Kolmogorov equations on spaces of measures associated to nonlinear filtering processes

Mattia Martini

Università degli Studi di Milano

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Overview

1. **Stochastic filtering**
   - Nonlinear filtering problem
   - Nonlinear filtering equations

2. **Kolmogorov equations associated to filtering equations**
   - Itô formula
   - Backward equation associated to the Zakai equation
   - Backward equation associated to the K.-S. equation
We want to introduce and study a class of backward Kolmogorov equations on

- $\mathcal{M}_2^+(\mathbb{R}^d)$, $\mathcal{P}_2(\mathbb{R}^d)$: positive and probability measures with finite second moment;
- $\langle \mu, \psi \rangle = \mu(\psi) = \int_{\mathbb{R}^d} \psi(x) \mu(dx)$;
We want to introduce and study a class of backward Kolmogorov equations on

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SDEs for measure-valued processes arise naturally in the stochastic filtering framework.

- Many results when there is a density, using stochastic calculus on Hilbert spaces (e.g. Rozovsky [9], Pardoux [8]).
- New tools for calculus on spaces of (probability) measures (e.g. Ambrosio, Gigli & Savarè [1], P.-L. Lions [5], Carmona & Delarue [3]).
- Optimal control with partial observation (e.g. Gozzi & Świąch [4] in the Hilbert setting, or recently Bandini, Cosso, Fuhrman & Pham [2] on $\mathcal{P}_2(\mathbb{R}^d)$).
Stochastic filtering

The problem

Signal process

\[ dX_t = b(X_t) \, ds + \sigma(X_t) \, dW_t, \quad X_0 \in L^2(\Omega, \mathcal{F}_0), \quad t \in [0, T]. \] (1)

Observation process

For every \( t \in [0, T] \),

\[ dY_t = h(X_t) \, dt + dB_t, \quad Y_0 = Y_0, \quad FY_t = \sigma(Y_s, s \leq t) \lor N, \] (two.pnum)

where \( N \) are \( \mathbb{P} \)-negligible sets.

Goal

• The signal \( X \) is not directly observed;
• The available information is given by \( Y \);
• We want to provide an approximation of \( X \) given the observation \( Y \).
Stochastic filtering The problem

### Signal process

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### Observation process

For every \( t \in [0, T] \),

\[ dY_t = h(X_t) \, dt + dB_t, \quad Y_0 = 0, \]

\[ \mathcal{F}_t^Y = \sigma(Y_s, 0 \leq s \leq t) \cup \mathcal{N}, \]  \hspace{1cm} (2)

where \( \mathcal{N} \) are \( \mathbb{P} \)-negligible sets.
**Stochastic filtering** The problem

### Signal process

\[ dX_t = b(X_t) \, ds + \sigma(X_t) \, dW_t, \quad X_0 \in L^2(\Omega, \mathcal{F}_0), \quad t \in [0, T]. \]  

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### Goal

- The signal \( X \) is not directly observed;
- The available information is given by \( Y \);
- We want to provide an approximation of \( X \) given the observation \( Y \).
• Given the information $\mathcal{F}_t^Y$, the best estimate for $\varphi(X_t)$ is

$$\mathbb{E} \left[ \varphi(X_t) | \mathcal{F}_t^Y \right];$$
Stochastic filtering  

The filter

- Given the information $\mathcal{F}_t^Y$, the best estimate for $\varphi(X_t)$ is

  $$E \left[ \varphi(X_t) | \mathcal{F}_t^Y \right];$$

- Let $\Pi_t$ be the regular conditional probability distribution of $X_t$ given $\mathcal{F}_t^Y$: for any $A \in \mathcal{B}(\mathbb{R}^d)$

  $$\Pi_t(A, \omega) = \mathbb{P} \left( X_t \in A \mid \mathcal{F}_t^Y \right)(\omega), \quad \text{a.e.} \ \omega.$$
Stochastic filtering \textbf{The filter}

