European options
in a non-linear incomplete market with default

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BSDEs and Mean Field Systems
Market with imperfections

- Market with default.
  Ref: M. Jeanblanc, C. Blanchet-Scaillet, S. Crepey...
- The market is non-linear: the dynamics of the wealth process are non-linear.
  (Ex: funding costs...)
- The market is incomplete
Market with imperfections

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- The market is non-linear: the dynamics of the wealth process are non-linear.
  (Ex: funding costs...)
- The market is incomplete
- Our goal: study of the superhedging price of a European option.
The model

- Let \((\Omega, \mathcal{G}, \mathcal{P})\) be a complete probability space.
- Let \(\mathcal{W}\) be a one-dimensional Brownian motion.
- **default time** : \(\vartheta\) (random variable)
The model

- Let $\left( \Omega, \mathcal{G}, \mathcal{P} \right)$ be a complete probability space.
- Let $W$ be a one-dimensional Brownian motion.
- **default time**: $\vartheta$ (random variable)
- Let $N$ be the **default jump process**:
  $$N_t := 1_{\vartheta \leq t}$$
- Let $\mathcal{G} = \left\{ \mathcal{G}_t, t \geq 0 \right\}$ be the filtration associated with $W$ and $N$.
- **Hyp**: $W$ is a $\mathcal{G}$-Brownian motion.
- We have a $\mathcal{G}$-martingale representation theorem w.r.t. $W$ and $M$ (cf. Jeanblanc-Song (2015)).
**Hyp**: the $\mathcal{G}$-predictable compensator of $N_t$ is $\int_0^t \lambda_s \, ds$. $(\lambda_s)$ is called the **intensity** process, and is supposed to be bounded. It vanishes after $\vartheta$. 

Let $T > 0$. 

$$H_2 := \{ \text{predictable processes } Z \text{ s.t. } E\left[ \int_0^T Z_t^2 \lambda_t \, dt \right] < \infty \}$$
Hyp: the $\mathbb{G}$-predictable compensator of $N_t$ is: $\int_0^t \lambda_s ds$. $(\lambda_s)$ is called the **intensity** process, and is supposed to be bounded. It vanishes after $\vartheta$.

The compensated martingale of $(N_t)$ is thus given by

$$M_t := N_t - \int_0^t \lambda_s ds$$
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(\( \lambda_s \)) is called the **intensity** process, and is supposed to be bounded. It vanishes after \( \vartheta \).

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Let \( T > 0 \).

\( \mathcal{H}^2 := \{ \text{predictable processes } Z \text{ s.t. } \mathbb{E} \left[ \int_0^T Z_t^2 dt \right] < \infty \} \)

\( \mathcal{H}^2_{\lambda} := \{ \text{predictable processes } K \text{ s.t. } \mathbb{E} \left[ \int_0^T K_t^2 \lambda_t dt \right] < \infty \} \)
The market

One risky asset:

\[ dS_t = S_t - (\mu_t dt + \sigma_t dW_t + \beta_t dM_t) \]
with \( S_0 > 0 \).

- \( \sigma, \mu, \) and \( \beta \) are \( \mathcal{G} \)-predictable and bounded.
- Hyp: \( \sigma_t > 0 \) and \( \beta_\varphi > -1 \).
- To simplify the presentation, suppose \( \sigma_t = 1 \).

- investor with \text{initial} wealth \( x \).
  \( Z_t \) = amount invested in the risky asset at \( t \) (where \( Z \in \mathbb{H}^2 \)).
- Let \( V_t^{x,Z} \) the value of the portfolio at time \( t \).
The market

One risky asset:

\[ dS_t = S_t \left( \mu_t \, dt + \sigma_t \, dW_t + \beta_t \, dM_t \right) \] with \( S_0 > 0 \).

