - s, \mu_s^N)(Y_s^{i,N}) a(Y_s^{i,N}, \mu_s^N) \partial_\mu \mathcal{V}(t - s, \mu_s^N)(Y_s^{i,N}) ds \]

\[ \lim_{N \to \infty} \int_0^t \int_{\mathbb{R}^d} \partial_\mu \mathcal{V}^*(t - s, \mu_s^\infty)(y) a(y, \mu_s^\infty) \mathcal{V}(t - s, \mu_s^\infty)(y) \mu_s^\infty(dy) ds. \]
Suppose $m_0 \in \mathcal{P}_{12}(\mathbb{R}^d)$, $\Phi \in \mathcal{M}_5(\mathcal{P}_2(\mathbb{R}^d))$, $b, \sigma \in \mathcal{M}_5(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ and uniformly bounded. Then, in $C(\mathbb{R}_+, \mathbb{R})$, the fluctuations process
\[
(\sqrt{N}[\Phi(\mu^N_t) - \Phi(\mu^\infty_t)])_{t \geq 0}
\]
converges weakly to a centered Gaussian process $L$ with covariance
\[
\text{Cov}(L_t, L_u) = \text{Cov}\left(\frac{\delta \nu}{\delta m}(t, m_0, \xi_1), \frac{\delta \nu}{\delta m}(u, m_0, \xi_1)\right) + \int_0^{t \wedge u} \int_{\mathbb{R}^d} \partial_\mu \nu^*(t - s, \mu^\infty_s)(y) a(y, \mu^\infty_s) \nu(u - s, \mu^\infty_s)(y) \mu^\infty_s(dy) ds.
\]