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- For every $\varphi \in C_b(\mathbb{R}^d)$ and $t \in [0, T]$,

$$\langle \Pi_t, \varphi \rangle = E \left[ \varphi(X_t) | \mathcal{F}_t^Y \right], \quad \text{a.s.}$$
Stochastic filtering The filter

- Given the information $\mathcal{F}_t^Y$, the best estimate for $\varphi(X_t)$ is
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- For every $\varphi \in C_b(\mathbb{R}^d)$ and $t \in [0, T]$, 
  $$\langle \Pi_t, \varphi \rangle = \mathbb{E} \left[ \varphi(X_t) | \mathcal{F}_t^Y \right], \text{ a.s.}$$

$\{\Pi_t = \text{Law}(X_t | \mathcal{F}_t^Y)\}_{t \in [0, T]}$ is a $\mathcal{P}(\mathbb{R}^d)$-valued process called filter.
Define $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = M_t^{-1} = \exp\left\{ -\frac{1}{2} \int_0^t |h(X_s)|^2 \, ds - \int_0^t h(X_s) \, dB_s \right\}$. 

Theorem (Kallianpur-Striebel formula)

The filter $\Pi$ can be represented as

$$\langle \Pi_t, \phi \rangle = \langle \rho_t, \phi \rangle \langle \rho_t, \cdot \rangle, \quad t \in [0, T], \phi \in C_b(\mathbb{R}^d),$$

where $\langle \rho_t, \phi \rangle = \mathbb{E}_{\mathbb{Q}}[M_t \phi(X_t) | \mathcal{F}_Y]$. 

$\{\rho_t\}$ is a $\mathbb{P}^+\left(\mathbb{R}^d\right)$-valued process called the unnormalized filter. $Y$ is a brownian motion under $\mathbb{Q}$.

By Itô formula applied to $M_t \phi(X_t)$ we obtain

The Zakai equation (Z)

The unnormalized filter satisfies, for every test $\phi$,

$$\frac{d}{dt}\langle \rho_t, \phi \rangle = \langle \rho_t, A\phi \rangle \, dt + \langle \rho_t, h\phi \rangle \, dB_t, \quad t \in (0, T),$$

where $A$ is the infinitesimal generator of $X$. 

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Stochastic filtering  The unnormalized filter

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Theorem (Kallianpur-Striebel formula)

*The filter $\Pi$ can be represented as*

\[
\langle \Pi_t, \varphi \rangle = \frac{\langle \rho_t, \varphi \rangle}{\langle \rho_t, 1 \rangle}, \quad t \in [0, T], \varphi \in C_b(\mathbb{R}^d),
\]

Where $\langle \rho_t, \varphi \rangle = \mathbb{E}_\mathbb{Q}^\mathbb{Q} \left[ M_t \varphi(X_t) | \mathcal{F}_t^Y \right]$. 

$\{\rho_t\}_{t \in [0, T]}$ is a $\mathcal{M}^+(\mathbb{R}^d)$-valued process called unnormalized filter.
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where $\langle \rho_t, \varphi \rangle = \mathbb{E}^\mathbb{Q} \left[ M_t \varphi(X_t) | \mathcal{F}_t^Y \right]$.

\{\rho_t\}_{t \in [0, T]} is a $\mathcal{M}^+(\mathbb{R}^d)$-valued process called **unnormalized filter**.

$Y$ is a brownian motion under $\mathbb{Q}$.
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**The Zakai equation (Z)**

The unnormalized filter satisfies, for every test $\phi$,
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d\langle \rho_t, \phi \rangle = \langle \rho_t, A\phi \rangle \, dt + \langle \rho_t, h\phi \rangle \, dY_t, \quad t \in (0, T],
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Let $A$ be the generator of $X$: $A\varphi = b^T (D_x \varphi) + \frac{1}{2} \text{tr}\{(D_x^2 \varphi)\sigma\sigma^T\}$.

