#### A GLOBAL STOCHASTIC MAXIMUM PRINCIPLE FOR FULLY COUPLED FORWARD-BACKWARD STOCHASTIC SYSTEMS

#### Shaolin Ji Joint work with Mingshang Hu, Xiaole Xue

Institute for Financial Studies, Shandong University

9th colloquium on BSDE and mean filed systems,Annecy, France July 1st, 2022

Problem formulation

Main results

General case

References

Problem formulation

Main results

General case

References

- Maximum principle, the necessary conditions that must satisfied by any optimal control, is an important approach in solving optimization problems.
- Boltyanski-Gamkrelidze-Pontryagin announced the Pontryagin's maximum principle for the first time for deterministic control systems in 1956.

Idea: "the spike variation" + "the first-order of Taylor's expansion".

The classical stochastic optimal control problem (Yong and Zhou 1999):

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t), \\ X(0) = x_0, \end{cases}$$
(1)

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t))dt + h(X(T))\right].$$
 (2)

 J. Yong and X. Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York, (1999)

- If the diffusion terms depend on the controls and control domain is nonconvex, one can't follow the idea for deterministic control systems.
- Peng(1990) first introduced the second-order term in the Taylor expansion of the variation and obtained the global maximum principle.
- S. Peng, A general stochastic maximum principle for optimal control problems. SIAM Journal on control and optimization, 28(4) (1990):pp. 966-979

The stochastic recursive optimal control problem:

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t), \\ dY(t) = -g(t, X(t), Y(t), Z(t), u(t))dt + Z(t)dB(t), \\ X(0) = x_0, Y(T) = \phi(X(T)), \end{cases}$$

$$J(u(\cdot))=Y(0).$$

- D. Duffie and L. Epstein, Stochastic differential utility, Econometrica, 60(1992), pp. 353–394
- N. El Karoui, S. Peng and MC. Quenez, Backward stochastic differential equations in finance. Mathematical Finance, 7(1) (1997):pp. 1-71

- When the control domain is nonconvex, one encounters an essential difficulty when trying to derive the first-order and second-order expansions and it is proposed as an open problem in Peng (1998).
- ▶ Yong (2010),Wu (2013) studied this kind of problem.
- ► Hu (2017) constructed first-order and second-order variational equation and obtained a novel global maximum principle.
- What about the fully coupled case?

- S. Peng, Open problems on backward stochastic differential equations. In: Chen, S, Li, X, Yong, J, Zhou, XY (eds.) Control of distributed parameter and stocastic systems, pp. 265–273, Boston: Kluwer Acad. Pub. (1998).
- Z. Wu, A general maximum principle for optimal control of forward-backward stochastic systems. Automatica, 49(5) (2013):pp. 1473-1480.
- J. Yong, Optimality variational principle for controlled forward-backward stochastic differential equations with mixed initial-terminal conditions. SIAM Journal on Control and Optimization, 48(6) (2010):pp. 4119-4156.
- M. Hu, Stochastic global maximum principle for optimization with recursive utilities. Probability, Uncertainty and Quantitative Risk, 2(1) (2017):pp 1-20.

#### Problem formulation

Main results

General case

References

Consider the following fully coupled stochastic control system:

$$dX(t) = b(t, X(t), Y(t), Z(t), u(t))dt + \sigma(t, X(t), Y(t), Z(t), u(t))dB(t),$$

$$dY(t) = -g(t, X(t), Y(t), Z(t), u(t))dt + Z(t)dB(t),$$

$$X(0) = x_0, Y(T) = \phi(X(T)).$$
(3)

There are much literatures on the well-posedness of fully coupled FBSDE.

- Ma, Protter, and Yong (1994) first proposed the four-step scheme.
- Under some monotonicity conditions, Hu and Peng (1995) obtained an existence and uniqueness result.
- ► A unified approach, Ma, Wu, Zhang and Zhang (2015).

The readers may refer to Ma and Yong (1999), Cvitanić and Zhang (2013), Zhang (2017) for the FBSDE theory.