- \( \sigma, \mu, \) and \( \beta \) are \( \mathbb{G} \)-predictable and bounded.
- Hyp : \( \sigma_t > 0 \) and \( \beta_\theta > -1 \).
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- Investor with initial wealth \( x \).
  - \( Z_t \) = amount invested in the risky asset at \( t \) (where \( Z \in \mathbb{H}^2 \)).
- Let \( V_t^{x,Z} \) the value of the portfolio at time \( t \).
- In the classical linear case:

\[ dV_t = (r_t \, V_t + \theta_t \, Z_t) \, dt + Z_t (dW_t + \beta_t \, dM_t); \quad V_0 = x, \]

where \( r_t \) = risk-free interest rate, and \( \theta_t : = \mu_t - r_t \).
Here, for \((x, Z) \in \mathbb{R} \times \mathbb{H}^2\), the wealth \(V_t^{x,Z}\) satisfies:

\[-dV_t = f(t, V_t, Z_t)\,dt - Z_t(dW_t + \beta_t\,dM_t); \quad V_0 = x.\]

where \(f : (t, \omega, y, z) \mapsto f(t, \omega, y, z)\) is a nonlinear Lipschitz driver (non-convex).
Examples

recall the dynamics of the wealth $V^{x,Z}$:

$$-dV_t = f(t, V_t, Z_t)dt - Z_t(dW_t + \beta_t dM_t); \quad V_0 = x.$$ 

- Classical linear case: $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t$, where $\theta_t = \mu_t - r_t$.
- borrowing rate $R \neq$ lending rate $r$:
  $$f(t, V_t, Z_t) = -r_t (V_t - Z_t)^+ + R_t (V_t - Z_t)^- - \mu_t Z_t$$
- a repo market on which the risky asset is traded:
  $$f(t, V_t, Z_t) = -l_t Z_t^- + b_t Z_t^+ - r_t V_t - \theta_t Z_t,$$
  where $b_t = borrowing repo rate$,
  $l_t = lending repo rate$.

(cf. Brigo et al. ...).

- large seller whose strategy impacts the default intensity (cf. Dum.-Grig.-Q.-Sul. (2018))
Pricing in a complete non-linear market
(Ref : El Karoui-P-Q 97) Brownian filtration : suppose $\mathcal{F} := \mathcal{F}^W$.

$$dS_t = S_t(\mu_t dt + dW_t)$$

Consider a European option with maturity $T$ and payoff $\eta \in L^2(\mathcal{F}_T)$.

$\exists! (X, Z)$ in $\mathbb{H}^2 \times \mathbb{H}^2$ /

$$-dX_t = f(t, X_t, Z_t) dt - Z_t dW_t; \quad X_T = \eta.$$

$\rightarrow X = V^{X_0, Z}$
Pricing in a complete non-linear market

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$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t; \quad X_T = \eta.$$ 

$\rightarrow X = V^{X_0, Z} \rightarrow X_0 = X_0(T, \eta)$ is the hedging price (for the seller).

This leads to a $f$-nonlinear pricing system, introduced in El Karoui-Que. 96: $(T, \eta) \mapsto X^f(T, \eta)$ satisfying the monotonicity property, consistency property /$\eta$, the No-Arbitrage property....

later denoted by $\mathcal{E}^f$ and called $f$-expectation by S. Peng 97 (actually under an additional assumption ensuring that $\mathcal{E}^f(0) = 0$):

$$\forall \eta \in L^2(\mathcal{F}_T), \quad \mathcal{E}^f_{s, T}(\eta) := X_s(T, \eta), s \in [0, T].$$
The **buyer's hedging price** in this complete non-linear market would be equal to

\[-\xi_{t,T}(-\eta) = -X_t(T, -\eta).\]

Remark: setting \(\tilde{X}_0 := X_0(T, -\eta)\) and \(\tilde{Z} = Z(T, -\eta)\), we have \(\nabla \tilde{X}_0, \tilde{Z} + \eta = 0\) a.s.
Here, our nonlinear market is **incomplete**. Indeed, let $\eta \in L^2(G_T)$. It might not be possible to find $(x, Z)$ in $\mathbb{R} \times \mathbb{H}^2$ such that

$$V_T^{x,Z} = \eta.$$
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$$V_T^{x,Z} = \eta.$$ 

In other words, there does not necessarily exist $(V, Z) \in \mathbb{H}^2 \times \mathbb{H}^2$ such that

$$-dV_t = f(t, V_t, Z_t)dt - Z_t dW_t - Z_t \beta_t dM_t; \quad V_T = \eta,$$

However, by the $G$-martingale representation w.r.t. $W, M$, $\exists! (Y, Z, K)$ in $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2$ solution of the BSDE with default (cf. G-Q-S 2018 for details)

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \eta.$$
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Indeed, let \( \eta \in L^2(G_T) \). It might not be possible to find \((x, Z)\) in \( \mathbb{R} \times \mathbb{H}^2 \) such that

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In general, $K \neq Z\beta$. 