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where $Y$ is a Brownian motion under $\mathbb{Q}$. 
Let $A$ be the generator of $X$: $A\varphi = b^\top (D_x\varphi) + \frac{1}{2} \text{tr}\{(D_x^2\varphi)\sigma\sigma^\top\}$.

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The unnormalized filter satisfies, for every test $\varphi$,

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where $Y$ is a Brownian motion under $\mathbb{Q}$.

Using the Kallianpur-Striebel formula

The Kushner-Stratonovich equation (KS)

The filter satisfies, for every test $\varphi$,

$$d\langle \Pi_t, \varphi \rangle = \langle \Pi_t, A\varphi \rangle \, dt + \left(\langle \Pi_t, h\varphi \rangle - \langle \Pi_t, \varphi \rangle \langle \Pi_t, h \rangle \right) \, dl_t, \quad t \in (0, T],$$

where $\{l_t\}_{t \in [0, T]}$ is called innovation process and is a Brownian motion under $\mathbb{P}$.
Stochastic filtering Example: Kalman-Bucy filter

Signal:

\[ dX_t = b_t X_t \, dt + \sigma_t \, dW_t, \quad a_t^{ij} = \sigma_t \sigma_t^\top, \]

\[ A_t \varphi(x) = D_x \varphi(x)^\top b_t x + \frac{1}{2} \sum_{i,j} a_t^{ij} \partial_{ij}^2 \varphi(x). \]
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Observation:

\[ dY_t = h_t X_t \, dt + dB_t, \quad Y_0 = 0. \]

\((X, Y)\) is a gaussian process.
Stochastic filtering Example: Kalman-Bucy filter

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\begin{align*}
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\begin{align*}
    d\langle \Pi_t, \varphi \rangle &= \langle \Pi_t, A_s \varphi \rangle \, dt + \langle \Pi_t, \varphi h_t^\top \nu \rangle \, dl_t - \langle \Pi_t, \varphi \rangle \langle \Pi_t, h^\top \nu \rangle \, dl_t, \\
    \nu(x) &= x.
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\]
\(\nu(x) = x\). Moreover, for \(\omega \in \Omega\) fixed, \(\Pi_t(\omega)\) is gaussian with

- Mean \(\hat{X}_t\) that solves the SDE
  \[
d\hat{X}_t = b_t \hat{X}_t \, dt + \gamma_t h_t \, dl_t, \quad l_t = Y_t - \int_0^t h_s \hat{X}_s \, ds.
  \]

- Deterministic variance that solves the Riccati equation
  \[
  \frac{d}{dt} \gamma_t = \gamma_t b_t^\top + b_t \gamma_t + a_t - \gamma_t (h^\top h) \gamma_t^\top.
  \]
Let \( \{ \rho_t \}_{t \in [0, T]} \) be a solution to (Z), i.e. for every test \( \varphi \)
\[
d\langle \rho_t, \varphi \rangle = \langle \rho_t, A \varphi \rangle \, dt + \langle \rho_t, h \varphi \rangle \, dY_t, \quad t \in (0, T].
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Itô formula for the Zakai equation

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\]

Hypotheses (H)

a. \( b, \sigma, h \) are Borel-measurable and bounded, \( b, \sigma \) are Lipschitz;
b. The matrix \( \sigma \sigma^\top (x) \) is positive definite for every \( x \in \mathbb{R}^d \).
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Proposition (M. [6])

Let \( u \) be in \( C^2_L (\mathcal{M}_2^+ (\mathbb{R}^d)) \) and let us assume (H). Then, for every \( t \in [0, T] \):

\[
\, du(\rho_t) = \langle \rho_t, A\delta_\mu u(\rho_t) \rangle \, dt + \langle \rho_t, h\delta_\mu u(\rho_t) \rangle \, dY_t + \frac{1}{2} \langle \rho_t \otimes \rho_t, h^\top h\delta^2_\mu u(\rho_t) \rangle \, dt.
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\]

