# Kolmogorov equations on spaces of measures

## associated to nonlinear filtering processes

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# Overview

We want to introduce and study a class of backward Kolmogorov equations on

- $\mathcal{M}_2^+(\mathbb{R}^d), \mathcal{P}_2(\mathbb{R}^d)$: positive and probability measures with finite second moment;
- $\langle \mu, \psi \rangle = \mu(\psi) = \int_{\mathbb{R}^d} \psi(x)\mu(\mathrm{d}x)$;

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SDEs for measure-valued processes arise naturally in the stochastic filtering framework.

- Many results when there is a density, using stochastic calculus on Hilbert spaces (e.g. Rozovsky [9], Pardoux [8]).
- New tools for calculus on spaces of (probability) measures (e.g. Ambrosio, Gigli & Savarè [1], P.-L. Lions [5], Carmona & Delarue [3] ).
- Optimal control with partial observation (e.g. Gozzi & Święch [4] in the Hilbert setting, or recently Bandini, Cosso, Fuhrman & Pham [2] on $\mathcal{P}_2(\mathbb{R}^d)$).

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## Signal process

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}s + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0 \in L^2(\Omega, \mathcal{F}_0), \quad t \in [0, T]. \tag{1}$$

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## Observation process

For every $t \in [0, T]$,

$$\begin{aligned} \mathrm{d}Y_t &= h(X_t)\,\mathrm{d}t + \mathrm{d}B_t, \quad Y_0 = 0, \\ \mathcal{F}_t^Y &= \sigma\left(Y_s, 0 \le s \le t\right) \vee \mathcal{N}, \end{aligned} \tag{2}$$

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## Goal

- The signal $X$ is not directly observed;
- The available information is given by $Y$;
- We want to provide an approximation of $X$ given the observation $Y$.

- Given the information $\mathcal{F}_t^Y$, the best estimate for $\varphi(X_t)$ is

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$$\Pi_t(A, \omega) = \mathbb{P}\left(X_t \in A|\mathcal{F}_t^Y\right)(\omega), \quad \text{a.e. } \omega.$$

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- For every $\varphi \in \mathrm{C_b}(\mathbb{R}^d)$ and $t \in [0,T]$,

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$\{\Pi_t = \mathrm{Law}(X_t|\mathcal{F}_t^Y)\}_{t\in[0,T]}$ is a $\mathcal{P}(\mathbb{R}^d)$-valued process called **filter**.

Define $\mathbb{Q}$ by $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}|_{\mathcal{F}_t} = M_t^{-1} = \exp\left\{-\frac{1}{2}\int_0^t |h(X_s)|^2\,\mathrm{d}s - \int_0^t h(X_s)\,\mathrm{d}B_s\right\}$.

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## Theorem (Kallianpur-Striebel formula)

*The filter* $\Pi$ *can be represented as*

$$\langle \Pi_t, \varphi\rangle = \frac{\langle \rho_t, \varphi\rangle}{\langle \rho_t, \mathbf{1}\rangle}, \quad t \in [0,T], \varphi \in \mathrm{C_b}(\mathbb{R}^d), \tag{3}$$

*where* $\langle \rho_t, \varphi\rangle = \mathbb{E}^{\mathbb{Q}}\left[M_t \varphi(X_t) | \mathcal{F}_t^Y\right]$.

$\{\rho_t\}_{t\in[0,T]}$ is a $\mathcal{M}^+(\mathbb{R}^d)$-valued process called **unnormalized filter**.

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$Y$ is a brownian motion under $\mathbb{Q}$. By Itô formula applied to $M_t\varphi(X)$ we obtain

### The Zakai equation (Z)

The unnormalized filter satisfies, for every test $\varphi$,

$$\mathrm{d}\langle \rho_t, \varphi\rangle = \langle \rho_t, A\varphi\rangle\,\mathrm{d}t + \langle \rho_t, h\varphi\rangle\,\mathrm{d}Y_t, \quad t \in (0, T], \tag{4}$$

where $A$ is the infinitesimal generator of $X$.