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Notation: if \((Y, Z, K)\) is the solution of the \(\mathbb{G}\)-BSDE

\[-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \eta,\]

we set \(\mathcal{E}^{f}_{s,T}(\eta) := Y_s\) for all \(s \in [0, T]\), called \(f\)-evaluation/expectation of \(\eta\) under \(P\).

It might be a possible price but it does not necessarily allow the seller to be hedged (except if \(K = Z\beta\)).

**Definition**

**seller’s superhedging price** at time 0:

\[v_0 := \inf\{x \in \mathbb{R} : \exists Z \in \mathbb{H}^2 \text{ with } V^x,Z_T \geq \eta \text{ a.s.}\}.\]

Dual representation formula for this price?
The classical linear (incomplete) case

Up to discounting, we may suppose \( r = 0 \), so

- \textit{In this case, } \( f(t, y, z) := -\mu_t z \)

**Definition**: Let \( R \sim P \).

\( R \) is called a martingale probability measure if

\( \forall x \in \mathbb{R}, \forall Z \in \mathbb{H}^2 \), the wealth \( (V_t^x, Z) \) is an \( R \)-martingale

**Dual representation of the seller’s superhedging price** (ref: EL Karoui-Qu.(91-95)):

\[
\nu_0 = \sup_{R \in \mathcal{P}} E_R(\eta),
\]

where \( \mathcal{P} := \{ \text{martingale probability measures}\} \).

Recall the proof: using the martingale property of the wealths under \( R \) for all \( R \in \mathcal{P} \), we get \( V_0 \geq \ldots \).
Recall that the proof of the other inequality \( v_0 \leq \ldots \) relied on:

**Optional decomposition Theorem**: (ref: EL Karoui-Qu.(91-95)), generalized by Föllmer...:

*If \((Y_t)\) is a càdlàg supermartingale under \(R\), for all \(R \in \mathcal{P}\), then, \(\exists Z \in \mathbb{H}^2\), and a càdlàg nondecreasing optional process \(h\), with \(h_0 = 0\) such that*

\[
Y_t = V^Y_{t;Z} - h_t \quad 0 \leq t \leq T.
\]

*that is,*

\[
Y_t = Y_0 + \int_0^t \mu_s Z_s ds + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t.
\]

Remark:

\(\forall R \in \mathcal{P}\),

\[E_R(\eta|F_S) = v_0 - E_R(h_T)\].

Hence

\[
\inf_{R \in \mathcal{P}} E_R(h_T) = 0.
\]

(Here, this is clear since we have \(h_T = V^Y_{v_0,Z}, Z_T - \xi = \) terminal profit for the seller, which does not hold in the non-linear case).
Recall that the proof of the other inequality $v_0 \leq \ldots$ relied on:

**Optional decomposition Theorem**: (ref: EL Karoui-Qu.(91-95)), generalized by Föllmer...)

*If $(Y_t)$ is a càd-làg **supermartingale** under $R$, for all $R \in \mathcal{P}$, then, $\exists Z \in \mathbb{H}^2$, and a càd-làg nondecreasing optional process $h$, with $h_0 = 0$ such that*

$$Y_t = V_t^{Y_0,Z} - h_t, \quad 0 \leq t \leq T.$$  

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**Proof of the dual representation**: let $X_S := \text{ess sup}_{R \in \mathcal{P}} E_R(\eta|\mathcal{F}_S)$. By the above theorem, we show $X_t = V_t^{X_0,Z} - h_t, \forall t \in [0, T]$. Hence,

$$X_T = \eta = V_T^{X_0,Z} - h_T \Rightarrow V_T^{X_0,Z} \geq \eta \Rightarrow X_0 \geq v_0 \ldots X_0 = v_0. \text{ QED}$$  