- \( \delta_\mu u \) is a notions of derivatives for \( u : \mathcal{M}_2^+ (\mathbb{R}^d) \rightarrow \mathbb{R} \):

\[
\delta_\mu u : \mathcal{M}_2^+ (\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}, \quad \delta^2_\mu u : \mathcal{M}_2^+ (\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} ;
\]
- Proof by cylindrical approximation: \( u(\mu) := g (\langle \mu, \psi_1 \rangle, \ldots, \langle \mu, \psi_n \rangle) \).
The generator $\mathcal{L}: C^2_L(M^+_{2}(\mathbb{R}^d)) \rightarrow C_b(M^+_{2}(\mathbb{R}^d))$
The infinitesimal generator of the Zakai equation

The generator $\mathcal{L} : C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \to C_b(\mathcal{M}_2^+(\mathbb{R}^d))$

$$(\mathcal{L}u)(\mu) = \langle \mu, A\delta_\mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h\delta^2_\mu u(\mu) \rangle$$

$$= \int_{\mathbb{R}^d} (A\delta_\mu u)(\mu, x)\mu(\text{d}x) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x)^\top h(y)\delta^2_\mu u(\mu, x, y)\mu(\text{d}x)\mu(\text{d}y).$$
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= \int_{\mathbb{R}^d} (A\delta_\mu u)(\mu, x) \mu(\,dx) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x)^\top h(y) \delta^2_\mu u(\mu, x, y) \mu(\,dx) \mu(\,dy).
\]

**Remark**

- Formally \( d\rho_t = A^* \rho_t \, dt + h^\top \rho_t \, dY_t \), so:

\[
du(\rho_t) = \langle A^* \rho_t, \delta_\mu u(\rho_t) \rangle \, dt + \langle h^\top \rho_t, \delta_\mu u(\rho_t) \rangle \, dY_t + \frac{1}{2} \langle h^\top \rho_t \otimes h \rho_t, \delta^2_\mu u(\rho_t) \rangle \, dt.
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**Remark**

- Formally $d \rho_t = A^* \rho_t \, dt + h^\top \rho_t \, dY_t$, so:

  $$d u(\rho_t) = \langle A^* \rho_t, \delta_{\mu} u(\rho_t) \rangle \, dt + \langle h^\top \rho_t, \delta_{\mu} u(\rho_t) \rangle \, dY_t + \frac{1}{2} \langle h^\top \rho_t \otimes h \rho_t, \delta_{\mu}^2 u(\rho_t) \rangle \, dt.$$  

- On $\mathbb{R}$, if $dX_t = bX_t \, dt + \sigma X_t \, dB_t$, then

  $$d u(X_t) = bX_t \, D_x u(X_t) \, dt + \sigma X_t \, D_x u(X_t) \, dB_t + \frac{1}{2} \sigma^2 X_t^2 \, D_x^2 u(X_t) \, dt.$$
Let

\[(Lu)(\mu) = \langle \mu, A\delta_\mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^T h \delta^2_\mu u(\mu) \rangle.\]  

(5)
The backward Kolmogorov equation

Existence and uniqueness

Let

\[(Lu)(\mu) = \langle \mu, A\delta_\mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h\delta_\mu^2 u(\mu) \rangle.\] (5)

Given \(\Phi: \mathcal{M}_2^+(\mathbb{R}^d) \to \mathbb{R},\) the Backward Kolmogorov equation (BEZ) reads as

\[
\begin{cases}
\partial_s u(\mu, s) + Lu(\mu, s) = 0, & (\mu, s) \in \mathcal{M}_2^+(\mathbb{R}^d) \times [0, T], \\
u(\mu, T) = \Phi(\mu), & \mu \in \mathcal{M}_2^+(\mathbb{R}^d).
\end{cases}
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Let \(\{\rho^{s, \mu}_t\}_{t \in [s, T]}\) be a solution to (Z) starting at time \(s\) from \(\mu \in \mathcal{M}^+_2(\mathbb{R}^d)\).
Let
\[(Lu)(\mu) = \langle \mu, A\delta \mu u(\mu) \rangle + \frac{1}{2} \langle \mu \otimes \mu, h^\top h \delta^2 \mu u(\mu) \rangle.\] (5)

