European options in a non-linear incomplete market with default

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BSDEs and Mean Field Systems

Market with imperfections

- Market with default .
  Ref : M. Jeanblanc, C. Blanchet-Scaillet, S. Crepey...
- The market is non-linear : the dynamics of the wealth process are non-linear.
  (Ex : funding costs...)
- The market is incomplete

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- The market is incomplete
- Our goal : study of the superhedging price of a European option.

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- Let W be a one-dimensional Brownian motion.
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- Let W be a one-dimensional Brownian motion.
- default time : ϑ (random variable)
- Let **N** be the default jump process :
  $\mathbf{N_t} := \mathbf{1}_{\vartheta \leq \mathbf{t}}$
- Let $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ be the filtration associated with W and N.
- **Hyp :** W is a $\mathbb{G}$-Brownian motion.‘
- We have a $\mathbb{G}$-martingale representation theorem w.r.t. W and M (cf. Jeanblanc-Song (2015)).

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Let $T > 0$.

- $\mathbb{H}^2 := \{$ predictable processes Z s.t. $\mathbb{E}\left[\int_0^T Z_t^2 dt\right] < \infty\}$
- $\mathbb{H}^2_\lambda := \{$ predictable processes K s.t. $\mathbb{E}\left[\int_0^T K_t^2 \lambda_t dt\right] < \infty\}$

The market

One risky asset :

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t d\mathbf{M}_t) \text{ with } S_0 > 0.$$

- $\sigma_.$, $\mu_.$, and $\beta_.$ are $\mathbb{G}$- predictable and bounded.
- **Hyp :** $\sigma_t > 0$ and $\beta_\vartheta > -1$.
- To **simplify** the presentation, suppose $\sigma_t = \mathbf{1}$.

- investor with **initial** wealth x.
  $\mathbf{Z_t}$= amount invested in the risky asset at t (where $\mathbf{Z}_. \in \mathbb{H}^2$).
- Let $V_t^{x,Z}$ the value of the portfolio at time t.

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- Let $V_t^{x,Z}$ the value of the portfolio at time t.
- In the classical **linear** case :

$$dV_t = (r_t V_t + \theta_t Z_t)dt + Z_t(dW_t + \beta_t dM_t); \quad V_0 = x,$$

where r_t = risk-free interest rate, and $\theta_t := \mu_t - r_t$.

Here, for $(x, Z) \in \mathbb{R} \times \mathbb{H}^2$, the wealth $V_t^{x,Z}$ satisfies :

$$-dV_t = \mathbf{f}(t, V_t, Z_t)dt - Z_t(dW_t + \beta_t dM_t); \quad V_0 = x.$$

where $\mathbf{f} : (t, \omega, y, z) \mapsto \mathbf{f}(t, \omega, y, z)$ is a **nonlinear** Lipschitz driver (non-convex).

Examples

recall the dynamics of the wealth $V^{x,Z}$:

$$-dV_t = \mathbf{f}(t, V_t, Z_t)dt - Z_t(dW_t + \beta_t dM_t); \quad V_0 = x.$$

- Classical linear case : $f(t, V_t, Z_t) = -r_t V_t - \theta_t Z_t$, where $\theta_t = \mu_t - r_t$.
- borrowing rate $\mathbf{R} \neq$ lending rate $\mathbf{r}$:
  $f(t, V_t, Z_t) = -r_t(V_t - Z_t)^+ + \mathbf{R_t}(V_t - Z_t)^- - \mu_t Z_t$
- **a repo market** on which the risky asset is traded :
  $f(t, V_t, Z_t) = -\mathbf{l_t Z_t^-} + \mathbf{b_t Z_t^+} - r_t V_t - \theta_t Z_t$,
  $\mathbf{b_t}$ = borrowing repo rate,
  $\mathbf{l_t}$ = lending repo rate.
  (cf. Brigo et al. ...).
- **large seller** whose strategy impacts the default intensity (cf. Dum.-Grig.-Q.-Sul. (2018))

Pricing in a complete non-linear market

(Ref : El Karoui-P-Q 97) Brownian filtration : suppose $\mathcal{F} := \mathcal{F}^W$.

$$dS_t = S_t(\mu_t dt + dW_t)$$

Consider a European option with maturity T and payoff $\eta \in L^2(\mathcal{F}_T)$.
$\exists!$ (X, Z) in $\mathbb{H}^2 \times \mathbb{H}^2$ /

$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t; \quad X_T = \eta.$$

$\rightarrow X = V^{X_0, Z}$

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$$-dX_t = f(t, X_t, Z_t)dt - Z_t dW_t; \quad X_T = \eta.$$

$\rightarrow X = V^{X_0, Z} \rightarrow X_0 = X_0(T, \eta)$ is the hedging price(for the seller).