# Problem Formulation

- J. Ma, P. Protter, J. Yong, Solving forward-backward stochastic differential equations explicitly—a four step scheme, Probab. Theory Related Fields 98 (2) (1994), pp. 339–359.
- Y. Hu, S Peng, Solution of forward-backward stochastic differential equations. Probability Theory and Related Fields, 103(2) (1995), pp.273-283.
- J. Ma, Z. Wu, D. Zhang and J. Zhang, On well-posedness of forward-backward SDEs-A unified approach. The Annals of Applied Probability, 25(4) (2015):pp. 2168-2214.
- J. Ma and J. Yong, Forward-backward stochastic differential equations and their applications. Springer Science & Business Media, (1999).
- J. Cvitanić and J.Zhang. Contract theory in continuous-time models. Springer-Verlag, 2013.
- J. Zhang. Backward Stochastic Differential Equations: From Linear to Fully Nonlinear Theory (Vol. 86). Springer, (2017).

The optimal control problem is to minimize the cost functional

 $J(u(\cdot)) = Y(0)$ 

over  $\mathcal{U}[0, T]$ :  $\inf_{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot)).$ 

Difficulty:  $\sigma(\cdot)$  depends on Z and u, the regularity/integrability of process Z() seems to be not enough in the case when the first and second order expansions are necessary.

### Assumptions

To guarantee the existence and uniqueness of FBSDEs, we impose the following conditions from Cvitanić and Zhang (2013).

For  $\psi = b$ ,  $\sigma$ , g and  $\phi$ , (i)  $\psi$ ,  $\psi_x$ ,  $\psi_y$ ,  $\psi_z$  are continuous in (x, y, z, u);  $\psi_x$ ,  $\psi_y$ ,  $\psi_z$  are bounded; there exists a constant L > 0 such that

$$\begin{aligned} |\psi(t, x, y, z, u)| &\leq L \left( 1 + |x| + |y| + |z| + |u| \right), \\ |\sigma(t, 0, 0, z, u) - \sigma(t, 0, 0, z, u')| &\leq L (1 + |u| + |u'|). \end{aligned}$$

(ii) For any  $2 \leq \beta \leq 8$ ,

$$\Lambda_{\beta} := C_{\beta} 2^{\beta+1} (1+T^{\beta}) c_1^{\beta} < 1,$$

where  $c_1$  and  $C_\beta$  are related to the norm of the derivatives of the coefficients with respect to x, y, z.

Some Remarks:

- Various conditions to guarantee the well-posedness of fully coupled FBSDE
- Does the derived global SMP still hold?
- It essentially depends on whether the L<sup>p</sup>-estimates of the solution of FBSDE is valid
- Our approach still applies

Problem formulation

#### Main results

General case

References

Let  $\bar{u}(\cdot)$  be optimal and  $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$  be the corresponding state processes.

Since the control domain is not necessarily convex, we resort to spike variation method.

For any  $u(\cdot) \in \mathcal{U}[0, T]$  and  $0 < \epsilon < T$ , define

$$u^{\epsilon}(t) = \left\{ egin{array}{cc} ar{u}(t), & t\in [0,T]ar{E}_{\epsilon}, \ u(t), & t\in E_{\epsilon}, \end{array} 
ight.$$

where  $E_{\epsilon} \subset [0, T]$  is a measurable set with  $|E_{\epsilon}| = \epsilon$ .

### A heuristic derivation

- We have  $X^{\epsilon}(t) \bar{X}(t) \sim O(\sqrt{\epsilon})$ ,  $Y^{\epsilon}(t) \bar{Y}(t) \sim O(\sqrt{\epsilon})$ and  $Z^{\epsilon}(t) - \bar{Z}(t) \sim O(\sqrt{\epsilon})$ .
- Suppose that

$$\begin{aligned} X^{\epsilon}(t) - \bar{X}(t) &= X_1(t) + X_2(t) + o(\epsilon), \\ Y^{\epsilon}(t) - \bar{Y}(t) &= Y_1(t) + Y_2(t) + o(\epsilon), \\ Z^{\epsilon}(t) - \bar{Z}(t) &= Z_1(t) + Z_2(t) + o(\epsilon), \end{aligned}$$

where  $X_1(t) \sim O(\sqrt{\epsilon})$ ,  $X_2(t) \sim O(\epsilon)$ ,  $Y_1(t) \sim O(\sqrt{\epsilon})$ ,  $Y_2(t) \sim O(\epsilon)$ ,  $Z_1(t) \sim O(\sqrt{\epsilon})$  and  $Z_2(t) \sim O(\epsilon)$ .

M. Hu, S. Ji and X. Xue, A global stochastic maximum principle for fully coupled forward-backward stochastic systems, SIAM J. Control Optim., 56(6) (2018), pp. 4309-4335. When deriving the variational equation of X, the diffusion term of the variational equation includes the term  $\delta\sigma(t)I_{E_{\epsilon}}(t)$ .