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u(\mu, T) = \Phi(\mu), & \mu \in \mathcal{M}^+_2(\mathbb{R}^d).
\end{cases}
\]

Let \(\{\rho_t^{s,\mu}\}_{t \in [s, T]}\) be a solution to (Z) starting at time \(s\) from \(\mu \in \mathcal{M}^+_2(\mathbb{R}^d)\).

**Theorem (M. [6])**

Let \(\Phi \in C^2_L(\mathcal{M}^+_2(\mathbb{R}^d))\). Let \((H)\) holds and let us set
\[u(\mu, s) := \mathbb{E} \left[ \Phi(\rho_t^{s,\mu}) \right], \quad (\mu, s) \in \mathcal{M}^+_2(\mathbb{R}^d) \times [0, T].\] (6)

Then \(u\) is the unique classical solution to (BEZ).
Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form

\[ u(\mu, s) = \mathbb{E} \left[ \Phi(\rho_{s,\mu}) \right]. \]
Proof (key steps)

Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form

$$u(\mu, s) = E \left[ \Phi(\rho_{T}^{s, \mu}) \right].$$

Existence:

- Prove that $$\mu \mapsto u(\mu, s) := E \left[ \Phi(\rho_{T}^{s, \mu}) \right]$$ is in $$C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d})).$$
Proof (key steps)

Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form

$$u(\mu, s) = \mathbb{E} \left[ \Phi(\rho_{T}^{s, \mu}) \right].$$

Existence:

- Prove that $\mu \mapsto u(\mu, s) := \mathbb{E} \left[ \Phi(\rho_{T}^{s, \mu}) \right]$ is in $C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d}))$:
  - given a suitable notion of derivative for functions from $C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d}))$ to $C_{L}^{2}(\mathcal{M}_{2}^{+}(\mathbb{R}^{d}))$, we show that $\mu \mapsto \rho_{T}^{s, \mu}$ is twice differentiable;
Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form

\[ u(\mu, s) = \mathbb{E} \left[ \Phi(\rho_{\tau}^{s, \mu}) \right]. \]

Existence:

- Prove that \( \mu \mapsto u(\mu, s) := \mathbb{E} \left[ \Phi(\rho_{\tau}^{s, \mu}) \right] \) is in \( C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \):
  - given a suitable notion of derivative for functions from \( C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \) to \( C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \), we show that \( \mu \mapsto \rho_{\tau}^{s, \mu} \) is twice differentiable;
  - since \( \Phi \in C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \) and by the previous point, we conclude by a chain rule.
Proof (key steps)

Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form
  \[ u(\mu, s) = \mathbb{E} \left[ \Phi(\rho^T_s, \mu) \right]. \]

Existence:

- Prove that \( \mu \mapsto u(\mu, s) := \mathbb{E} \left[ \Phi(\rho^T_s, \mu) \right] \) is in \( C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \):
  - given a suitable notion of derivative for functions from \( C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \) to \( C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \), we show that \( \mu \mapsto \rho^s_T, \mu \) is twice differentiable;
  - since \( \Phi \in C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \) and by the previous point, we conclude by a chain rule.
- By Itô formula and Markov property
  \[
  \lim_{h \to 0} \frac{1}{h} \left[ u(\mu, s + h) - u(\mu, s) \right] = -\lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \int_{s}^{s+h} \mathcal{L}u(\rho^s_T, \mu, s + h) \, d\tau \right] = -\mathcal{L}u(\mu, s).
  \]
The Kushner-Stratonovich equation case