This leads to a f-nonlinear pricing system, introduced in El Karoui-Que. 96 :
$(T, \eta) \mapsto X^f(T, \eta)$ satisfying the **monotonicity** property, **consistency** property /η, the **No-Arbitrage** property....
later denoted by $\mathcal{E}^f$ and called f-expectation by S.Peng 97 (actually under an additional assumption ensuring that $\mathcal{E}^f(0) = 0$) :

$$\forall \eta \in L^2(\mathcal{F}_T), \quad \mathcal{E}^f_{s,T}(\eta) := X_s(T, \eta), s \in [0, T].$$

The buyer's hedging price in this complete non-linear market would be equal to

$$-\mathscr{E}^{f}_{t,T}(-\eta) = -X_t(T,-\eta).$$

Remark : setting $\tilde{X}_0 := X_0(T,-\eta)$ and $\tilde{Z} = Z(T,-\eta)$, we have $V_T^{\tilde{X}_0,\tilde{Z}} + \eta = 0$ a.s.

Here, our nonlinear market is **incomplete**.
Indeed, let $\eta \in L^2(\mathcal{G}_T)$. It might not be possible to find (x, Z) in $\mathbb{R} \times \mathbb{H}^2$ such that

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However, by the $\mathbb{G}$-martingale representation w.r.t. W, M, $\exists!$ $(Y, Z, \mathbf{K})$ in $\mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ solution of the BSDE with default (cf. G-Q-S 2018 for details)

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - \mathbf{K_t} dM_t; \quad Y_T = \eta.$$

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In general, $\mathbf{K} \neq Z\beta$.

Notation : if (Y, Z, K) is the solution of the $\mathbb{G}$-BSDE

$$-dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t - K_t dM_t; \quad Y_T = \eta,$$

we set $\mathcal{E}^f_{s,T}(\eta) := Y_s$ for all $s \in [0, T]$, called *f*-evaluation/expectation of η under P.

It might be a possible price but it does not necessarily allow the seller to be hedged (except if $\mathbf{K} = Z\beta$).

Definition

seller's superhedging price at time 0 :

$$v_0 := \inf\{x \in \mathbb{R} : \exists Z \in \mathbb{H}^2 \text{ with } V_T^{x,Z} \geq \eta \ a.s.\}.$$

Dual representation formula for this price ?

The classical linear (incomplete) case

Up to discounting, we may suppose $r = 0$, so

- *In this case*, $f(t, y, z) := -\mu_t z$
- **Definition** : Let $R \sim P$.
  R is called a martingale probability measure if
  $\forall\, x \in \mathbb{R}$, $\forall\, Z \in \mathbb{H}^2$, the **wealth** $(V_t^{x,Z})$ is an R**-martingale**
- **Dual representation of the seller's superhedging price**
  (ref : EL Karoui-Qu.(91-95)) :

$$v_0 = \sup_{R \in \mathscr{P}} E_R(\eta),$$

where $\mathscr{P} := \{$ martingale probability measures$\}$.
Recall the proof : using the martingale property of the wealths under R for all $\mathbf{R} \in \mathscr{P}$,we get $V_0 \geq$ ....

Recall that the proof of the other inequality $v_0 \leq$ ... relied on :

Optional decomposition Theorem : (ref : EL Karoui-Qu.(91-95)), generalized by Föllmer...) :

If (Y_t) *is a càd-làg* **supermartingale** *under* $\mathbf{R}$, *for all* $\mathbf{R} \in \mathscr{P}$, *then,* $\exists\, Z \in \mathbb{H}^2$, *and a càd-làg nondecreasing optional process* $\mathbf{h}.$, *with* $\mathbf{h}_0 = 0$ *such that*

$$Y_t = V_t^{Y_0, Z} - \mathbf{h}_t \quad 0 \leq t \leq T.$$

that is,

$$Y_t = Y_0 + \int_0^t \mu_s Z_s ds + \int_0^t Z_s (dW_s + \beta_s dM_s) - \mathbf{h}_t.$$