This inspires us that  $Z_1(t)$  should have the following form

$$Z_1(t) = \Delta(t) I_{\boldsymbol{E}_{\boldsymbol{\varepsilon}}}(t) + Z_1'(t).$$

where  $\Delta(t)$  is an  $\mathbb{F}$ -adapted process to be determined and  $Z'_1(t)$  has good estimates similarly as  $X_1(t)$ .

•  $\Delta(t)$  is determined by an algebra equation.

 $\Delta(t) = p(t)(\sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t) + \Delta(t), u(t)) - \sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t))).$ 

where p(t) is the adjoint process.

• Do Taylor's expansions at  $\overline{Z}(t) + \Delta(t)I_{E_{\epsilon}}(t)$ .

### A heuristic derivation

To illustrate this, the expansion for  $\sigma$  with respect to Z:

$$\begin{split} &\sigma(Z^{\epsilon}(t)) - \sigma(\bar{Z}(t)) \\ &= \sigma(\bar{Z}(t) + \Delta(t)I_{E_{\epsilon}}(t) + Z'_{1}(t) + Z_{2}(t)) - \sigma(\bar{Z}(t)) + o(\epsilon) \\ &= \sigma(\bar{Z}(t) + \Delta(t)I_{E_{\epsilon}}(t) + Z'_{1}(t) + Z_{2}(t) - \sigma(\bar{Z}(t) + \Delta(t)I_{E_{\epsilon}}(t)) \\ &+ \sigma(\bar{Z}(t) + \Delta(t)I_{E_{\epsilon}}(t)) - \sigma(\bar{Z}(t)) + o(\epsilon) \\ &= \sigma_{z}(\bar{Z}(t) + \Delta(t)I_{E_{\epsilon}}(t))(Z'_{1}(t) + Z_{2}(t)) \\ &+ \frac{1}{2}\sigma_{zz}(\bar{Z}(t) + \Delta(t)I_{E_{\epsilon}}(t))Z'_{1}(t)^{2} + \sigma(\bar{Z}(t) + \Delta(t)I_{E_{\epsilon}}(t)) - \sigma(\bar{Z}(t)) + o(\epsilon) \\ &= \sigma_{z}(\bar{Z}(t))(Z'_{1}(t) + Z_{2}(t)) + \frac{1}{2}\sigma_{zz}(\bar{Z}(t))Z'_{1}(t)^{2} \\ &+ [\sigma_{z}(\bar{Z}(t) + \Delta(t)) - \sigma_{z}(\bar{Z}(t))]Z'_{1}(t)I_{E_{\epsilon}}(t) \\ &+ [\sigma(\bar{Z}(t) + \Delta(t)) - \sigma(\bar{Z}(t))]I_{E_{\epsilon}}(t) + o(\epsilon). \end{split}$$

### The first order variational equation

$$\begin{aligned} dX_{1}(t) &= \left[ b_{x}(t)X_{1}(t) + b_{y}(t)Y_{1}(t) + b_{z}(t)(Z_{1}(t) - \Delta(t)I_{E_{e}}(t)) \right] d\\ &+ \left[ \sigma_{x}(t)X_{1}(t) + \sigma_{y}(t)Y_{1}(t) + \sigma_{z}(t)(Z_{1}(t) - \Delta(t)I_{E_{e}}(t)) \right] d\\ &+ \delta\sigma(t,\Delta)I_{E_{e}}(t) \right] dB(t), \end{aligned}$$

$$\begin{aligned} dY_{1}(t) &= - \left[ g_{x}(t)X_{1}(t) + g_{y}(t)Y_{1}(t) + g_{z}(t)(Z_{1}(t) - \Delta(t)I_{E_{e}}(t)) \right] \\ &- q(t)\delta\sigma(t,\Delta)I_{E_{e}}(t) \right] dt + Z_{1}(t)dB(t), \end{aligned}$$

$$\begin{aligned} X_{1}(0) &= 0, \\ Y_{1}(T) &= \phi_{x}(\bar{X}(T))X_{1}(T). \end{aligned}$$