Let $A$ be the generator of $X$: $A\varphi = b^\top(\mathrm{D}_x\varphi) + \frac{1}{2}\operatorname{tr}\{(\mathrm{D}_x^2\varphi)\sigma\sigma^\top\}$.

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Using the Kallianpur-Striebel formula

### The Kushner-Stratonovich equation (KS)

The filter satisfies, for every test $\varphi$,

$$\mathrm{d}\langle\Pi_t, \varphi\rangle = \langle\Pi_t, A\varphi\rangle\,\mathrm{d}t + \left(\langle\Pi_t, h\varphi\rangle - \langle\Pi_t, \varphi\rangle\langle\Pi_t, h\rangle\right)\,\mathrm{d}I_t, \quad t \in (0, T],$$

where $\{I_t\}_{t\in[0,T]}$ is called **innovation process** and is a Brownian motion under $\mathbb{P}$.

Signal:

$$\mathrm{d}X_t = b_t X_t \,\mathrm{d}t + \sigma_t \,\mathrm{d}W_t, \quad a_t^{ij} = \sigma_t \sigma_t^\top,$$

$$A_t \varphi(x) = \mathrm{D}_x \varphi(x)^\top b_t x + \frac{1}{2} \sum_{i,j} a_t^{ij} \partial_{ij}^2 \varphi(x).$$

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Observation:

$$\mathrm{d}Y_t = h_t X_t \,\mathrm{d}t + \,\mathrm{d}B_t, \quad Y_0 = 0.$$

$(X, Y)$ is a gaussian process.

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$\iota(x) = x$. Moreover, for $\omega \in \Omega$ fixed, $\Pi_t(\omega)$ is gaussian with

- Mean $\hat{X}_t$ that solves the SDE

$$\mathrm{d}\hat{X}_t = b_t \hat{X}_t \,\mathrm{d}t + \gamma_t h_t \,\mathrm{d}I_t, \quad I_t = Y_t - \int_0^t h_s \hat{X}_s \,\mathrm{d}s.$$

- Deterministic variance that solves the Riccati equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \gamma_t = \gamma_t b_t^\top + b_t \gamma_t + a_t - \gamma_t (h^\top h) \gamma_t^\top.$$

Let $\{\rho_t\}_{t\in[0,T]}$ be a solution to (Z), i.e. for every test $\varphi$

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## Hypotheses (H)

a. $b, \sigma, h$ are Borel-measurable and bounded, $b, \sigma$ are Lipschitz;
b. The matrix $\sigma\sigma^\top(x)$ is positive definite for every $x \in \mathbb{R}^d$.

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## Proposition (M. [6])

Let $u$ be in $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}^+_2(\mathbb{R}^d))$ and let us assume (H). Then, for every $t \in [0,T]$:

$$\mathrm{d}u(\rho_t) = \langle \rho_t, A\delta_\mu u(\rho_t)\rangle \,\mathrm{d}t + \langle \rho_t, h\delta_\mu u(\rho_t)\rangle \,\mathrm{d}Y_t + \frac{1}{2}\langle \rho_t \otimes \rho_t, h^\top h\delta^2_\mu u(\rho_t)\rangle \,\mathrm{d}t.$$

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- $\delta_\mu u$ is a notions of derivatives for $u\colon \mathcal{M}^+(\mathbb{R}^d) \to \mathbb{R}$:

$$\delta_\mu u\colon \mathcal{M}^+(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}, \quad \delta^2_\mu u\colon \mathcal{M}^+(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R};$$

- Proof by cylindrical approximation: $u(\mu) := g\left(\langle \mu, \psi_1\rangle, \dots, \langle \mu, \psi_n\rangle\right)$.