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proof of the dual representation : let $X_S := ess\sup_{R \in \mathscr{P}} E_R(\eta | \mathcal{F}_S)$.
By the above theorem, we show $X_t = V_t^{X_0,Z} - h_t, \forall t \in [0,T]$. Hence,

$$X_T = \eta = \mathbf{V_T^{X_0,Z}} - \mathbf{h_T} \Rightarrow V_T^{X_0,Z} \geq \eta \Rightarrow X_0 \geq v_0 \ldots X_0 = v_0. \textbf{ QED}$$

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Remark : $\forall R \in \mathscr{P}$, $E_R(\eta) = \mathbf{v_0} - \mathbf{E_R}(\mathbf{h_T})$. Hence $\mathbf{inf_{R \in P} E_R}(\mathbf{h_T}) = \mathbf{0}$.

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Remark : $\forall R \in \mathscr{P}$, $E_R(\eta) = \mathbf{v_0} - \mathbf{E_R}(\mathbf{h_T})$. Hence $\mathbf{\inf_{R \in P} E_R(h_T) = 0}$.
(Here, this is clear since we have $h_T = V_T^{v_0,Z} - \xi$ = terminal profit for the seller, which does not hold in the non-linear case).

- Question : what is the analogous of martingale probability measures in the case when f is non-linear ?
- First, we define the non-linear f-expectation under Q for $\mathbf{Q} \sim P$.

Let $Q \sim P$. From the $\mathbb{G}$-martingale representation theorem, its density process (ζ_t) satisfies

$$d\zeta_t = \zeta_{t^-}(\alpha_t dW_t + \nu_t dM_t); \zeta_0 = 1,$$

where (α_t) and (ν_t) are $\mathbb{G}$-predictable processes with $\nu_{\vartheta \wedge T} > -1$ a.s.
By Girsanov's theorem,

- $\mathbf{W^Q}_t := W_t - \int_0^t \alpha_s ds$ is a Q-Brownian motion, and
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We have a Q-martingale representation for Q-martingales w.r.t. W^Q and M^Q. We can thus consider Q-BSDEs driven by W^Q, M^Q.

Let $Q \sim P$. Let $(X,Z,K) \in \mathbb{H}^2_Q \times \mathbb{H}^2_Q \times \mathbb{H}^2_{Q,\lambda}$ be the sol. of the Q-**BSDE**

$$-dX_t = f(t, X_t, Z_t)dt - Z_t d\mathbf{W_t^Q} - K_t d\mathbf{M_t^Q}; \quad X_T = \eta.$$

We call Q-pricing system or f-evaluation under Q, denoted by $\mathscr{E}^f_Q$ or more simply $\mathscr{E}_Q$ (or $\mathscr{E}^Q$), the operator defined by : for $\eta \in L^2_Q(\mathcal{G}_T)$,

$$\mathscr{E}^{\mathbf{Q}}_{\mathbf{s},T}(\eta) := X_s, \qquad s \in [0,T]$$

It can be a possible price (see the last slide for details).

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Definition

(Peng) Let $Y \in S^2_Q$. The process (Y_t) is said to be a (strong) $\mathscr{E}_Q$-martingale (or $\mathscr{E}^f$-martingale under Q), if $\forall s, t$ stopping times with $s \leq t$,

$$\mathscr{E}^Q_{s,t}(Y_t) = Y_s \quad \text{a.s..}$$

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Question : what is the analogous of martingale probability measures in the non-linear case ?

Definition

A probability $Q \sim P$ is called an $\mathscr{E}^f$-martingale probability measure if : $\forall\, x \in \mathbb{R}$ and $\forall\, Z \in \mathbb{H}^2_Q$, the wealth $V^{x,Z}$ is a $\mathscr{E}^f$-martingale under Q.

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Remarks :

- $P \in \mathscr{Q}$.
- $Q \in \mathscr{Q} \Leftrightarrow W + \int \beta_s dM_s$ is a Q-martingale.
- $\mathscr{Q}$ is equipotent to $\mathscr{P}$

Dual representation of the seller's price

Theorem

Let $\eta \in L^2_Q(\mathcal{G}_T)$*, for all* $\mathbf{Q} \in \mathcal{Q}$*. Under an appropriate integrability condition (see next slide), we have* $v_0 < \infty$ *and*

$$v_0 = \sup_{\mathbf{Q} \in \mathcal{Q}} \mathcal{E}^{\mathbf{Q}}_{0,T}(\eta),$$

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Proposition : The supremum is attained if and only if the option is replicable.
In this case, $\mathcal{E}^{\mathbf{Q}}_{0,T}(\eta) = \mathcal{E}^{P}_{0,T}(\eta)\ \forall\ \mathbf{Q} \in \mathcal{Q}$.