# The second order variational equation

$$\begin{split} dX_{2}(t) \\ &= \{b_{x}(t)X_{2}(t) + b_{y}(t)Y_{2}(t) + b_{z}(t)Z_{2}(t) + \delta b(t, \Delta)I_{E_{c}(t)} \\ &+ \frac{1}{2} [X_{1}(t), Y_{1}(t), Z_{1}(t) - \Delta(t)I_{E_{c}}(t)]^{\mathsf{T}} \} dt \\ &+ [X_{1}(t), Y_{1}(t), Z_{1}(t) - \Delta(t)I_{E_{c}}(t)]^{\mathsf{T}} \} dt \\ &+ \{\sigma_{x}(t)X_{2}(t) + \sigma_{y}(t)Y_{2}(t) + \sigma_{z}(t)Z_{2}(t) + \delta\sigma_{x}(t, \Delta)X_{1}(t)I_{E_{c}}(t) \\ &+ \delta\sigma_{y}(t, \Delta)Y_{1}(t) + \delta\sigma_{z}(t, \Delta) (Z_{1}(t) - \Delta(t)I_{E_{c}}(t)) \\ &+ \frac{1}{2} [X_{1}(t), Y_{1}(t), Z_{1}(t) - \Delta(t)I_{E_{c}}(t)] D^{2}\sigma(t) \\ &\cdot [X_{1}(t), Y_{1}(t), Z_{1}(t) - \Delta(t)I_{E_{c}}(t)]^{\mathsf{T}} \} dB(t), \\ dY_{2}(t) \\ &= - \{g_{x}(t)X_{2}(t) + g_{y}(t)Y_{2}(t) + g_{z}(t)Z_{2}(t) + [q(t)\delta\sigma(t, \Delta) + \delta g(t, \Delta)]I_{E_{c}}(t) \\ &+ \frac{1}{2} [X_{1}(t), Y_{1}(t), Z_{1}(t) - \Delta(t)I_{E_{c}}(t)] D^{2}g(t) \\ &\cdot [X_{1}(t), Y_{1}(t), Z_{1}(t) - \Delta(t)I_{E_{c}}(t)]^{\mathsf{T}} \} dt \\ &+ Z_{2}(t)dB(t), \\ X_{2}(0) = 0, Y_{2}(T) = \phi_{x}(\bar{X}(T))X_{2}(T) + \frac{1}{2}\phi_{xx}(\bar{X}(T))X_{1}^{2}(T). \end{split}$$

### The estimates of the first order variational equation

Lemma (Hu, J., Xue, 2018)

For any  $2 \le \beta \le 8$ , we have the following estimates

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(|X_{1}(t)|^{\beta}+|Y_{1}(t)|^{\beta}\right)\right]+\mathbb{E}\left[\left(\int_{0}^{T}|Z_{1}(t)|^{2}dt\right)^{\beta/2}\right]=O(\epsilon^{\beta/2}),\\\mathbb{E}\left[\sup_{t\in[0,T]}\left(|X^{\epsilon}(t)-\bar{X}(t)-X_{1}(t)|^{2}+|Y^{\epsilon}(t)-\bar{Y}(t)-Y_{1}(t)|^{2}\right)\right]\\+\mathbb{E}\left[\int_{0}^{T}|Z^{\epsilon}(t)-\bar{Z}(t)-Z_{1}(t)|^{2}dt\right]=O(\epsilon^{2}),\\\mathbb{E}\left[\sup_{t\in[0,T]}\left(|X^{\epsilon}(t)-\bar{X}(t)-X_{1}(t)|^{4}+|Y^{\epsilon}(t)-\bar{Y}(t)-Y_{1}(t)|^{4}\right)\right]\\+\mathbb{E}\left[\left(\int_{0}^{T}|Z^{\epsilon}(t)-\bar{Z}(t)-Z_{1}(t)|^{2}dt\right)^{2}\right]=o(\epsilon^{2}).$$

<ロト < 回 ト < 巨 ト < 巨 ト < 巨 ト 三 の < C 25 / 41 Lemma (Hu, J., Xue, 2018)

For any  $2 \le \beta \le 4$  we have the following estimates

$$\mathbb{E}\left[\sup_{t\in[0,T]} (|X_{2}(t)|^{2} + |Y_{2}(t)|^{2})\right] + \mathbb{E}\left[\int_{0}^{T} |Z_{2}(t)|^{2} dt\right] = O(\epsilon^{2}),$$
  
$$\mathbb{E}\left[\sup_{t\in[0,T]} (|X_{2}(t)|^{\beta} + |Y_{2}(t)|^{\beta})\right] + \mathbb{E}\left[\left(\int_{0}^{T} |Z_{2}(t)|^{2} dt\right)^{\frac{\beta}{2}}\right] = o(\epsilon^{\frac{\beta}{2}}),$$
  
$$Y^{\epsilon}(0) - \bar{Y}(0) - Y_{1}(0) - Y_{2}(0) = o(\epsilon).$$

The first-order and second-order adjoint equations are introduced as follows.

