Mean field stochastic differential equations with a discontinuous diffusion

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General setting

- A stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,\infty)})$ satisfying usual conditions, $B = (B_t)_{t\geq 0}$ a d-dimensional Brownian motion adapted to $(\mathcal{F}_t)_{t\geq 0}$.
- Main equation:

$$
\begin{cases}
X_t = \overbrace{x_0}^{\in L_p(\Omega)} + \displaystyle\sum_{i=1}^{n} \int_0^t 1_{\{g(s)\in\mathcal{A}_i\}} \overbrace{\sigma_i(s, X_s, \mathbb{P}_{X_s})}^{\in\mathbb{R}^{d\times d}} \mathrm{d}B_s + \int_0^t \overbrace{b(s, X_s, \mathbb{P}_{X_s})}^{\in\mathbb{R}^d} \mathrm{d}s, \\
\underbrace{g(t) = \mathbb{E}\,\|X_t - z\|^p}_{\text{moment function}},
\end{cases}
$$

where

- $n \in \mathbb{N}$ (or $n = \infty$) and $p \geq 2$,
- $z \in \mathbb{R}^d$ is a fixed reference point,
- $(\mathcal{A}_i)_{i=1}^n$, pairwise disjoint Borel sets on $[0,\infty)$ such that $\bigcup_{i=1}^n \mathcal{A}_i = [0,\infty)$,
- $\sigma_i(t, x, \mu)$, $b(t, x, \mu)$ jointly measurable, Lipschitz continuous in (x, μ), satisfy the linear growth condition, uniformly in t.

Related literature

- **Huang and Wang** (2021): Mckean-Vlasov SDEs with drifts discontinuous under Wasserstein distance
- **Bauer, Meyer-Brandis and Proske** (2018): Strong solutions of mean-field stochastic differential equations with irregular drift
- **Mishura and Veretennikov** (2020): Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations
- **Mehri and Stannat** (2019): Weak solutions to Vlasov-McKean equations under Lyapunov-type conditions
- **Zhang** (2019): A discretized version of Krylov's estimate and its applications
- **Lejay and Pichot** (2012): Simulating diffusion processes in discontinuous media: A numerical scheme with constant time steps

We discuss the following topics:

- Example of an equation with multiple solutions
- Uniqueness when the drift is strong enough
- Non-existence of a global solution

Example

$$\begin{cases} X_t = 1 + \int_0^t 1_{\{g(s) \neq 1+a\}} \mathrm{d}B_s + \int_0^t 0 \cdot 1_{\{g(s)=1+a\}} \mathrm{d}B_s \\ g(t) = \mathbb{E}\,|X_t|^2, \end{cases} \tag{1}$$

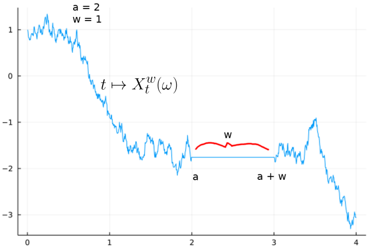
where

- $a \geq 0$ a fixed constant.

Solutions:

- $X_t^w := 1 + B_{t \wedge a} + \left(B_{t \vee (a+w)} - B_{a+w}\right)$, $w > 0$, with moment function:

$$g_w(t) = \begin{cases} 1 + t, & t < a, \\ 1 + a, & t \in [a, a+w], \\ t - (1+a) & t > a + w. \end{cases}$$



- $X_t := 1 + B_t$ also a solution.
  - $g(t) = 1 + t$.

Observation 1

Let $Y = (Y_t)_{t \geq 0}$ be the unique strong solution to the equation

$$Y_t = x_0 + \int_0^t \sigma_1(s, Y_s, \mathbb{P}_{Y_s})\mathrm{d}B_s + \int_0^t b(s, Y_s, \mathbb{P}_{Y_s})\mathrm{d}s.$$

Let $h(t) := \mathbb{E}\,\|Y_t - z\|^p$ and $\mu := \lambda \circ h^{-1}$. Then for any $\mathcal{N} \in \mathcal{B}([0, \infty))$ with $\mu(\mathcal{N}) = 0$ the process Y also solves the equation

$$\begin{cases} X_t = x_0 + \displaystyle\int_0^t b(s, X_s, \mathbb{P}_{X_s})\mathrm{d}s + \int_0^t 1_{\{g(s) \in [0,\infty) \setminus \mathcal{N}\}} \sigma_1(s, X_s, \mathbb{P}_{X_s})\mathrm{d}B_s \\ \qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_0^t 1_{\{g(s) \in \mathcal{N}\}} \sigma_2(s, X_s, \mathbb{P}_{X_s})\mathrm{d}B_s, \\ g(t) = \mathbb{E}\,\|X_t - z\|^p. \end{cases}$$

- Assume that $\mathcal{A}_i = [y_{i-1}, y_i)$, where $0 =: y_0 < y_1 < \ldots < y_n := \infty$.