The operator $\mathcal{L}^{KS}: C^2_L(\mathcal{P}_2(\mathbb{R}^d)) \to C_b(\mathcal{P}_2(\mathbb{R}^d))$

$$\mathcal{L}^{KS} u(\pi) = \langle \pi, A\delta_\mu u(\pi) \rangle + \frac{1}{2} \langle \pi \otimes \pi, (h - \pi(h)) \mathbf{1} (h - \pi(h)) \delta_\mu u(\pi) \rangle.$$
The Kushner-Stratonovich equation case

The operator \( L^{KS} : C^2_{L}(\mathcal{P}_2(\mathbb{R}^d)) \to C_b(\mathcal{P}_2(\mathbb{R}^d)) \)

\[
L^{KS}u(\pi) = \langle \pi, A\delta_\mu u(\pi) \rangle + \frac{1}{2} \langle \pi \otimes \pi, (h - \pi(h))^{\top}(h - \pi(h))\delta_\mu^2 u(\pi) \rangle.
\]

Given \( \Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \), the Backward Kolmogorov equation (BEKS) reads as

\[
\begin{cases}
\partial_s u(\pi, s) + L^{KS}u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T], \\
u(\pi, T) = \Phi(\pi), & \pi \in \mathcal{P}_2(\mathbb{R}^d).
\end{cases}
\]
The Kushner-Stratonovich equation case

The operator $L^{KS} : C^2_L(\mathcal{P}_2(\mathbb{R}^d)) \to C_b(\mathcal{P}_2(\mathbb{R}^d))$

$$L^{KS}u(\pi) = \langle \pi, A \delta_\mu u(\pi) \rangle + \frac{1}{2} \langle \pi \otimes \pi, (h - \pi(h)) \top (h - \pi(h)) \delta_\mu^2 u(\pi) \rangle.$$  

Given $\Phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, the Backward Kolmogorov equation (BEKS) reads as

$$\begin{cases} 
\partial_s u(\pi, s) + L^{KS}u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T], \\
u(\pi, T) = \Phi(\pi), & \pi \in \mathcal{P}_2(\mathbb{R}^d).
\end{cases}$$

Let $\{\Pi_t^{s, \pi}\}_{t \in [s, T]}$ be a solution to (KS) starting at time $s$ from $\pi \in \mathcal{P}_2(\mathbb{R}^d)$:

$$d \langle \Pi_t, \psi \rangle = \langle \Pi_t, A \psi \rangle \, dt + (\langle \Pi_t, h \psi \rangle - \langle \Pi_t, \psi \rangle \langle \Pi_t, h \rangle) \cdot dI_t, \quad t \in (0, T]. \quad (7)$$

**Theorem (M. [6])**

Let $\Phi \in C^2_L(\mathcal{P}_2(\mathbb{R}^d))$. Let (H) holds and let us set

$$u(\pi, s) = \mathbb{E} \left[ \Phi(\Pi_T^{s, \pi}) \right], \quad (\pi, s) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T].$$

Then $u$ is the unique classical solution to (BEKS).
**The Kushner-Stratonovich equation case**

**Viscosity approach**

\( K \subset \mathbb{R}^d \) compact, \( \Phi \in C_b(\mathcal{P}_2(K)) \):

\[
\begin{aligned}
\partial_s u(\pi, s) + L^{KS} u(\pi, s) &= 0, \quad (\pi, s) \in \mathcal{P}_2(K) \times (0, T], \\
u(\pi, T) &= \Phi(\pi), \quad \pi \in \mathcal{P}_2(K).
\end{aligned}
\]

Let \( \{\Pi_t^{s, \pi}\}_{t \in [s, T]} \) be a solution to (KS) confined in \( \mathcal{P}_2(K) \).

**Theorem (M. [7])**

Let \( \Phi \in C_b(\mathcal{P}_2(K)) \). Let (H) holds and let us set

\[
u(\pi, s) = \mathbb{E} \left[ \Phi(\Pi_t^{s, \pi}) \right], \quad (\pi, s) \in \mathcal{P}_2(K) \times (0, T].
\]

Then \( u \) is the unique viscosity solution to (BEKS).
Proof of the comparison principle (Key steps)

Let $u_1$ and $u_2$ be respectively a subsolution and a supersolution to (BEKS). Moreover, let $u(\pi, s) := \mathbb{E} \left[ \Phi(\Pi_s^\pi) \right]$. We want to show that $u_1 \leq u_2$.

- Show: $u_1 \leq u$ and $u \leq u_2$.
- Introduce a family of approximated problems:

$$
\begin{cases}
\partial_s u(\pi, s) + \mathcal{L}^{KS} u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(K) \times (0, T], \\
u(\pi, T) = \Phi_n(\pi) \in C^2_L(\mathcal{P}_2(K)), & \pi \in \mathcal{P}_2(K).
\end{cases}
$$

- $u^n(\pi, s) := \mathbb{E} \left[ \Phi_n(\Pi_s^\pi) \right]$ is a classical solution to the approximated problem which converges to $u$.
- Using the Borwein-Preiss variational principle with a suitable smooth gauge-type function, we introduce a suitable test function that allows us to conclude.
Thank you!
References


Linear functional derivative

\( u : \mathcal{M}^+ (\mathbb{R}^d) \to \mathbb{R} \) is in \( C^1_b (\mathcal{M}^+ (\mathbb{R}^d)) \) if it is continuous, bounded and if exists

\[ \delta_\mu u : \mathcal{M}^+ (\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, x) \mapsto \delta_\mu u (\mu, x) \in \mathbb{R}, \]

bounded, continuous and such that for all \( \mu \) and \( \mu' \) in \( \mathcal{M}^+ (\mathbb{R}^d) \), it holds:

\[ u(\mu') - u(\mu) = \int_0^1 \int_{\mathbb{R}^d} \delta_\mu u (t\mu' + (1 - t)\mu, x) \left[ \mu' \right. - \mu \left.] (dx) \, dt. \quad (8) \]

Similarly we can define \( C^k_b (\mathcal{M}^+ (\mathbb{R}^d)) \), \( k \in \mathbb{N} \).

Example

Let \( g \in C^2_b (\mathbb{R}) \) and let \( \psi \in C_b (\mathbb{R}^d) \). We define

\[ u : \mathcal{M}^+ (\mathbb{R}^d) \ni \mu \mapsto g (\langle \mu, \psi \rangle) \in \mathbb{R}. \]

Then \( u \in C^2_b (\mathcal{M}^+ (\mathbb{R}^d)) \) and it holds:

\[ \delta_\mu u (\mu, x) = g' (\langle \mu, \psi \rangle) \psi (x), \quad \delta^2_\mu u (\mu, x, y) = g'' (\langle \mu, \psi \rangle) \psi (x) \psi (y). \]
The space $C^2_L(\mathcal{M}^+(\mathbb{R}^d))$

$u: \mathcal{M}^+(\mathbb{R}^d) \rightarrow \mathbb{R}$ is in $C^2_L(\mathcal{M}^+(\mathbb{R}^d))$ if:

a. $u$ is in $C^2_b(\mathcal{M}^+(\mathbb{R}^d))$;

b. $\mathbb{R}^d \ni x \mapsto \delta_\mu u(\mu, x) \in \mathbb{R}$ is twice differentiable, with continuous and bounded derivatives on $\mathcal{M}^+(\mathbb{R}^d) \times \mathbb{R}^d$;

We set

$$D_\mu u(\mu, x) := D_x \delta_\mu u(\mu, x) \in \mathbb{R}^d,$$

Remark

On $\mathcal{P}_2(\mathbb{R}^d)$, the derivative $D_\mu u$ coincides with the one introduced by P.-L. Lions through the lifting procedure in the context of mean field games ([5, 3]).
1. Prove the formula for functions of the form

\[ u : \mathcal{M}_2^+(\mathbb{R}^d) \ni \mu \mapsto g(\langle \mu, \psi_1 \rangle, \ldots, \langle \mu, \psi_n \rangle), \]

exploiting classical Itô formula and the Zakai equation.