$$\begin{cases} dp(t) = -\{g_{x}(t)+g_{y}(t)p(t)+g_{z}(t)K_{1}(t)+b_{x}(t)p(t) \\ +b_{y}(t)p^{2}(t)+b_{z}(t)K_{1}(t)p(t)+\sigma_{x}(t)q(t) \\ +\sigma_{y}(t)p(t)q(t)+\sigma_{z}(t)K_{1}(t)q(t)\}dt+q(t)dB(t), \\ p(T) = \phi_{x}(\bar{X}(T)), \end{cases}$$
(4)

where

$$\mathcal{K}_{1}(t) = (1 - p(t)\sigma_{z}(t))^{-1} \left[\sigma_{x}(t)p(t) + \sigma_{y}(t)p^{2}(t) + q(t)\right].$$

<ロト < 部 ト < 言 ト < 言 ト 言 の < C 27 / 41 Theorem (Hu, J., Xue, 2018)

The equation (4) has a solution  $(p(\cdot), q(\cdot))$ .

Uniqueness

- Case I:  $(p(\cdot), q(\cdot))$  is bounded;
- Case II: q(·) is unbounded and σ(t, x, y, z, u) takes some special forms.

### The second-order adjoint equation

$$\begin{aligned} -dP(t) \\ &= \{P(t) \left[ (D\sigma(t)^{\mathsf{T}} [1, p(t), \mathcal{K}_{1}(t)]^{\mathsf{T}})^{2} + 2Db(t)^{\mathsf{T}} [1, p(t), \mathcal{K}_{1}(t)]^{\mathsf{T}} + \mathcal{H}_{y}(t) \right] \\ &+ 2Q(t) D\sigma(t)^{\mathsf{T}} [1, p(t), \mathcal{K}_{1}(t)]^{\mathsf{T}} + [1, p(t), \mathcal{K}_{1}(t)] D^{2} \mathcal{H}(t) [1, p(t), \mathcal{K}_{1}(t)]^{\mathsf{T}} \\ &+ \mathcal{H}_{z}(t) \mathcal{K}_{2}(t) \} dt - Q(t) dB(t), \end{aligned}$$

$$P(T) = \phi_{xx}(\bar{X}(T)), \tag{5}$$

where

$$H(t, x, y, z, u, p, q) = g(t, x, y, z, u) + pb(t, x, y, z, u) + q\sigma(t, x, y, z, u),$$

$$\begin{split} \mathcal{K}_{2}(t) &= (1 - p(t)\sigma_{z}(t))^{-1} \left\{ p(t)\sigma_{y}(t) + 2 \left[\sigma_{x}(t) + \sigma_{y}(t)p(t) + \sigma_{z}(t)\mathcal{K}_{1}(t)\right] \right\} \mathcal{P}(t) \\ &+ (1 - p(t)\sigma_{z}(t))^{-1} \left\{ Q(t) + p(t)[1, p(t), \mathcal{K}_{1}(t)]D^{2}\sigma(t)[1, p(t), \mathcal{K}_{1}(t)]^{\mathsf{T}} \right\}. \end{split}$$

### Stochastic Maximum Principle

Define

$$\begin{aligned} \mathcal{H}(t, x, y, z, u, p, q, P) \\ &= pb(t, x, y, z + \Delta(t), u) + q\sigma(t, x, y, z + \Delta(t), u) \\ &+ \frac{1}{2}P(\sigma(t, x, y, z + \Delta(t), u) - \sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)))^2 \\ &+ g(t, x, y, z + \Delta(t), u), \end{aligned}$$

where  $\Delta(t)$  is defined by, for  $t \in [0, T]$  $\Delta(t) = p(t)(\sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t) + \Delta(t), u) - \sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t))).$ (6)

> <ロト < 回 ト < 巨 ト < 巨 ト < 巨 ト 三 の < (で 30 / 41

Theorem (Hu, J., Xue, 2018)

Let  $\bar{u}(\cdot) \in \mathcal{U}[0, T]$  be optimal and  $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$  be the corresponding state processes of (3). Then the following stochastic maximum principle holds:

 $\begin{aligned} \mathcal{H}(t,\bar{X}(t),\bar{Y}(t),\bar{Z}(t),u,p(t),q(t),P(t)) \\ \geq \mathcal{H}(t,\bar{X}(t),\bar{Y}(t),\bar{Z}(t),\bar{u}(t),p(t),q(t),P(t)), \quad \forall u \in U, \text{ a.e., a.s.} \end{aligned}$ 

where  $(p(\cdot), q(\cdot))$ ,  $(P(\cdot), Q(\cdot))$  satisfy (4), (5) respectively, and  $\Delta(\cdot)$  satisfies (6).

Problem formulation

Main results

#### General case

References

There are several different conditions to guarantee the existence and uniqueness of the solution to fully coupled FBSDEs.

- Our assumptions are mainly from Cvitanić and Zhang (2013)
- Does our method still work under other assumptions?
- Does the established stochastic maximum principle still hold?

Our approach essentially depends on the following assumptions:

There exists a unique solution to FBSDE

and its solution has  $L^p$ -estimates ( $p \ge 8$ ).

### General case: The key issues to derive SMP

• The solution to the following linear FBSDE (variational equation) has  $L^p$ -estimates ( $p \ge 8$ ).

$$\begin{cases}
d\hat{X}(t) = \left[ \alpha_{1}(t)\hat{X}(t) + \beta_{1}(t)\hat{Y}(t) + \gamma_{1}(t)\hat{Z}(t) + L_{1}(t) \right] dt \\
+ \left[ \alpha_{2}(t)\hat{X}(t) + \beta_{2}(t)\hat{Y}(t) + \gamma_{2}(t)\hat{Z}(t) + L_{2}(t) \right] dB(t), \\
d\hat{Y}(t) = -\left[ \langle \alpha_{3}(t), \hat{X}(t) \rangle + \beta_{3}(t)\hat{Y}(t) + \gamma_{3}(t)\hat{Z}(t) + L_{3}(t) \right] dt + \hat{Z}(t) dB(t), \\
\hat{X}(0) = x_{0}, \ \hat{Y}(T) = \langle \kappa, \hat{X}(T) \rangle + \varsigma.
\end{cases}$$
(8)

where  $\alpha_1(t) = \tilde{b}_x^{\varepsilon}(t) = \int_0^1 b_x(t, \Theta(t) + \theta(\Theta^{\varepsilon}(t) - \Theta(t)), u^{\varepsilon}(t))d\theta$  for  $0 \le \varepsilon < T$ , other terms are similar.

Other assumptions, roughly speaking,

- There exist unique solutions to the adjoint equations.
- The algebra equation of  $\Delta(t)$  has a unique solution.

#### Conclusion:

- ► For any well-posedness conditions which can guarantee the above assumptions hold, our SMP does hold.
- Our approach to derive SMP is essentially independent of different well-posedness conditions.
- M. Hu, S. Ji and X. Xue, A note on the global stochastic maximum principle for fully coupled forward-backward stochastic systems, arXiv.1812.10469, (2018).

In more details, our approach holds under the following two kinds of assumptions in Hu, Ji and Xue (2018).

#### The first kind of assumptions is:

- There exists a unique  $L^p$ -solution ( $p \ge 8$ ) to FBSDE (7);
- There exists a unique bounded solution to the first-order adjoint equation;
- ► There exists a unique solution to the algebra equation;
- There exists a unique  $L^p$ -solution  $(p \ge 8)$  to FBSDE (8).

#### The second kind of assumptions is:

- There exists a unique  $L^p$ -solution  $(p \ge 8)$  to FBSDE (7);
- There exists a unique solution to the first-order adjoint equation;
- There exists a unique solution to the algebra equation;
- The solution to linear FBSDE (8) has  $L^p$ -estimates( $p \ge 8$ );
- $\sigma$  is linear in z, and  $\|\sigma_z\|$  is small enough.

Problem formulation

Main results

General case

#### References

- M. Hu, S. Ji and X. Xue, A global stochastic maximum principle for fully coupled forward-backward stochastic systems, SIAM J. Control Optim., 56(6) (2018), pp. 4309-4335.
- M. Hu, S. Ji and X. Xue, A note on the global stochastic maximum principle for fully coupled forward-backward stochastic systems, arXiv.1812.10469, (2018).

# Thank you!