Central limit theorems over nonlinear functionals of measures and fluctuations of mean-field interacting particle systems

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Central Limit Theorem

Let $\ell \geq 0$, $\mathcal{P}_\ell(\mathbb{R}^d) = \{$prob. meas. η on $\mathbb{R}^d$ s.t. $\int_{\mathbb{R}^d} |x|^\ell \eta(dx) < \infty\}$, and $U : \mathcal{P}_\ell(\mathbb{R}^d) \to \mathbb{R}$ be the linear functional defined by

$$U(\mu) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx),$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is a measurable s.t. $\sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{1+|x|^{\ell/2}} < \infty$.

Let $m^N = \frac{1}{N}\sum_{i=1}^N \delta_{\zeta_i}$ where $(\zeta_i)_{i\geq 1}$ are i.i.d. according to $m_0 \in \mathcal{P}_\ell(\mathbb{R}^d)$. We have

$$\sqrt{N}(U(m^N) - U(m_0)) = \sqrt{N}\left(\frac{1}{N}\sum_{i=1}^N \varphi(\zeta_i) - \mathbb{E}[\varphi(\zeta_1)]\right)$$

$$\xrightarrow{(d)} \mathcal{N}_1(0, \mathrm{Var}(\varphi(\zeta_1))) \text{ as } N \to \infty.$$

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1. Generalization to nonlinear functionals

2. Fluctuations of interacting particle systems

Linear functional derivative

Notion first introduced by *Cardaliaguet, Delarue, Lasry, Lions* and used in the literature on mean-field games.

Definition

Let $\ell \geq 0$. A functional $U : \mathcal{P}_\ell(\mathbb{R}^d) \to \mathbb{R}$ admits a linear functional derivative at $\mu \in \mathcal{P}_\ell(\mathbb{R}^d)$ if there exists a measurable function $\mathbb{R}^d \ni y \mapsto \frac{\delta U}{\delta m}(\mu, y)$ such that $\sup_{y \in \mathbb{R}^d} \left|\frac{\delta U}{\delta m}(\mu, y)\right| / (1 + |y|^\ell) < \infty$ and

$$\forall \nu \in \mathcal{P}_\ell(\mathbb{R}^d), \lim_{\varepsilon \to 0^+} \frac{U(\mu + \varepsilon(\nu - \mu)) - U(\mu)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(\mu, y) \, (\nu - \mu)(dy).$$

For $\varphi : \mathbb{R}^d \to \mathbb{R}$ measurable such that $\sup_{x \in \mathbb{R}^d} \frac{|\varphi(x)|}{1+|x|^\ell} < \infty$, the linear functional $U(\mu) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx)$ admits a linear functional derivative at each $\mu \in \mathcal{P}_\ell(\mathbb{R}^d)$ given by $\frac{\delta U}{\delta m}(\mu, y) = \varphi(y)$.

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Related integral calculus

Lemma

Let $\ell \geq 0$, $m, m' \in \mathcal{P}_\ell(\mathbb{R}^d)$, and suppose that the linear functional derivative of a functional $U : \mathcal{P}_\ell(\mathbb{R}^d) \to \mathbb{R}$ exists on the segment $(m_s := sm' + (1-s)m)_{s\in[0,1]}$ and that

$$\sup_{(s,y)\in[0,1]\times\mathbb{R}^d} \left|\frac{\delta U}{\delta m}(m_s, y)\right| /(1+|y|^\ell) < \infty.$$

Then

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_s, y)(m' - m)(dy)\, ds.$$

Generalization of the CLT to nonlinear functionals

For $i \in \{1, \cdots, N\}$ and $s \in [0,1]$, write

$$m_s^{N,i} := \frac{N-(i-1)-s}{N} m_0 + \frac{1}{N}\sum_{j=1}^{i-1} \delta_{\zeta_j} + \frac{s}{N}\delta_{\zeta_i}.$$

Using $m_0^{N,i} = m_1^{N,i-1}$, one has the telescoping sum

$$U(m^N) - U(m_0) = U(m_1^{N,N}) - U(m_0^{N,1}) = \sum_{i=1}^{N} \left(U(m_1^{N,i}) - U(m_0^{N,i}) \right)$$

$$= \sum_{i=1}^{N} \int_{s=0}^{1} \int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_s^{N,i}, x) \frac{\delta_{\zeta_i} - m_0}{N}(dx) ds.$$

Note that $m_0^{N,i} = m_1^{N,i-1} = \frac{N-(i-1)}{N} m_0 + \frac{1}{N}\sum_{j=1}^{i-1} \delta_{\zeta_j}$ is indep. from ζ_i.