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(\mathcal{L}u)(\mu) &= \langle \mu, A\delta_\mu u(\mu)\rangle + \frac{1}{2}\langle \mu \otimes \mu, h^\top h \delta_\mu^2 u(\mu)\rangle \\
&= \int_{\mathbb{R}^d} (A\delta_\mu u)(\mu, x)\mu(\,\mathrm{d}x) + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(x)^\top h(y)\delta_\mu^2 u(\mu, x, y)\mu(\,\mathrm{d}x)\mu(\,\mathrm{d}y).
\end{aligned}
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# The infinitesimal generator of the Zakai equation

The generator $\mathcal{L}\colon \mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d)) \to \mathrm{C}_{\mathrm{b}}(\mathcal{M}_2^+(\mathbb{R}^d))$

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## Remark

- Formally $\mathrm{d}\rho_t = A^*\rho_t\,\mathrm{d}t + h^\top\rho_t\,\mathrm{d}Y_t$, so:

$$
\mathrm{d}u(\rho_t) = \langle A^*\rho_t, \delta_\mu u(\rho_t)\rangle\,\mathrm{d}t + \langle h^\top\rho_t, \delta_\mu u(\rho_t)\rangle\,\mathrm{d}Y_t + \frac{1}{2}\langle h^\top\rho_t\otimes h\rho_t, \delta_\mu^2 u(\rho_t)\rangle\,\mathrm{d}t.
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The generator $\mathcal{L}\colon \mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d)) \to \mathrm{C}_{\mathrm{b}}(\mathcal{M}_2^+(\mathbb{R}^d))$

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- On $\mathbb{R}$, if $\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t$, then

$$\mathrm{d}u(X_t) = bX_t\,\mathrm{D}_x u(X_t)\,\mathrm{d}t + \sigma X_t\,\mathrm{D}_x u(X_t)\,\mathrm{d}B_t + \frac{1}{2}\sigma^2 X_t^2\,\mathrm{D}_x^2 u(X_t)\,\mathrm{d}t.$$

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Given $\Phi\colon \mathcal{M}_2^+(\mathbb{R}^d) \to \mathbb{R}$, **the Backward Kolmogorov equation** (BEZ) reads as

$$\begin{cases} \partial_s u(\mu, s) + \mathcal{L}u(\mu, s) = 0, & (\mu, s) \in \mathcal{M}_2^+(\mathbb{R}^d) \times [0, T], \\ \\ u(\mu, T) = \Phi(\mu), & \mu \in \mathcal{M}_2^+(\mathbb{R}^d). \end{cases}$$

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Let $\{\rho_t^{s,\mu}\}_{t\in[s,T]}$ be a solution to (Z) starting at time $s$ from $\mu \in \mathcal{M}_2^+(\mathbb{R}^d)$.

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Let $\{\rho_t^{s,\mu}\}_{t\in[s,T]}$ be a solution to (Z) starting at time $s$ from $\mu \in \mathcal{M}_2^+(\mathbb{R}^d)$.

**Theorem (M. [6])**

*Let $\Phi \in \mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$. Let (H) holds and let us set*

$$u(\mu, s) := \mathbb{E}\left[\Phi(\rho_T^{s,\mu})\right], \quad (\mu, s) \in \mathcal{M}_2^+(\mathbb{R}^d) \times [0, T]. \tag{6}$$

*Then $u$ is the unique classical solution to (BEZ).*

# Proof (key steps)

Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form

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Existence:

- Prove that $\mu \mapsto u(\mu, s) := \mathbb{E}\left[\Phi(\rho_T^{s,\mu})\right]$ is in $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$:

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  - given a suitable notion of derivative for functions from $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$ to $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$, we show that $\mu \mapsto \rho_T^{s,\mu}$ is twice differentiable;

# Proof (key steps)

Uniqueness:

- By the Itô formula, every classical solution to (BEZ) has the form

$$u(\mu, s) = \mathbb{E}\left[\Phi(\rho_T^{s,\mu})\right].$$

Existence:

- Prove that $\mu \mapsto u(\mu, s) := \mathbb{E}\left[\Phi(\rho_T^{s,\mu})\right]$ is in $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$:
  - given a suitable notion of derivative for functions from $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$ to $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$, we show that $\mu \mapsto \rho_T^{s,\mu}$ is twice differentiable;
  - since $\Phi \in \mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$ and by the previous point, we conclude by a chain rule.