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Sketch of the proof of the theorem :
First, using the $\mathcal{E}_Q$-martingale property of the wealths for all $\mathbf{Q} \in \mathcal{Q}$, we get (quite easily) :

$$v_0 \geq \sup_{\mathbf{Q} \in \mathcal{Q}} \mathcal{E}^{Q}_{0,T}(\eta)$$

In order to show the inequality $\leq$, we first show :

Theorem (non-linear optional decomposition) :

Let $(Y_t) \in S_Q^2 \; \forall \; Q \in \mathcal{Q}$.

If (Y_t) is a strong $\mathscr{E}_{\mathbf{Q}}$-supermartingale $\forall \; \mathbf{Q} \in \mathcal{Q}$, In order to show the inequality $\leq$, we first show :

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Let $(Y_t) \in S^2_Q \; \forall \; Q \in \mathcal{Q}$.

If (Y_t) is a strong $\mathcal{E}_{\mathbf{Q}}$-supermartingale $\forall \; \mathbf{Q} \in \mathcal{Q}$,

then, there exists $Z \in \mathbb{H}^2$, and a nondecreasing optional càd-làg process $\mathbf{h}$, with $\mathbf{h}_0 = 0$ /

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s)dt + \int_0^t Z_s(dW_s + \beta_s dM_s) - \mathbf{h}_t, \quad 0 \leq t \leq T.$$

Remark : in the **linear** *case (with* $r = 0$*),* $f(t,z) = -\mu_t z$*. Hence,*

$Y_t = Y_0 + \int_0^t Z_s(\mu_s ds + dW_s + \beta_s dM_s) - \mathbf{h}_t$ *,*

which corresponds to the classical optional decomposition theorem.

End of the proof of the dual representation :
$\exists\ (X_t) \in S^2 /$ for all S,

$$X_S = ess \sup_{\mathbf{Q} \in \mathscr{Q}} \mathscr{E}^Q_{S,T}(\eta) \quad a.s.$$

(recall that it remained to show that $X_0 \geq v_0$)

- (X_t) is an $\mathscr{E}_{\mathbf{Q}}$-supermartingale for each $\mathbf{Q} \in \mathscr{Q}$ (with $X(T) = \eta$).
- By the optional $\mathscr{E}^f$-decomposition theorem, $\exists\ Z, h... /$

$$X_t = X_0 - \int_0^t f(s, X_s, Z_s)dt + \int_0^t Z_s(dW_s + \beta_s dM_s) - \mathbf{h}_t, \quad 0 \leq t \leq T.$$

- By the comparison theorem for **forward** SDEs, $(\eta =)\ X_T \leq V_T^{X_0, Z}$.
  Hence, $X_0 \geq v_0$. Hence, $X_0 = v_0$. **QED**

Note that (v_0, Z) is a **superhedging strategy** for the seller (since $\eta \leq V_T^{v_0, Z}$).

For each $S \in \mathcal{T}$, set

$$X(S) := \mathop{ess\,\sup}_{\mathbf{Q} \in \mathcal{Q}} \mathcal{E}^{Q}_{S,T}(\eta)$$

Proposition :

$$v_0 < \infty \quad \Leftrightarrow \quad E_{\mathbf{Q}}[\mathop{ess\,\sup}_{S \in \mathcal{T}} X(S)^2] < +\infty, \; \forall \mathbf{Q} \in \mathcal{Q}$$

Remark : for example, this condition is satisfied if $\eta = (S_T - K)^+$.

- Let (x,φ) in $\mathbb{R}\times\mathbb{H}^2$ be a *superhedging strategy* in the sense that $V_T^{x,\varphi} \geq \eta$ a.s.
- $\rightarrow V_T^{x,\varphi} - \eta =$ *terminal profit* realized by the seller.
- **Prop** : $x = v_0 \quad \Leftrightarrow \quad \inf_{Q\in\mathcal{Q}} \mathbb{E}_Q(V_T^{x,\varphi} - \eta) = 0.$
  In particular, this minimality condition is satisfied by (v_0, Z).