Generalization of the CLT to nonlinear functionals U

$$\sqrt{N}(U(m^N)-U(m_0))=\frac{1}{\sqrt{N}}\sum_{i=1}^N\overbrace{\int_{\mathbb{R}^d}\frac{\delta U}{\delta m}(m_0^{N,i},x)(\delta_{\zeta_i}-m_0)(dx)}^{\text{martingale increment}}+R_N$$

$$R_N=\frac{1}{\sqrt{N}}\sum_{i=1}^N\int_{s=0}^1\int_{\mathbb{R}^d}\underbrace{\left(\frac{\delta U}{\delta m}(m_s^{N,i},x)-\frac{\delta U}{\delta m}(m_0^{N,i},x)\right)}_{TV(m_s^{N,i},m_0^{N,i})\leq\frac{s}{N}}(\delta_{\zeta_i}-m_0)(dx)ds$$

Find assumptions on U ensuring that

- by the Central Limit Theorem for martingales and using

$$\sup_{1\leq i\leq N,s\in[0,1]}W_\ell(m_s^{N,i},m_0)\underset{N\to\infty}{\longrightarrow}0,$$

  the first term converges in law to $\mathcal{N}_1(0,\mathrm{Var}(\frac{\delta U}{\delta m}(m_0,\zeta_1)))$,
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Remarks

- For linear functionals $U(\mu) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx)$, $\frac{\delta U}{\delta m}(\mu, x) = \varphi(x)$ and $\mathrm{Var}(\frac{\delta U}{\delta m}(m_0, \zeta_1)) = \mathrm{Var}\left(\varphi(\zeta_1)\right)$.
- To prove the CLT, our decomposition requires less regularity on U than the one previously considered by *Delarue, Lacker, Ramanan EJP 19* (see also *Szpruch and Tse AAP 21*) to prove that $\sup_{N\in\mathbb{N}} N^2\mathbb{E}[(U(m^N) - U(m_0))^4] < \infty$:

$$U(m^N) - U(m_0) = \frac{1}{N}\sum_{i=1}^{N} \overbrace{\int_{\mathbb{R}^d} \frac{\delta U}{\delta m}(m_0, x)(\delta_{\zeta_i} - m_0)(dx)}^{\text{independent increments}}$$

$$+ \frac{1}{N}\sum_{i=1}^{N}\int_{s=0}^{1}\int_{\mathbb{R}^d} \underbrace{\left(\frac{\delta U}{\delta m}(m_s^{N,i}, x) - \frac{\delta U}{\delta m}(m_0, x)\right)}_{TV(m_s^{N,i}, m_0) \leq \frac{i+s-1}{N}}(\delta_{\zeta_i} - m_0)(dx)ds$$

Remarks

- For linear functionals $U(\mu)=\int_{\mathbb{R}^d}\varphi(x)\mu(dx)$, $\frac{\delta U}{\delta m}(\mu,x)=\varphi(x)$ and $\mathrm{Var}(\frac{\delta U}{\delta m}(m_0,\zeta_1))=\mathrm{Var}\left(\varphi(\zeta_1)\right)$.
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Theorem

Let $\ell \geq 0$, $m_0 \in \mathcal{P}_\ell(\mathbb{R}^d)$ and $m^N = \frac{1}{N}\sum_{i=1}^N \delta_{\zeta_i}$, with $(\zeta_i)_{i\geq 1}$ i.i.d. $\sim m_0$. Suppose that there exists $r > 0$ such that