Remark: $\forall R \in \mathcal{P}, E_R(\eta) = v_0 - E_R(h_T)$. Hence $\inf_{R \in \mathcal{P}} E_R(h_T) = 0$. (Here, this is clear since we have $h_T = V_{v_0}, Z_T - \xi_T = \text{terminal profit for the seller}$, which does not hold in the non-linear case).
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$$Y_t = V_t^{Y_0,Z} - h_t \quad 0 \leq t \leq T.$$*

that is,

$$Y_t = Y_0 + \int_0^t \mu_s Z_s ds + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t.$$

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**Remark**: $\forall R \in \mathcal{P}$, $E_R(\eta) = v_0 - E_R(h_T)$. Hence $\inf_{R \in \mathcal{P}} E_R(h_T) = 0$. 

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Non-linear incomplete market with default
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*If $(Y_t)$ is a càd-làg supermartingale under $R$, for all $R \in \mathcal{P}$, then, $\exists Z \in \mathbb{H}^2$, and a càd-làg nondecreasing optional process $h$, with $h_0 = 0$ such that*

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By the above theorem, we show $X_t = V^{X_0,Z}_t - h_t$, $\forall t \in [0, T]$. Hence,

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**Remark**: $\forall R \in \mathcal{P}$, $E_R(\eta) = v_0 - E_R(h_T)$. Hence $\inf_{R \in \mathcal{P}} E_R(h_T) = 0$.

(Here, this is clear since we have $h_T = V^{v_0,Z}_T - \xi = \text{terminal profit for the seller, which does not hold in the non-linear case}$.)

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Marie-Claire Quenez (LPSM) | Non-linear incomplete market with default | 29 June 2022
• Question: what is the analogous of martingale probability measures in the case when \( f \) is non-linear?

• First, we define the non-linear \( f \)-expectation under \( Q \) for \( Q \sim P \).
Let $Q \sim P$. From the $\mathcal{G}$-martingale representation theorem, its density process $(\zeta_t)$ satisfies

$$d\zeta_t = \zeta_t - (\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1,$$

where $(\alpha_t)$ and $(\nu_t)$ are $\mathcal{G}$-predictable processes with $\nu_{\vartheta \wedge \tau} > -1$ a.s. By Girsanov’s theorem,

- $W^Q_t := W_t - \int_0^t \alpha_s ds$ is a $Q$-Brownian motion, and
- $M^Q_t := M_t - \int_0^t \nu_s \lambda_s ds$ is a $Q$-martingale.
Let $Q \sim P$. From the $\mathcal{G}$-martingale representation theorem, its density process $(\zeta_t)$ satisfies

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where $(\alpha_t)$ and $(\nu_t)$ are $\mathcal{G}$-predictable processes with $\nu_{\Theta \wedge T} > -1$ a.s.

By Girsanov’s theorem,

- $\mathcal{W}^Q_t := \mathcal{W}_t - \int_0^t \alpha_s ds$ is a $Q$-Brownian motion, and
- $\mathcal{M}^Q_t := \mathcal{M}_t - \int_0^t \nu_s \lambda_s ds$ is a $Q$-martingale.

We have a $Q$-martingale representation for $Q$-martingales w.r.t. $\mathcal{W}^Q$ and $\mathcal{M}^Q$. We can thus consider $Q$-BSDEs driven by $\mathcal{W}^Q, \mathcal{M}^Q$. 
Let $Q \sim P$. Let $(X, Z, K) \in \mathbb{H}^2_Q \times \mathbb{H}^2_Q \times \mathbb{H}^2_Q, \lambda$ be the sol. of the $Q$-BSDE

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t^Q - K_t dM_t^Q; \quad X_T = \eta.$$  

We call $Q$-pricing system or $f$-evaluation under $Q$, denoted by $\mathcal{E}_Q^f$ or more simply $\mathcal{E}_Q$ (or $\mathcal{E}^Q$), the operator defined by: for $\eta \in L^2_Q(G_T)$,

$$\mathcal{E}_{s,T}^Q(\eta) := X_s, \quad s \in [0, T]$$

It can be a possible price (see the last slide for details).
Let $Q \sim P$. Let $(X, Z, K) \in \mathbb{H}_Q^2 \times \mathbb{H}_Q^2 \times \mathbb{H}_Q^2, \lambda$ be the sol. of the $Q$-BSDE

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t^Q - K_t dM_t^Q; \quad X_T = \eta.$$ 

We call $Q$-pricing system or $f$-evaluation under $Q$, denoted by $\mathcal{E}_Q^f$ or more simply $\mathcal{E}_Q$ (or $\mathcal{E}_Q^Q$), the operator defined by: for $\eta \in L^2_Q(G_T)$,

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It can be a possible price (see the last slide for details).