Theorem

There exists a sequence $(Q_n) \in \mathscr{Q}$ *s.t.*
(i)

$$\mathbf{v_0} = \lim_{n\rightarrow+\infty} \mathscr{E}_{0,T}^{\mathbf{Q_n}}(\eta),$$

(ii) $Q_n \rightarrow Q^*(\ll P)$ weakly *as* $n\rightarrow\infty$ *, and* η *is replicable under* Q^* *since*

$$\eta = V_T^{\mathbf{v_0},Z} \quad Q^* - a.e.,$$

where Z *is the process from the non-linear optional decomposition of* X*.*

Here, $Q_n \rightarrow Q^*$ *weakly* in the sense that $\frac{dQ_n}{dP} \rightarrow \frac{dQ^*}{dP}$ P-a.s.

In the **linear** case, we can even prove the following (new) result : there exists a non-negative measure $R^* \ll P$, which is the *weak* limit of a sequence (R_n) of *martingale probability measures*, such that

$$\mathbf{v_0} = \text{“}E_{R^*}(\eta)\text{”} \left(= \int_\Omega \eta dR^*\right), \tag{0.1}$$

and η is replicable under R^*, more precisely

$$\eta = V_T^{\mathbf{v_0},Z} \quad R^* - a.e.,$$

where Z is the process from the (linear) optional decomposition of the dual value process $X_S := ess \sup_{R \in \mathscr{P}} E_R(\eta | \mathcal{G}_S)$.
Remark : in the non-linear case, we cannot have an analogous equality to (0.1) *since $\mathscr{E}_{\mathbf{R}^*}$ does not make sense (we do not even know if R^* is a probability measure).*

Characterization of v_0 via a constrained BSDE

Theorem : $v_0 = X_0$, where the process X is characterized as the (minimal) **supersolution** of the constrained BSDE with default, that is, such that $\exists\ (Z,K) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$, and a **predictable** nondecreasing process $\mathbf{A}$ satisfying

$$
\begin{aligned}
& -dX_t = f(t, X_t, Z_t)dt - Z_t dW_t - K_t dM_t + d\mathbf{A}_t; \quad X_T = \eta\,; \\
& \mathbf{A}_\cdot + \int_0^\cdot (K_s - \beta_s Z_s)\lambda_s ds \quad \text{is nondecreasing} \\
& (K_t - \beta_t Z_t)\lambda_t \leq 0,\ dP \otimes dt - \text{a.e.}\,;
\end{aligned}
$$

Remark : Z_t and $\mathbf{h}_t := \mathbf{A}_t - \int_0^t (K_s - \beta_s Z_s) dM_s$ correspond to the processes from the non-linear optional decomposition of the dual value process (X_t).

Definition (buyer's superhedging price)

$$\tilde{v}_0 := \sup\{x \in \mathbb{R} : \ \exists Z \in \mathbb{H}^2 \text{ with } V_T^{-x,Z} + \eta \geq 0\}.$$

Remark : Note that superhedging price $\tilde{v}_0$ for the buyer is equal to the opposite of the superhedging price for the seller of the option with payoff $-\eta$.

$$\tilde{v}_0 = -\sup_{\mathbf{Q} \in \mathcal{Q}} \mathscr{E}^f_{\mathbf{Q},0,T}(-\eta).$$

The interval $(\tilde{v}_0, v_0)$ (open of closed, it depends on η) can be interpreted as an arbitrage-free interval (set of arbitrage-free prices for the European option η) in the sense of Karatzas and Kou. It can be empty for particular f and η (see our paper for details).

Our present paper :

Grigorova M., Quenez, M.-C., and Sulem, A., European options in a **non-linear incomplete** market with default, *SIAM Journal on Financial Mathematics*, 11(3), (2018-2020), 849-880.

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Remark :

For the notion of *non-linear f-pricing systems* in a **complete non-linear** market and its properties (notions of *consistency, no-arbitrage property*, *non-negativity* when $f(t,0,0) \geq 0$ ...), see :

N. El Karoui and M.C. Quenez, *Non-linear Pricing Theory and Backward Stochastic Differential Equations*, in Financial Mathematics, Lectures Notes in Math. 1656, Bressanone 1996, W.J. Runggaldier, ed., Springer, 1997, pp. 191–246.