- U admits a linear functional derivative on the ball $B_{W_\ell}(m_0, r)$,
- $\exists C < \infty,\ \forall(\mu, x) \in B_{W_\ell}(m_0, r) \times \mathbb{R}^d, \left|\frac{\delta U}{\delta m}(\mu, x)\right| \leq C\left(1 + |x|^{\ell/2}\right)$
- $\exists \alpha \in (1/2, 1],\ \exists C < \infty,\ \forall \mu_1, \mu_2 \in B_{W_\ell}(m_0, r)$, $\forall x \in \mathbb{R}^d, \left|\frac{\delta U}{\delta m}(\mu_2, x) - \frac{\delta U}{\delta m}(\mu_1, x)\right| \leq C(1 + |x|^\ell) TV^\alpha(\mu_2, \mu_1)$,
- $\sup_{x\in\mathbb{R}^d} \frac{\left|\frac{\delta U}{\delta m}(\mu,x) - \frac{\delta U}{\delta m}(m_0,x)\right|}{1+|x|^{\ell/2}}$ converges to 0 when $W_\ell(\mu, m_0) \to 0$.

Then the following convergence in distribution holds :

$$\sqrt{N}\Big(U(m^N) - U(m_0)\Big) \stackrel{d}{\Longrightarrow} \mathcal{N}\left(0, \mathrm{Var}\left(\frac{\delta U}{\delta m}(m_0, \zeta_1)\right)\right).$$

Wasserstein distances

- For $\ell > 0$, we endow $\mathcal{P}_\ell(\mathbb{R}^d)$ with the Wasserstein distance :

$$\text{for } \mu,\nu \in \mathcal{P}_\ell(\mathbb{R}^d),\ W_\ell(\mu,\nu) = \left(\inf_{\pi\in\mathcal{P}(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |y-x|^\ell \pi(dx,dy)\right)^{1/\ell\vee 1}$$

where $\mathcal{P}(\mu,\nu) = \{\pi \in \mathcal{P}_\ell(\mathbb{R}^{2d}) : \forall A \in \mathcal{B}(\mathbb{R}^d), \pi(A\times\mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d \times A) = \nu(A)\}$

- We endow $\mathcal{P}_0(\mathbb{R}^d)$ with

$$W_0(\mu,\nu) = \inf_{\pi\in\mathcal{P}(\mu,\nu)} \int_{\mathbb{R}^d\times\mathbb{R}^d} (|y-x|\wedge 1)\,\pi(dx,dy),$$

which metricizes the weak convergence topology.

- When m_0 is discrete, for $D_\ell(\mu_2,\mu_1) := \int_{\mathbb{R}^d}(1+|y|^\ell)|\mu_2-\mu_1|(dy)$, $\sup_{1\le i\le N, s\in[0,1]} D_\ell(m_s^{N,i}, m_0) \underset{N\to\infty}{\longrightarrow} 0$, and we may replace W_ℓ by the stronger metric D_ℓ in the hypotheses.

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Related results in the statistical literature

- *Boos Serfling Ann. Stat 80* dimension $d=1$, existence of a Gateaux differential of U at m_0 linear in the measure such that

$$U(m^N)-U(m_0)-\frac{1}{N}\sum_{i=1}^{N}dU(m_0,\delta_{\zeta_i}-m_0)=o\left(\left\|(m^N-m_0)((-\infty,\cdot])\right\|_\infty\right)$$

(almost amounts to Fréchet diff. at m_0 w.r.t. Kolmogorov dist.).

- *Dudley 90 :* "Gateaux derivative considered too weak"
  Existence of a class $\mathcal{F}$ of measurable functions s.t.
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$$\|\mu-m_0\|=\sup_{f\in\mathcal{F}}\left|\int f(x)(\mu-m_0)(dx)\right|,$$

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Integral calculus related to the (Gateaux) linear functional derivative
$\rightarrow$ versatile tool permitting to go beyond the i.i.d. case (*Flenghi J. 22*)

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Interacting particle system

$$\begin{cases} Y_t^{i,N} = \zeta_i + \int_0^t b(Y_s^{i,N},\mu_s^N)\,ds + \int_0^t \sigma(Y_s^{i,N},\mu_s^N)\,dW_s^i, \quad 1\le i\le N, \quad t\ge 0,\\ \mu_s^N := \frac{1}{N}\sum_{i=1}^N \delta_{Y_s^{i,N}}, \end{cases}$$

where

- $b:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^d$, $\sigma:\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}^{d\times d'}$ Lipschitz,
- $(W^i,\zeta_i)_{i\ge1}$ i.i.d. with W^i a d'-dimensional Brownian motion independent from the $\mathbb{R}^d$-valued initial random vector $\zeta_i\sim m_0$.