**Definition**

(Peng) Let $Y \in S^2_Q$. The process $(Y_t)$ is said to be a (strong) $\mathcal{E}_Q$-martingale (or $\mathcal{E}_Q^f$-martingale under $Q$), if $\forall s, t$ stopping times with $s \leq t$,

$$\mathcal{E}_{s,t}^Q(Y_t) = Y_s \quad \text{a.s.}$$
Let $Q \sim P$. Let $(X, Z, K) \in \mathbb{H}^2_Q \times \mathbb{H}^2_Q \times \mathbb{H}^2_{Q,\lambda}$ be the sol. of the $Q$-BSDE

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We call $Q$-pricing system or $f$-evaluation under $Q$, denoted by $\mathcal{E}^f_Q$ or more simply $\mathcal{E}_Q$ (or $\mathcal{E}^Q$), the operator defined by: for $\eta \in L^2_Q(\mathcal{G}_T),$

$$\mathcal{E}^Q_{s,T}(\eta) := X_s, \quad s \in [0, T]$$

It can be a possible price (see the last slide for details).

**Definition**

(Peng) Let $Y \in S^2_Q$. The process $(Y_t)$ is said to be a (strong) $\mathcal{E}_Q$-martingale (or $\mathcal{E}^f$-martingale under $Q$), if $\forall s, t$ stopping times with $s \leq t$,

$$\mathcal{E}^Q_{s,t}(Y_t) = Y_s \quad \text{a.s.}..$$

Question: what is the analogous of martingale probability measures in the non-linear case?
Definition

A probability $Q \sim P$ is called an $\mathcal{E}^f$-martingale probability measure if:

$\forall \ x \in \mathbb{R}$ and $\forall \ Z \in \mathbb{H}^2_Q$, the wealth $V^{x,Z}$ is a $\mathcal{E}^f$-martingale under $Q$.

We denote by $\mathcal{Q} := \{ \mathcal{E}^f$-martingale probabilities $\}$
Definition

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Remarks:

- $P \in \mathcal{Q}$.
- $Q \in \mathcal{Q} \iff W + \int \beta_s dM_s$ is a $Q$-martingale.
- $\mathcal{Q}$ is equipotent to $\mathcal{P}$.
Theorem

Let $\eta \in L^2_Q(\mathcal{G}_T)$, for all $Q \in \mathcal{Q}$. Under an appropriate integrability condition (see next slide), we have $v_0 < \infty$ and

$$v_0 = \sup_{Q \in \mathcal{Q}} \mathcal{E}^Q_{0,T}(\eta),$$

Proposition:
The supremum is attained if and only if the option is replicable. In this case, $\mathcal{E}^Q_{0,T}(\eta) = \mathcal{E}^P_{0,T}(\eta)$ for all $Q \in \mathcal{Q}$. 

Sketch of the proof of the theorem:
First, using the $\mathcal{E}^Q$-martingale property of the wealths for all $Q \in \mathcal{Q}$, we get (quite easily): $v_0 \geq \sup_{Q \in \mathcal{Q}} \mathcal{E}^Q_{0,T}(\eta)$.
Dual representation of the seller’s price

**Theorem**

Let \( \eta \in L^2_Q(G_T) \), for all \( Q \in \mathcal{Q} \). Under an appropriate integrability condition (see next slide), we have \( v_0 < \infty \) and

\[
v_0 = \sup_{Q \in \mathcal{Q}} E_Q^0 \left( \eta \right),
\]

**Proposition**: The supremum is attained if and only if the option is replicable.

In this case, \( E_Q^{\mathcal{Q}}(\eta) = E_P^{\mathcal{Q}}(\eta) \) \( \forall Q \in \mathcal{Q} \).
Dual representation of the seller’s price

**Theorem**

Let $\eta \in L^2_Q(\mathcal{G}_T)$, for all $Q \in \mathcal{Q}$. Under an appropriate integrability condition (see next slide), we have $v_0 < \infty$ and

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**Proposition**: The supremum is attained if and only if the option is replicable.