2. Prove the formula for functions of the form

\[ u(\mu) = \langle \frac{\mu^r}{\mu(\mathbb{R}^d)^r}, \varphi(\cdot, \ldots, \cdot, \mu(\mathbb{R}^d)) \rangle \]

by approximation, where \( \varphi : \mathbb{R}^{d \times r + 1} \rightarrow \mathbb{R} \) is symmetrical in the first \( r \) arguments.

3. Prove the formula for functions in \( C^2_L(\mathcal{M}_2^+(\mathbb{R}^d)) \) by approximation.
The backward Kolmogorov equation  Existence and uniqueness

Theorem (M. [6])

Let us set

\[ u(\mu, s) = \mathbb{E} \left[ \Phi(\rho^s_T, \mu) \right], \tag{9} \]

where \( \rho^s_T, \mu \) is the weak solution to the Zakai equation starting at time \( s \) from \( \mu \in \mathcal{M}_2^+ (\mathbb{R}^d) \), \( \Phi \in C^2_L(\mathcal{M}_2^+ (\mathbb{R}^d)) \) and let (H) hold. Then \( u \) is the unique classical solution to the backward Kolmogorov equation (BEZ).

Proof (uniqueness)

We show that if \( u \) is a classical solution to (BEZ), then \( u(\mu, s) = \mathbb{E} \left[ \Phi(\rho^s_T, \mu) \right] \).

- By the Itô formula

\[
u(\rho^s_T, T) - u(\rho^s_s, s) = \int_s^T \{ \partial_s u(\rho^s_T, \tau) + \mathcal{L} u(\rho^s_T, \tau) \} \, d\tau + \int_s^T \mathcal{G} u(\rho^s_T, \tau) \cdot dY_\tau.
\]

- By taking the expectation and since \( u \) solves (BEZ)

\[
\mathbb{E} \left[ \Phi(\rho^s_T, \mu) \right] - u(\mu, s) = \mathbb{E} \left[ \int_s^T \mathcal{G} u(\rho^s_T, \tau) \cdot dY_\tau \right].
\]

- The rhs is zero since the integral is a martingale, thus \( u(\mu, s) = \mathbb{E} \left[ \Phi(\rho^s_T, \mu) \right]. \)
The backward Kolmogorov equation Existence and uniqueness

Proof (existence)

Let $u(\mu, s) = \mathbb{E} \left[ \Phi(\rho^{s,\mu}_T) \right]$ be our candidate solution.

1. Prove that $\mu \mapsto u(\mu, s)$ is in $C^2_{L}(\mathcal{M}^+_2(\mathbb{R}^d))$:
   - given a suitable notion of derivative for functions from $C^2_{L}(\mathcal{M}^+_2(\mathbb{R}^d))$ to $C^2_{L}(\mathcal{M}^+_2(\mathbb{R}^d))$, we show that $\mu \mapsto \rho^{s,\mu}_T$ is twice differentiable;
   - since $\Phi \in C^2_{L}(\mathcal{M}^+_2(\mathbb{R}^d))$ and by the previous point, we conclude by a chain rule.

2. Prove the continuity of
   
   $[0, T] \ni s \mapsto \mathcal{L}u(\mu, s), \quad [s, T] \times [0, T] \ni (\tau, \sigma) \mapsto \mathcal{L}u(\rho^{s,\mu}_{\tau}, \sigma) \in L^2(\Omega)$.

3. By the Itô formula and the Markov property
   
   $$\lim_{h \to 0} \frac{1}{h} [u(\mu, s + h) - u(\mu, s)] = - \lim_{h \to 0} \frac{1}{h} \mathbb{E} \left[ \int_s^{s+h} \mathcal{L}u(\rho^{s,\mu}_{\tau}, s + h) \, d\tau \right] = -\mathcal{L}u(\mu, s).$$