Mean-field limit as $N\to\infty$: SDE nonlinear in the sense of McKean

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Question : for $\Phi:\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R}$ a nonlinear functional, limiting behaviour of the fluctuations process

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Regularity class

Definition

A function $f : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ belongs to class $\mathcal{M}_k(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, if the derivatives $D^{(j,n,\beta)} f(x, \mu, y_1, \dots, y_n)$ exist for every multi-index (j, n, β) such that $|(j, n, \beta)| \leq k$ and satisfy

$$\forall x, y_1, \dots, y_n \in \mathbb{R}^d,\ \forall \mu \in \mathcal{P}_2(\mathbb{R}^d),\ \left|D^{(j,n,\beta)} f(x, \mu, y_1, \dots, y_n)\right| \leq C$$

$$\text{and } \forall x', y'_1, \dots, y'_n \in \mathbb{R}^d,\ \forall \mu' \in \mathcal{P}_2(\mathbb{R}^d),$$

$$\left|D^{(j,n,\beta)} f(x, \mu, y_1, \dots, y_n) - D^{(j,n,\beta)} f(x', \mu', y'_1, \dots, y'_n)\right| \leq C\left(|x - x'| + \sum_{i=1}^{n} |y_i - y'_i| + W_2(\mu, \mu')\right)$$

The n derivatives w.r.t. μ are taken in the Lions sense

$$D^{(0,1,0)} f(x, \mu) = \partial_{y_1} \frac{\delta U}{\delta m}(x, \mu, y_1).$$

Master equation

Let $\mathcal{V} : \mathbb{R}_+ \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\mathcal{V}(t, \mathcal{L}(\theta)) = \Phi\big(\mathcal{L}(X_t^\theta)\big)$$

where, for θ a square integrable $\mathbb{R}^d$-valued initial random vector $\perp W$,

$$X_t^\theta = \theta + \int_0^t b(X_s^\theta, \mathcal{L}(X_s^\theta))\,ds + \int_0^t \sigma(X_s^\theta, \mathcal{L}(X_s^\theta))\,dW_s, \qquad t \geq 0.$$

Buckdahn, Li, Peng, Rainer AP 17 $\longrightarrow$ if $\Phi \in \mathcal{M}_2(\mathcal{P}_2(\mathbb{R}^d))$, $b, \sigma \in \mathcal{M}_2(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, then $\mathcal{V}$ satisfies the *master equation* :

$$\begin{cases} \partial_s \mathcal{V}(s,\mu) = \int_{\mathbb{R}^d} \left[\partial_\mu \mathcal{V}(s,\mu)(x) \cdot b(x,\mu) + \frac{1}{2}\mathrm{Tr}\big(\partial_x \partial_\mu \mathcal{V}(s,\mu)(x) a(x,\mu)\big)\right] \mu(dx) \\ \mathcal{V}(0,\mu) = \Phi(\mu), \end{cases}$$

where $a(x,\mu) := \sigma(x,\mu)\sigma(x,\mu)^*$.

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where $a(x,\mu) := \sigma(x,\mu)\sigma(x,\mu)^*$.

$$\begin{aligned}
\sqrt{N}\Big[\Phi(\mu^N_t)-\Phi(\mu^\infty_t)\Big] &= \overbrace{\sqrt{N}\big(\mathcal{V}(t,m^N)-\mathcal{V}(t,m_0)\big)}^{\text{fluctuations of the initial emp. meas.}}\\
&+\frac{1}{\sqrt{N}}\sum_{i=1}^N\int_0^t \partial_\mu\mathcal{V}^*(t-s,\mu^N_s)(Y^{i,N}_s)\sigma(Y^{i,N}_s,\mu^N_s)dW^i_s \quad := I^N_t\\
&+\int_0^t\frac{1}{2}\Bigg[\underbrace{\frac{1}{N^{3/2}}\sum_{i=1}^N}_{\mathcal{O}(N^{-1/2})}\mathrm{Tr}\Big(a(Y^{i,N}_s,\mu^N_s)\partial^2_\mu\mathcal{V}(t-s,\mu^N_s)(Y^{i,N}_s,Y^{i,N}_s)\Big)\Bigg]ds.
\end{aligned}$$

$$\begin{aligned}
\langle I^N_\cdot\rangle_t &= \frac{1}{N}\sum_{i=1}^N\int_0^t \partial_\mu\mathcal{V}^*(t-s,\mu^N_s)(Y^{i,N}_s)a(Y^{i,N}_s,\mu^N_s)\partial_\mu\mathcal{V}(t-s,\mu^N_s)(Y^{i,N}_s)ds\\
&\xrightarrow[N\to\infty]{} \int_0^t\int_{\mathbb{R}^d}\partial_\mu\mathcal{V}^*(t-s,\mu^\infty_s)(y)a(y,\mu^\infty_s)\mathcal{V}(t-s,\mu^\infty_s)(y)\mu^\infty_s(dy)ds.
\end{aligned}$$

$$\sqrt{N}\Big[\Phi(\mu_t^N)-\Phi(\mu_t^\infty)\Big]=\overbrace{\sqrt{N}\big(\mathcal{V}(t,m^N)-\mathcal{V}(t,m_0)\big)}^{\text{fluctuations of the initial emp. meas.}}$$

$$+\frac{1}{\sqrt{N}}\sum_{i=1}^N\int_0^t\partial_\mu\mathcal{V}^*(t-s,\mu_s^N)(Y_s^{i,N})\sigma(Y_s^{i,N},\mu_s^N)dW_s^i \quad := I_t^N$$

$$+\int_0^t\frac{1}{2}\Bigg[\underbrace{\frac{1}{N^{3/2}}\sum_{i=1}^N}_{\mathcal{O}(N^{-1/2})}\mathrm{Tr}\Big(a(Y_s^{i,N},\mu_s^N)\partial_\mu^2\mathcal{V}(t-s,\mu_s^N)(Y_s^{i,N},Y_s^{i,N})\Big)\Bigg]ds.$$

$$\langle I_\cdot^N\rangle_t=\frac{1}{N}\sum_{i=1}^N\int_0^t\partial_\mu\mathcal{V}^*(t-s,\mu_s^N)(Y_s^{i,N})a(Y_s^{i,N},\mu_s^N)\partial_\mu\mathcal{V}(t-s,\mu_s^N)(Y_s^{i,N})ds$$

$$\underset{N\to\infty}{\longrightarrow}\int_0^t\int_{\mathbb{R}^d}\partial_\mu\mathcal{V}^*(t-s,\mu_s^\infty)(y)a(y,\mu_s^\infty)\mathcal{V}(t-s,\mu_s^\infty)(y)\mu_s^\infty(dy)ds.$$

Theorem

Suppose $m_0 \in \mathcal{P}_{12}(\mathbb{R}^d)$, $\Phi \in \mathcal{M}_5(\mathcal{P}_2(\mathbb{R}^d))$, $b, \sigma \in \mathcal{M}_5(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ and uniformly bounded. Then, in $C(\mathbb{R}_+, \mathbb{R})$, the fluctuations process

$$(\sqrt{N}\big[\Phi(\mu_t^N) - \Phi(\mu_t^\infty)\big])_{t\geq 0}$$

converges weakly to a centered Gaussian process L with covariance

$$Cov(L_t, L_u) = Cov\left(\frac{\delta \mathcal{V}}{\delta m}(t, m_0, \xi_1), \frac{\delta \mathcal{V}}{\delta m}(u, m_0, \xi_1)\right)$$
$$+ \int_0^{t\wedge u} \int_{\mathbb{R}^d} \partial_\mu \mathcal{V}^*\left(t - s, \mu_s^\infty\right)(y) a(y, \mu_s^\infty) \mathcal{V}\left(u - s, \mu_s^\infty\right)(y) \mu_s^\infty(dy) ds.$$