In this case, $\mathcal{E}^Q_{0,T}(\eta) = \mathcal{E}^P_{0,T}(\eta)$ $\forall$ $Q \in \mathcal{Q}$.

**Sketch of the proof of the theorem**: First, using the $\mathcal{E}^Q$-martingale property of the wealths for all $Q \in \mathcal{Q}$, we get (quite easily):

$$v_0 \geq \sup_{Q \in \mathcal{Q}} \mathcal{E}^Q_{0,T}(\eta)$$
In order to show the inequality \( \leq \), we first show:

**Theorem (non-linear optional decomposition):** 
Let \((Y_t) \in S^2_Q \forall Q \in \mathcal{Q}\).

If \((Y_t)\) is a strong \(\mathcal{F}_Q\)-supermartingale \(\forall Q \in \mathcal{Q}\),

\[
(Y_t) = (Y_0) - \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t, \quad 0 \leq t \leq T.
\]

**Remark:** in the linear case (with \(r = 0\)), \(f(t, z) = -\mu t z\). Hence,

\[
(Y_t) = (Y_0) + \int_0^t Z_s (\mu s ds + dW_s + \beta_s dM_s) - h_t,
\]

which corresponds to the classical optional decomposition theorem.
In order to show the inequality \( \leq \), we first show:

**Theorem (non-linear optional decomposition):**

Let \( (Y_t) \in S^2_Q \ \forall \ Q \in \mathcal{Q} \).

If \( (Y_t) \) is a strong \( \mathcal{E}_Q \)-supermartingale \( \forall \ Q \in \mathcal{Q} \),
then, there exists \( Z \in H^2 \), and a nondecreasing optional càdlàg process \( h \), with \( h_0 = 0 \) / \[
Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t, \quad 0 \leq t \leq T.
\]

**Remark:** in the **linear** case (with \( r = 0 \)), \( f(t, z) = -\mu_t z \). Hence,

\[
Y_t = Y_0 + \int_0^t Z_s (\mu_s ds + dW_s + \beta_s dM_s) - h_t,
\]

**which corresponds to the classical optional decomposition theorem.**
End of the proof of the dual representation:
\[ \exists \ (X_t) \in S^2/ \text{for all } S, \]
\[ X_S = \text{ess sup}_{Q \in \mathcal{Q}} E_Q^{S,T}(\eta) \ a.s. \]

(recall that it remained to show that \( X_0 \geq v_0 \))

- \((X_t)\) is an \( \mathcal{E}_Q \)-supermartingale for each \( Q \in \mathcal{Q} \) (with \( X(T) = \eta \)).
- By the optional \( \mathcal{E}^f \)-decomposition theorem, \( \exists Z, h... / \)

\[ X_t = X_0 - \int_0^t f(s, X_s, Z_s) \, dt + \int_0^t Z_s (dW_s + \beta_s dM_s) - h_t, \quad 0 \leq t \leq T. \]

- By the comparison theorem for **forward** SDEs, (\( \eta = ) \) \( X_T \leq V_T^{X_0,Z} \). Hence, \( X_0 \geq v_0 \). Hence, \( X_0 = v_0 \). **QED**

Note that \((v_0, Z)\) is a **superhedging strategy** for the seller (since \( \eta \leq V_T^{v_0,Z} \)).
For each $S \in \mathcal{T}$, set

$$X(S) := \text{ess sup}_{Q \in \mathcal{Q}} \mathcal{E}_{S,T}^Q(\eta)$$

**Proposition:**

$$v_0 < \infty \iff E_Q[\text{ess sup}_{S \in \mathcal{T}} X(S)^2] < +\infty, \quad \forall Q \in \mathcal{Q}$$

**Remark:** for example, this condition is satisfied if $\eta = (S_T - K)^+$. 
Let $(x, \varphi)$ in $\mathbb{R} \times \mathbb{H}^2$ be a superhedging strategy in the sense that $V^x_{T,\varphi} \geq \eta$ a.s.