Theorem 2

Assume that the drift is strong, that is, $\langle x - z, b(t,x,\mu)\rangle \geq 0$ for all $(t,x,\mu) \in [0,\infty) \times \mathbb{R}^d \times \mathcal{P}_p(\mathbb{R}^d)$, and the inequality is strict whenever $x \neq z$. Then there exists a unique global strong solution to the equation

$$\begin{cases} X_t = x_0 + \sum_{i=1}^{n} \int_0^t 1_{\{g(s)\in[y_{i-1},y_i)\}} \sigma_i(s, X_s, \mathbb{P}_{X_s}) \mathrm{d}B_s + \int_0^t b(s, X_s, \mathbb{P}_{X_s}) \mathrm{d}s, \\ g(t) = \mathbb{E} \left\| X_t - z \right\|^p, \end{cases}$$

when $p \geq 2$ and $\mathbb{P}(x_0 \neq z) > 0$.

Question

What if $\langle x - z, b(t,x,\mu)\rangle < 0$ for some (t,x,μ)?

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Assume that $z = 0$ and $d = 1$.

$$\begin{cases} X_t = x_0 + \int_0^t \left[1_{\{g(s)<y\}} C_{\sigma_1}(s) + 1_{\{g(s)\geq y\}} C_{\sigma_2}(s)\right] X_s \mathrm{d}B_s - \int_0^t C_b(s) X_s \mathrm{d}s, \\ g(t) = \mathbb{E}\left|X_t\right|^p, \end{cases} \tag{2}$$

where $y > 0$ and $C_{\sigma_i}, C_b : [0,\infty) \to [0,\infty)$ are continuous and bounded.

Theorem 3

Assume that $\mathbb{P}(x_0 \neq 0) > 0$ *and*

$$\inf_{t\geq 0}\left[\frac{p-1}{2} C_{\sigma_1}(t)^2 - C_b(t)\right] > 0, \quad \sup_{t\geq 0}\left[\frac{p-1}{2} C_{\sigma_2}(t)^2 - C_b(t)\right] < 0.$$

Then (2) *has no global solution.*

Idea of the proof

Assumption

$$\inf_{t\geq 0}\left[\frac{p-1}{2}C_{\sigma_1}(t)^2 - C_b(t)\right] > 0, \quad \sup_{t\geq 0}\left[\frac{p-1}{2}C_{\sigma_2}(t)^2 - C_b(t)\right] < 0$$

- If a solution exists, then

$$g(t) = \underbrace{\mathbb{E}\,|x_0|^p}_{=g(0)} \exp\left(p\int_0^t \left[\frac{p-1}{2}\left(1_{\{g(s)<y\}}C_{\sigma_1}(s)^2 + 1_{\{g(s)\geq y\}}C_{\sigma_2}(s)^2\right) - C_b(s)\right]\mathrm{d}s\right).$$

- Show that there is a local solution on $[0, r]$ such that $g(r) = y$ (if $g(0) = y$, then $r = 0$!) .

Assumption

$$\inf_{t\geq 0}\left[\frac{p-1}{2}C_{\sigma_1}(t)^2 - C_b(t)\right] > 0, \quad \sup_{t\geq 0}\left[\frac{p-1}{2}C_{\sigma_2}(t)^2 - C_b(t)\right] < 0$$

- Take any $t_1 > r$.
- Assume that $g(t_1) > y$.
- g continuous $\Rightarrow$ there is a $t_0 \in [r, t_1)$ such that $g(t_0) = y$ and $g(t) > y$ for $t \in (t_0, t_1)$.
- Then for any $t \in (t_0, t_1]$:

$$y \leq g(t) = \underbrace{g(0)\exp\left(\int_0^{t_0} \dots \mathrm{d}s\right)}_{=g(t_0)=y} \exp\underbrace{\left(p\int_{t_0}^{t}\left[\frac{p-1}{2}C_{\sigma_2}(s)^2 - C_b(s)\right]\mathrm{d}s\right)}_{<0} < y$$

$\Rightarrow$ Contradiction $\Rightarrow$ cannot hold that $g(t_1) > y$.

- $g(t_1) < y$ similarly.
- Remaining: $g(t) = y$ for all $t \in [r, t_1]$.
- **Conclusion:** no solution on $[0, r+\delta]$ for any $\delta > 0$.

Thank you for your attention!