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  - since $\Phi \in \mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$ and by the previous point, we conclude by a chain rule.
- By Itô formula and Markov property

$$\begin{aligned}\lim_{h\to 0} \frac{1}{h}\left[u(\mu, s+h) - u(\mu, s)\right] \\ = -\lim_{h\to 0} \frac{1}{h}\mathbb{E}\left[\int_s^{s+h} \mathcal{L}u(\rho_\tau^{s,\mu}, s+h)\,\mathrm{d}\tau\right] = -\mathcal{L}u(\mu, s).\end{aligned}$$

The operator $\mathcal{L}^{KS} \colon \mathrm{C}^2_{\mathrm{L}}(\mathcal{P}_2(\mathbb{R}^d)) \to \mathrm{C}_{\mathrm{b}}(\mathcal{P}_2(\mathbb{R}^d))$

$$\mathcal{L}^{KS}u(\pi) = \langle \pi, A\delta_\mu u(\pi)\rangle + \frac{1}{2}\langle \pi \otimes \pi, (h - \pi(h))^\top (h - \pi(h))\delta^2_\mu u(\pi)\rangle.$$

# The Kushner-Stratonovich equation case

The operator $\mathcal{L}^{KS} \colon \mathrm{C}^2_{\mathrm{L}}(\mathcal{P}_2(\mathbb{R}^d)) \to \mathrm{C}_{\mathrm{b}}(\mathcal{P}_2(\mathbb{R}^d))$

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Given $\Phi \colon \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, **the Backward Kolmogorov equation** (BEKS) reads as

$$\begin{cases} \partial_s u(\pi, s) + \mathcal{L}^{KS} u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(\mathbb{R}^d) \times [0, T], \\ \\ u(\pi, T) = \Phi(\pi), & \pi \in \mathcal{P}_2(\mathbb{R}^d). \end{cases}$$

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Given $\Phi\colon \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, **the Backward Kolmogorov equation** (BEKS) reads as

$$\begin{cases} \partial_s u(\pi,s) + \mathcal{L}^{KS}u(\pi,s) = 0, & (\pi,s)\in\mathcal{P}_2(\mathbb{R}^d)\times[0,T], \\ u(\pi,T) = \Phi(\pi), & \pi\in\mathcal{P}_2(\mathbb{R}^d). \end{cases}$$

Let $\{\Pi^{s,\pi}_t\}_{t\in[s,T]}$ be a solution to (KS) starting at time $s$ from $\pi\in\mathcal{P}_2(\mathbb{R}^d)$:

$$\mathrm{d}\langle \Pi_t,\psi\rangle = \langle \Pi_t, A\psi\rangle\,\mathrm{d}t + (\langle \Pi_t, h\psi\rangle - \langle\Pi_t,\psi\rangle\langle\Pi_t,h\rangle)\cdot \mathrm{d}I_t, \quad t\in(0,T]. \tag{7}$$

**Theorem (M. [6])**

*Let $\Phi\in\mathrm{C}^2_{\mathrm{L}}(\mathcal{P}_2(\mathbb{R}^d))$. Let (H) holds and let us set*

$$u(\pi,s) = \mathbb{E}\left[\Phi(\Pi^{s,\pi}_T)\right], \quad (\pi,s)\in\mathcal{P}_2(\mathbb{R}^d)\times[0,T].$$

*Then $u$ is the unique classical solution to (BEKS).*

$K \subset \mathbb{R}^d$ compact, $\Phi \in \mathrm{C_b}(\mathcal{P}_2(K))$:

$$\begin{cases} \partial_s u(\pi, s) + \mathcal{L}^{KS} u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(K) \times (0, T], \\ u(\pi, T) = \Phi(\pi), & \pi \in \mathcal{P}_2(K). \end{cases}$$

Let $\{\Pi_t^{s,\pi}\}_{t \in [s,T]}$ be a solution to (KS) confined in $\mathcal{P}_2(K)$.

**Theorem (M. [7])**

*Let $\Phi \in \mathrm{C_b}(\mathcal{P}_2(K))$. Let (H) holds and let us set*

$$u(\pi, s) = \mathbb{E}\left[\Phi(\Pi_T^{s,\pi})\right], \quad (\pi, s) \in \mathcal{P}_2(K) \times (0, T].$$

*Then $u$ is the unique viscosity solution to (BEKS).*

# Proof of the comparison principle (Key steps)

Let $u_1$ and $u_2$ be respectively a subsolution and a supersolution to (BEKS). Moreover, let $u(\pi, s) := \mathbb{E}\left[\Phi(\Pi_T^{s,\pi})\right]$. We want to show that $u_1 \leq u_2$.

- Show: $u_1 \leq u$ and $u \leq u_2$.
- Introduce a family of approximated problems:

$$\begin{cases} \partial_s u(\pi, s) + \mathcal{L}^{KS} u(\pi, s) = 0, & (\pi, s) \in \mathcal{P}_2(K) \times (0, T], \\ u(\pi, T) = \Phi_n(\pi) \in \mathrm{C}^2_{\mathrm{L}}(\mathcal{P}_2(K)), & \pi \in \mathcal{P}_2(K). \end{cases}$$

- $u^n(\pi, s) := \mathbb{E}\left[\Phi_n(\Pi_T^{s,\pi})\right]$ is a classical solution to the approximated problem which converges to $u$.
- Using the Borwein-Preiss variational principle with a suitable smooth gauge-type function, we introduce a suitable test function that allows us to conclude.

# Thank you!

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## Linear functional derivative

$u\colon \mathcal{M}^+(\mathbb{R}^d) \to \mathbb{R}$ is in $\mathrm{C}^1_\mathrm{b}(\mathcal{M}^+(\mathbb{R}^d))$ if it is continuous, bounded and if exists

$$\delta_\mu u\colon \mathcal{M}^+(\mathbb{R}^d) \times \mathbb{R}^d \ni (\mu, x) \mapsto \delta_\mu u(\mu, x) \in \mathbb{R},$$

bounded, continuous and such that for all $\mu$ and $\mu'$ in $\mathcal{M}^+(\mathbb{R}^d)$, it holds:

$$u(\mu') - u(\mu) = \int_0^1 \int_{\mathbb{R}^d} \delta_\mu u\left(t\mu' + (1-t)\mu, x\right) [\mu' - \mu](\mathrm{d}x)\,\mathrm{d}t. \tag{8}$$

Similarly we can define $\mathrm{C}^k_\mathrm{b}(\mathcal{M}^+(\mathbb{R}^d))$, $k \in \mathbb{N}$.

## Example

Let $g \in \mathrm{C}^2_\mathrm{b}(\mathbb{R})$ and let $\psi \in \mathrm{C}_\mathrm{b}(\mathbb{R}^d)$. We define

$$u\colon \mathcal{M}^+(\mathbb{R}^d) \ni \mu \mapsto g\left(\langle \mu, \psi\rangle\right) \in \mathbb{R}.$$

Then $u \in \mathrm{C}^2_\mathrm{b}(\mathcal{M}^+(\mathbb{R}^d))$ and it holds:

$$\delta_\mu u(\mu, x) = g'\left(\langle \mu, \psi\rangle\right)\psi(x), \quad \delta^2_\mu u(\mu, x, y) = g''\left(\langle \mu, \psi\rangle\right)\psi(x)\psi(y).$$

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## The space $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}^+(\mathbb{R}^d))$

$u\colon \mathcal{M}^+(\mathbb{R}^d) \to \mathbb{R}$ is in $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}^+(\mathbb{R}^d))$ if:

a. $u$ is in $\mathrm{C}^2_{\mathrm{b}}(\mathcal{M}^+(\mathbb{R}^d))$;

b. $\mathbb{R}^d \ni x \mapsto \delta_\mu u(\mu, x) \in \mathbb{R}$ is twice differentiable, with continuous and bounded derivatives on $\mathcal{M}^+(\mathbb{R}^d) \times \mathbb{R}^d$;

We set

$$\mathrm{D}_\mu u(\mu, x) := \mathrm{D}_x \delta_\mu u(\mu, x) \in \mathbb{R}^d,$$

## Remark

On $\mathcal{P}_2(\mathbb{R}^d)$, the derivative $\mathrm{D}_\mu u$ coincides with the one introduced by P.-L. Lions through the lifting procedure in the context of mean field games ([5, 3]).

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1. Prove the formula for functions of the form

$$u\colon \mathcal{M}_2^+(\mathbb{R}^d) \ni \mu \mapsto g\left(\langle \mu, \psi_1 \rangle, \dots, \langle \mu, \psi_n \rangle\right),$$

   exploiting classical Itô formula and the the Zakai equation.
2. Prove the formula for functions of the form

$$u(\mu) = \langle \frac{\mu^r}{\mu(\mathbb{R}^d)^r}, \varphi(\cdot, \dots, \cdot, \mu(\mathbb{R}^d)) \rangle$$

   by approximation, where $\varphi\colon \mathbb{R}^{d \times r+1} \to \mathbb{R}$ is symmetrical in the first $r$ arguments.
3. Prove the formula for functions in $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$ by approximation.

## Theorem (M. [6])

*Let us set*

$$u(\mu, s) = \mathbb{E}\left[\Phi(\rho_T^{s,\mu})\right], \tag{9}$$

*where* $\rho_T^{s,\mu}$ *is the weak solution to the Zakai equation starting at time s from* $\mu \in \mathcal{M}_2^+(\mathbb{R}^d)$, $\Phi \in \mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$ *and let (H) hold. Then u is the unique classical solution to the backward Kolmogorov equation (BEZ).*

## Proof (uniqueness)

We show that if $u$ is a classical solution to (BEZ), then $u(\mu, s) = \mathbb{E}\left[\Phi(\rho_T^{s,\mu})\right]$.

- By the Itô formula

$$u(\rho_T^{s,\mu}, T) - u(\rho_s^{s,\mu}, s) = \int_s^T \{\partial_s u(\rho_\tau^{s,\mu}, \tau) + \mathcal{L}u(\rho_\tau^{s,\mu}, \tau)\}\, \mathrm{d}\tau + \int_s^T \mathcal{G}u(\rho_\tau^{s,\mu}, \tau) \cdot \mathrm{d}Y_\tau.$$

- By taking the expectation and since $u$ solves (BEZ)

$$\mathbb{E}\left[\Phi\left(\rho_T^{s,\mu}\right)\right] - u(\mu, s) = \mathbb{E}\left[\int_s^T \mathcal{G}u(\rho_\tau^{s,\mu}, \tau) \cdot \mathrm{d}Y_\tau\right].$$

- The rhs is zero since the integral is a martingale, thus $u(\mu, s) = \mathbb{E}\left[\Phi(\rho_T^{s,\mu})\right]$.

### Proof (existence)

Let $u(\mu, s) = \mathbb{E}\left[\Phi(\rho_T^{s,\mu})\right]$ be our candidate solution.

1. Prove that $\mu \mapsto u(\mu, s)$ is in $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$:
   - given a suitable notion of derivative for functions from $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$ to $\mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$, we show that $\mu \mapsto \rho_T^{s,\mu}$ is twice differentiable;
   - since $\Phi \in \mathrm{C}^2_{\mathrm{L}}(\mathcal{M}_2^+(\mathbb{R}^d))$ and by the previous point, we conclude by a chain rule.
2. Prove the continuity of
$$[0, T] \ni s \mapsto \mathcal{L}u(\mu, s), \quad [s, T] \times [0, T] \ni (\tau, \sigma) \mapsto \mathcal{L}u(\rho_\tau^{s,\mu}, \sigma) \in L^2(\Omega).$$
3. By the Itô formula and the Markov property
$$\lim_{h\to 0} \frac{1}{h}\left[u(\mu, s+h) - u(\mu, s)\right] = -\lim_{h\to 0} \frac{1}{h}\mathbb{E}\left[\int_s^{s+h} \mathcal{L}u(\rho_\tau^{s,\mu}, s+h)\,\mathrm{d}\tau\right] = -\mathcal{L}u(\mu, s).$$