$\rightarrow V^x_{T,\varphi} - \eta = \text{terminal profit}$ realized by the seller.

**Prop:** $x = v_0 \iff \inf_{Q \in Q} \mathbb{E}_Q (V^x_{T,\varphi} - \eta) = 0$.

In particular, this minimality condition is satisfied by $(v_0, Z)$.

**Theorem**

*There exists a sequence $(Q_n) \in \mathcal{Q}$ s.t.*

(i) 

$v_0 = \lim_{n \to +\infty} \mathcal{E}^{Q_n}_{0,T} (\eta),$

(ii) $Q_n \to Q^* (\ll P)$ weakly as $n \to \infty$, and $\eta$ is replicable under $Q^*$ since

$\eta = V_{T,\varphi}^{v_0,Z} Q^* - a.e.,$

where $Z$ is the process from the non-linear optional decomposition of $X$.

*Here, $Q_n \to Q^*$ weakly in the sense that $\frac{dQ_n}{dP} \to \frac{dQ^*}{dP}$ $P$-a.s.*
In the **linear** case, we can even prove the following (new) result: there exists a non-negative measure $R^* \ll P$, which is the *weak* limit of a sequence $(R_n)$ of *martingale probability measures*, such that

$$v_0 = "E_{R^*}(\eta)" (= \int_\Omega \eta dR^*), \quad (0.1)$$

and $\eta$ is replicable under $R^*$, more precisely

$$\eta = V_T^{v_0, Z} R^* - a.e.,$$

where $Z$ is the process from the (linear) optional decomposition of the dual value process $X_S := \text{ess sup}_{R \in \mathcal{P}} E_R(\eta|G_S)$.

**Remark**: *in the non-linear case, we cannot have an analogous equality to (0.1) since $E_{R^*}$ does not make sense* (we do not even know if $R^*$ is a probability measure).
Characterization of $v_0$ via a constrained BSDE

**Theorem**: $v_0 = X_0$, where the process $X$ is characterized as the (minimal) **supersolution** of the constrained BSDE with default, that is, such that $\exists (Z, K) \in H^2 \times H_\lambda^2$, and a **predictable** nondecreasing process $A$ satisfying

$$-dX_t = f(t, X_t, Z_t)dt - Z_tdW_t - K_t dM_t + dA_t; \quad X_T = \eta;$$

$$A_t + \int_0^t (K_s - \beta_s Z_s)\lambda_s ds \quad \text{is nondecreasing}$$

$$(K_t - \beta_t Z_t)\lambda_t \leq 0, \quad dP \otimes dt - \text{a.e.};$$

Remark: $Z_t$ and $h_t := A_t - \int_0^t (K_s - \beta_s Z_s)dM_s$ correspond to the processes from the non-linear optional decomposition of the dual value process $(X_t)$. 
Definition (buyer’s superhedging price)

\[ \tilde{v}_0 := \sup \{ x \in \mathbb{R} : \exists Z \in \mathbb{H}^2 \text{ with } V_T^{-x,Z} + \eta \geq 0 \} . \]

Remark : Note that superhedging price \( \tilde{v}_0 \) for the buyer is equal to the opposite of the superhedging price for the seller of the option with payoff \( -\eta \).

\[ \tilde{v}_0 = - \sup_{\mathcal{Q} \in \mathcal{Q}} \mathcal{E}^f_{\mathcal{Q},0,T}(-\eta) . \]

The interval \((\tilde{v}_0, v_0)\) (open of closed, it depends on \( \eta \)) can be interpreted as an arbitrage-free interval (set of arbitrage-free prices for the European option \( \eta \)) in the sense of Karatzas and Kou. It can be empty for particular \( f \) and \( \eta \) (see our paper for details).
Our present paper:

Remark:
For the notion of non-linear $f$-pricing systems in a complete non-linear market and its properties (notions of consistency, no-arbitrage property, non-negativity when $f(t,0,0) \geq 0$ ...), see: