# A GLOBAL STOCHASTIC MAXIMUM PRINCIPLE FOR FULLY COUPLED FORWARD-BACKWARD STOCHASTIC SYSTEMS 

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# Introduction 

## Problem formulation

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## Introduction

- Maximum principle, the necessary conditions that must satisfied by any optimal control, is an important approach in solving optimization problems.
- Boltyanski-Gamkrelidze-Pontryagin announced the Pontryagin's maximum principle for the first time for deterministic control systems in 1956.

Idea: "the spike variation" + "the first-order of Taylor's expansion".

## Introduction

The classical stochastic optimal control problem (Yong and Zhou 1999):

$$
\left\{\begin{align*}
d X(t) & =b(t, X(t), u(t)) d t+\sigma(t, X(t), u(t)) d B(t),  \tag{1}\\
X(0) & =x_{0}  \tag{2}\\
J(u(\cdot)) & =\mathbb{E}\left[\int_{0}^{T} f(t, X(t), u(t)) d t+h(X(T))\right]
\end{align*}\right.
$$

- J. Yong and X. Y. Zhou, Stochastic Controls: Hamiltonian Systems and HJB Equations, Springer-Verlag, New York, (1999)


## Introduction

- If the diffusion terms depend on the controls and control domain is nonconvex, one can't follow the idea for deterministic control systems.
- Peng(1990) first introduced the second-order term in the Taylor expansion of the variation and obtained the global maximum principle.
- S. Peng, A general stochastic maximum principle for optimal control problems. SIAM Journal on control and optimization, 28(4) (1990):pp. 966-979


## Introduction

The stochastic recursive optimal control problem:

$$
\left\{\begin{array}{c}
d X(t)=b(t, X(t), u(t)) d t+\sigma(t, X(t), u(t)) d B(t) \\
d Y(t)=-g(t, X(t), Y(t), Z(t), u(t)) d t+Z(t) d B(t) \\
X(0)=x_{0}, Y(T)=\phi(X(T)) \\
J(u(\cdot))=Y(0)
\end{array}\right.
$$

- D. Duffie and L. Epstein, Stochastic differential utility, Econometrica, 60(1992), pp. 353-394
- N. El Karoui, S. Peng and MC. Quenez, Backward stochastic differential equations in finance. Mathematical Finance, 7(1) (1997):pp. 1-71


## Introduction

- When the control domain is nonconvex, one encounters an essential difficulty when trying to derive the first-order and second-order expansions and it is proposed as an open problem in Peng (1998).
- Yong (2010),Wu (2013) studied this kind of problem.
- Hu (2017) constructed first-order and second-order variational equation and obtained a novel global maximum principle.
- What about the fully coupled case?


## Introduction

- S. Peng, Open problems on backward stochastic differential equations. In: Chen, S, Li, X, Yong, J, Zhou, XY (eds.) Control of distributed parameter and stocastic systems, pp. 265-273, Boston: Kluwer Acad. Pub. (1998).
- Z. Wu, A general maximum principle for optimal control of forward-backward stochastic systems. Automatica, 49(5) (2013):pp. 1473-1480.
- J. Yong, Optimality variational principle for controlled forward-backward stochastic differential equations with mixed initial-terminal conditions. SIAM Journal on Control and Optimization, 48(6) (2010):pp. 4119-4156.
- M. Hu, Stochastic global maximum principle for optimization with recursive utilities. Probability, Uncertainty and Quantitative Risk, 2(1) (2017):pp 1-20.


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## Problem formulation

Consider the following fully coupled stochastic control system:

$$
\left\{\begin{align*}
d X(t)= & b(t, X(t), Y(t), Z(t), u(t)) d t \\
& +\sigma(t, X(t), Y(t), Z(t), u(t)) d B(t)  \tag{3}\\
d Y(t)= & -g(t, X(t), Y(t), Z(t), u(t)) d t+Z(t) d B(t) \\
X(0)= & x_{0}, Y(T)=\phi(X(T))
\end{align*}\right.
$$

## Problem Formulation

There are much literatures on the well-posedness of fully coupled FBSDE.

- Ma, Protter, and Yong (1994) first proposed the four-step scheme.
- Under some monotonicity conditions, Hu and Peng (1995) obtained an existence and uniqueness result.
- A unified approach, Ma, Wu, Zhang and Zhang (2015).

The readers may refer to Ma and Yong (1999), Cvitanić and Zhang (2013), Zhang (2017) for the FBSDE theory.

## Problem Formulation

- J. Ma, P. Protter, J. Yong, Solving forward-backward stochastic differential equations explicitly-a four step scheme, Probab. Theory Related Fields 98 (2) (1994), pp. 339-359.
- Y. Hu, S Peng, Solution of forward-backward stochastic differential equations. Probability Theory and Related Fields, 103(2) (1995), pp.273-283.
- J. Ma, Z. Wu, D. Zhang and J. Zhang, On well-posedness of forward-backward SDEs-A unified approach. The Annals of Applied Probability, 25(4) (2015):pp. 2168-2214.
- J. Ma and J. Yong, Forward-backward stochastic differential equations and their applications. Springer Science \& Business Media, (1999).
- J. Cvitanić and J.Zhang. Contract theory in continuous-time models. Springer-Verlag, 2013.
- J. Zhang. Backward Stochastic Differential Equations: From Linear to Fully Nonlinear Theory (Vol. 86). Springer, (2017).


## Problem Formulation

The optimal control problem is to minimize the cost functional

$$
J(u(\cdot))=Y(0)
$$

over $\mathcal{U}[0, T]$ :

$$
\inf _{u(\cdot) \in \mathcal{U}[0, T]} J(u(\cdot))
$$

Difficulty: $\sigma(\cdot)$ depends on $Z$ and $u$, the regularity/integrability of process $Z()$ seems to be not enough in the case when the first and second order expansions are necessary.

## Assumptions

To guarantee the existence and uniqueness of FBSDEs, we impose the following conditions from Cvitanić and Zhang (2013).

For $\psi=b, \sigma, g$ and $\phi$,
(i) $\psi, \psi_{x}, \psi_{y}, \psi_{z}$ are continuous in $(x, y, z, u) ; \psi_{x}, \psi_{y}, \psi_{z}$ are bounded; there exists a constant $L>0$ such that

$$
\begin{gathered}
|\psi(t, x, y, z, u)| \leq L(1+|x|+|y|+|z|+|u|) \\
\left|\sigma(t, 0,0, z, u)-\sigma\left(t, 0,0, z, u^{\prime}\right)\right| \leq L\left(1+|u|+\left|u^{\prime}\right|\right) .
\end{gathered}
$$

(ii) For any $2 \leq \beta \leq 8$,

$$
\Lambda_{\beta}:=C_{\beta} 2^{\beta+1}\left(1+T^{\beta}\right) c_{1}^{\beta}<1
$$

where $c_{1}$ and $C_{\beta}$ are related to the norm of the derivatives of the coefficients with respect to $x, y, z$.

Some Remarks:

- Various conditions to guarantee the well-posedness of fully coupled FBSDE
- Does the derived global SMP still hold?
- It essentially depends on whether the $L^{p}$-estimates of the solution of FBSDE is valid
- Our approach still applies


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## Spike Variation Method

Let $\bar{u}(\cdot)$ be optimal and $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ be the corresponding state processes.

Since the control domain is not necessarily convex, we resort to spike variation method.

For any $u(\cdot) \in \mathcal{U}[0, T]$ and $0<\epsilon<T$, define

$$
u^{\epsilon}(t)= \begin{cases}\bar{u}(t), & t \in[0, T] \backslash E_{\epsilon} \\ u(t), & t \in E_{\epsilon}\end{cases}
$$

where $E_{\epsilon} \subset[0, T]$ is a measurable set with $\left|E_{\epsilon}\right|=\epsilon$.

## A heuristic derivation

- We have $X^{\epsilon}(t)-\bar{X}(t) \sim O(\sqrt{\epsilon}), Y^{\epsilon}(t)-\bar{Y}(t) \sim O(\sqrt{\epsilon})$ and $Z^{\epsilon}(t)-\bar{Z}(t) \sim O(\sqrt{\epsilon})$.
- Suppose that

$$
\begin{aligned}
X^{\epsilon}(t)-\bar{X}(t) & =X_{1}(t)+X_{2}(t)+o(\epsilon), \\
Y^{\epsilon}(t)-\bar{Y}(t) & =Y_{1}(t)+Y_{2}(t)+o(\epsilon), \\
Z^{\epsilon}(t)-\bar{Z}(t) & =Z_{1}(t)+Z_{2}(t)+o(\epsilon),
\end{aligned}
$$

where $X_{1}(t) \sim O(\sqrt{\epsilon}), X_{2}(t) \sim O(\epsilon), Y_{1}(t) \sim O(\sqrt{\epsilon})$, $Y_{2}(t) \sim O(\epsilon), Z_{1}(t) \sim O(\sqrt{\epsilon})$ and $Z_{2}(t) \sim O(\epsilon)$.

- M. Hu, S. Ji and X. Xue, A global stochastic maximum principle for fully coupled forward-backward stochastic systems, SIAM J. Control Optim., 56(6) (2018), pp. 4309-4335.


## A heuristic derivation

When deriving the variational equation of $X$, the diffusion term of the variational equation includes the term $\delta \sigma(t) I_{E_{e}}(t)$.
This inspires us that $Z_{1}(t)$ should have the following form

$$
Z_{1}(t)=\Delta(t) I_{E_{e}}(t)+Z_{1}^{\prime}(t)
$$

where $\Delta(t)$ is an $\mathbb{F}$-adapted process to be determined and $Z_{1}^{\prime}(t)$ has good estimates similarly as $X_{1}(t)$.

## A heuristic derivation

- $\Delta(t)$ is determined by an algebra equation.
$\Delta(t)=p(t)(\sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t)+\Delta(t), u(t))-\sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)))$.
where $p(t)$ is the adjoint process.
- Do Taylor's expansions at $\bar{Z}(t)+\Delta(t) I_{E_{e}}(t)$.


## A heuristic derivation

To illustrate this, the expansion for $\sigma$ with respect to $Z$ :

$$
\begin{aligned}
& \sigma\left(Z^{\epsilon}(t)\right)-\sigma(\bar{Z}(t)) \\
&= \sigma\left(\bar{Z}(t)+\Delta(t) I_{E_{\epsilon}}(t)+Z_{1}^{\prime}(t)+Z_{2}(t)\right)-\sigma(\bar{Z}(t))+o(\epsilon) \\
&= \sigma\left(\bar{Z}(t)+\Delta(t) I_{E_{\epsilon}}(t)+Z_{1}^{\prime}(t)+Z_{2}(t)-\sigma\left(\bar{Z}(t)+\Delta(t) I_{E_{\epsilon}}(t)\right)\right. \\
&+\sigma\left(\bar{Z}(t)+\Delta(t) I_{E_{\epsilon}}(t)\right)-\sigma(\bar{Z}(t))+o(\epsilon) \\
&= \sigma_{z}\left(\bar{Z}(t)+\Delta(t) I_{E_{\epsilon}}(t)\right)\left(Z_{1}^{\prime}(t)+Z_{2}(t)\right) \\
&+\frac{1}{2} \sigma_{z z}\left(\bar{Z}(t)+\Delta(t) I_{E_{\epsilon}}(t)\right) Z_{1}^{\prime}(t)^{2}+\sigma\left(\bar{Z}(t)+\Delta(t) I_{E_{\epsilon}}(t)\right)-\sigma(\bar{Z}(t))+o(\epsilon \\
&= \sigma_{z}(\bar{Z}(t))\left(Z_{1}^{\prime}(t)+Z_{2}(t)\right)+\frac{1}{2} \sigma_{z z}(\bar{Z}(t)) Z_{1}^{\prime}(t)^{2} \\
&+\left[\sigma_{Z}(\bar{Z}(t)+\Delta(t))-\sigma_{z}(\bar{Z}(t))\right] Z_{1}^{\prime}(t) I_{E_{\epsilon}}(t) \\
&+[\sigma(\bar{Z}(t)+\Delta(t))-\sigma(\bar{Z}(t))] I_{E_{\epsilon}}(t)+o(\epsilon) .
\end{aligned}
$$

## The first order variational equation

$$
\left\{\begin{aligned}
d X_{1}(t)= & {\left[b_{x}(t) X_{1}(t)+b_{y}(t) Y_{1}(t)+b_{z}(t)\left(Z_{1}(t)-\Delta(t) I_{E_{\epsilon}}(t)\right)\right] d t } \\
& +\left[\sigma_{x}(t) X_{1}(t)+\sigma_{y}(t) Y_{1}(t)+\sigma_{z}(t)\left(Z_{1}(t)-\Delta(t) I_{E_{e}}(t)\right)\right. \\
& \left.+\delta \sigma(t, \Delta) I_{E_{c}}(t)\right] d B(t), \\
d Y_{1}(t)= & -\left[g_{x}(t) X_{1}(t)+g_{y}(t) Y_{1}(t)+g_{z}(t)\left(Z_{1}(t)-\Delta(t) I_{E_{e}}(t)\right)\right. \\
& \left.-q(t) \delta \sigma(t, \Delta) I_{E_{\epsilon}}(t)\right] d t+Z_{1}(t) d B(t), \\
X_{1}(0)= & 0, \\
Y_{1}(T)= & \phi_{x}(\bar{X}(T)) X_{1}(T) .
\end{aligned}\right.
$$

## The second order variational equation

$$
\begin{aligned}
& d X_{2}(t) \\
&=\left\{b_{x}(t) X_{2}(t)+b_{y}(t) Y_{2}(t)+b_{z}(t) Z_{2}(t)+\delta b(t, \Delta) I_{E_{\epsilon}(t)}\right. \\
&+\frac{1}{2}\left[X_{1}(t), Y_{1}(t), Z_{1}(t)-\Delta(t) I_{E_{e}}(t)\right] D^{2} b(t) \\
& \quad \cdot {\left.\left[X_{1}(t), Y_{1}(t), Z_{1}(t)-\Delta(t) I_{E_{\epsilon}}(t)\right]^{\top}\right\} d t } \\
&+\left\{\sigma_{x}(t) X_{2}(t)+\sigma_{y}(t) Y_{2}(t)+\sigma_{z}(t) Z_{2}(t)+\delta \sigma_{x}(t, \Delta) X_{1}(t) I_{E_{\epsilon}}(t)\right. \\
&+\delta \sigma_{y}(t, \Delta) Y_{1}(t)+\delta \sigma_{z}(t, \Delta)\left(Z_{1}(t)-\Delta(t) I_{E_{\epsilon}}(t)\right) \\
& \quad+\frac{1}{2}\left[X_{1}(t), Y_{1}(t), Z_{1}(t)-\Delta(t) I_{E_{\epsilon}}(t)\right] D^{2} \sigma(t) \\
& \quad \cdot {\left.\left[X_{1}(t), Y_{1}(t), Z_{1}(t)-\Delta(t) I_{E_{\epsilon}}(t)\right]^{\top}\right\} d B(t), } \\
& d Y_{2}(t) \\
&=-\left\{g_{x}(t) X_{2}(t)+g_{y}(t) Y_{2}(t)+g_{z}(t) Z_{2}(t)+[q(t) \delta \sigma(t, \Delta)+\delta g(t, \Delta)] I_{E_{\epsilon}}(t)\right. \\
& \quad+\frac{1}{2}\left[X_{1}(t), Y_{1}(t), Z_{1}(t)-\Delta(t) I_{E_{\epsilon}}(t)\right] D^{2} g(t) \\
&\left.\cdot\left[X_{1}(t), Y_{1}(t), Z_{1}(t)-\Delta(t) I_{E_{\epsilon}}(t)\right]^{\top}\right\} d t \\
& \quad+ Z_{2}(t) d B(t), \\
& X_{2}(0)=0, Y_{2}(T)=\phi_{x}(\bar{X}(T)) X_{2}(T)+\frac{1}{2} \phi_{x x}(\bar{X}(T)) X_{1}^{2}(T) .
\end{aligned}
$$

## The estimates of the first order variational equation

## Lemma (Hu, J., Xue, 2018)

For any $2 \leq \beta \leq 8$, we have the following estimates

$$
\begin{aligned}
& \mathbb{E} {\left[\sup _{t \in[0, T]}\left(\left|X_{1}(t)\right|^{\beta}+\left|Y_{1}(t)\right|^{\beta}\right)\right]+\mathbb{E}\left[\left(\int_{0}^{T}\left|Z_{1}(t)\right|^{2} d t\right)^{\beta / 2}\right]=O\left(\epsilon^{\beta / 2}\right) } \\
& \mathbb{E}\left[\sup _{t \in[0, T]}\left(\left|X^{\epsilon}(t)-\bar{X}(t)-X_{1}(t)\right|^{2}+\left|Y^{\epsilon}(t)-\bar{Y}(t)-Y_{1}(t)\right|^{2}\right)\right] \\
&+\mathbb{E}\left[\int_{0}^{T}\left|Z^{\epsilon}(t)-\bar{Z}(t)-Z_{1}(t)\right|^{2} d t\right]=O\left(\epsilon^{2}\right) \\
& \mathbb{E} {\left[\sup _{t \in[0, T]}\left(\left|X^{\epsilon}(t)-\bar{X}(t)-X_{1}(t)\right|^{4}+\left|Y^{\epsilon}(t)-\bar{Y}(t)-Y_{1}(t)\right|^{4}\right)\right] } \\
& \quad+\mathbb{E}\left[\left(\int_{0}^{T}\left|Z^{\epsilon}(t)-\bar{Z}(t)-Z_{1}(t)\right|^{2} d t\right)^{2}\right]=o\left(\epsilon^{2}\right)
\end{aligned}
$$

## The estimates of the second order variational equation

Lemma (Hu, J., Xue, 2018)
For any $2 \leq \beta \leq 4$ we have the following estimates

$$
\begin{aligned}
\mathbb{E}\left[\sup _{t \in[0, T]}\left(\left|X_{2}(t)\right|^{2}+\left|Y_{2}(t)\right|^{2}\right)\right]+\mathbb{E}\left[\int_{0}^{T}\left|Z_{2}(t)\right|^{2} d t\right] & =O\left(\epsilon^{2}\right), \\
\mathbb{E}\left[\sup _{t \in[0, T]}\left(\left|X_{2}(t)\right|^{\beta}+\left|Y_{2}(t)\right|^{\beta}\right)\right]+\mathbb{E}\left[\left(\int_{0}^{T}\left|Z_{2}(t)\right|^{2} d t\right)^{\frac{\beta}{2}}\right] & =o\left(\epsilon^{\frac{\beta}{2}}\right), \\
Y^{\epsilon}(0)-\bar{Y}(0)-Y_{1}(0)-Y_{2}(0) & =o(\epsilon) .
\end{aligned}
$$

## The first-order adjoint equation

The first-order and second-order adjoint equations are introduced as follows.

$$
\left\{\begin{align*}
d p(t)= & -\left\{g_{x}(t)+g_{y}(t) p(t)+g_{z}(t) K_{1}(t)+b_{x}(t) p(t)\right. \\
& +b_{y}(t) p^{2}(t)+b_{z}(t) K_{1}(t) p(t)+\sigma_{x}(t) q(t) \\
& \left.+\sigma_{y}(t) p(t) q(t)+\sigma_{z}(t) K_{1}(t) q(t)\right\} d t+q(t) d B(t), \\
p(T)= & \phi_{x}(\bar{X}(T)) \tag{4}
\end{align*}\right.
$$

where

$$
K_{1}(t)=\left(1-p(t) \sigma_{z}(t)\right)^{-1}\left[\sigma_{x}(t) p(t)+\sigma_{y}(t) p^{2}(t)+q(t)\right] .
$$

## The first-order adjoint equation

Theorem (Hu, J., Xue, 2018)
The equation (4) has a solution $(p(\cdot), q(\cdot))$.

Uniqueness

- Case I: $(p(\cdot), q(\cdot))$ is bounded;
- Case II: $q(\cdot)$ is unbounded and $\sigma(t, x, y, z, u)$ takes some special forms.


## The second-order adjoint equation

$$
\left\{\begin{align*}
&- d P(t)  \tag{5}\\
&=\left\{P(t)\left[\left(D \sigma(t) \top\left[1, p(t), K_{1}(t)\right]^{\top}\right)^{2}+2 D b(t) \top\left[1, p(t), K_{1}(t)\right]^{\top}+H_{y}(t)\right]\right. \\
&+2 Q(t) D \sigma(t) \top\left[1, p(t), K_{1}(t)\right]^{\top}+\left[1, p(t), K_{1}(t)\right] D^{2} H(t)\left[1, p(t), K_{1}(t)\right]^{\top} \\
&\left.+H_{z}(t) K_{2}(t)\right\} d t-Q(t) d B(t), \\
& P(T)=\phi_{x x}(\bar{X}(T)),
\end{align*}\right.
$$

where

$$
\begin{aligned}
& H(t, x, y, z, u, p, q)=g(t, x, y, z, u)+p b(t, x, y, z, u)+q \sigma(t, x, y, z, u) \\
& \begin{array}{l}
K_{2}(t)=\left(1-p(t) \sigma_{z}(t)\right)^{-1}\left\{p(t) \sigma_{y}(t)+2\left[\sigma_{x}(t)+\sigma_{y}(t) p(t)+\sigma_{z}(t) K_{1}(t)\right]\right\} P(t) \\
\quad+\left(1-p(t) \sigma_{z}(t)\right)^{-1}\left\{Q(t)+p(t)\left[1, p(t), K_{1}(t)\right] D^{2} \sigma(t)\left[1, p(t), K_{1}(t)\right] \top\right.
\end{array}
\end{aligned}
$$

## Stochastic Maximum Principle

Define

$$
\begin{aligned}
& \mathcal{H}(t, x, y, z, u, p, q, P) \\
& =p b(t, x, y, z+\Delta(t), u)+q \sigma(t, x, y, z+\Delta(t), u) \\
& \quad+\frac{1}{2} P(\sigma(t, x, y, z+\Delta(t), u)-\sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)))^{2} \\
& \quad+g(t, x, y, z+\Delta(t), u),
\end{aligned}
$$

where $\Delta(t)$ is defined by, for $t \in[0, T]$
$\Delta(t)=p(t)(\sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t)+\Delta(t), u)-\sigma(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t)))$.

## Stochastic Maximum Principle

## Theorem (Hu, J., Xue, 2018)

Let $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ be optimal and $(\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ be the corresponding state processes of (3). Then the following stochastic maximum principle holds:

$$
\begin{aligned}
& \mathcal{H}(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), u, p(t), q(t), P(t)) \\
& \geq \mathcal{H}(t, \bar{X}(t), \bar{Y}(t), \bar{Z}(t), \bar{u}(t), p(t), q(t), P(t)), \quad \forall u \in U, \text { a.e., a.s }
\end{aligned}
$$

where $(p(\cdot), q(\cdot)),(P(\cdot), Q(\cdot))$ satisfy (4), (5) respectively, and $\Delta(\cdot)$ satisfies (6).

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## General case: Different well-posedness conditions

There are several different conditions to guarantee the existence and uniqueness of the solution to fully coupled FBSDEs.

- Our assumptions are mainly from Cvitanić and Zhang (2013)
- Does our method still work under other assumptions?
- Does the established stochastic maximum principle still hold?


## General case: The key issues to derive SMP

Our approach essentially depends on the following assumptions:

- There exists a unique solution to FBSDE

$$
\left\{\begin{align*}
d X(t)= & b(t, X(t), Y(t), Z(t), u(t)) d t  \tag{7}\\
& +\sigma(t, X(t), Y(t), Z(t), u(t)) d B(t) \\
d Y(t)= & -g(t, X(t), Y(t), Z(t), u(t)) d t+Z(t) d B(t) \\
X(0)= & x_{0}, Y(T)=\phi(X(T))
\end{align*}\right.
$$

and its solution has $L^{p}$-estimates $(p \geq 8)$.

## General case: The key issues to derive SMP

- The solution to the following linear FBSDE (variational equation) has $L^{p}$-estimates $(p \geq 8)$.

$$
\left\{\begin{align*}
d \hat{X}(t)= & {\left[\alpha_{1}(t) \hat{X}(t)+\beta_{1}(t) \hat{Y}(t)+\gamma_{1}(t) \hat{Z}(t)+L_{1}(t)\right] d t } \\
& +\left[\alpha_{2}(t) \hat{X}(t)+\beta_{2}(t) \hat{Y}(t)+\gamma_{2}(t) \hat{Z}(t)+L_{2}(t)\right] d B(t), \\
d \hat{Y}(t)= & -\left[\left\langle\alpha_{3}(t), \hat{X}(t)\right\rangle+\beta_{3}(t) \hat{Y}(t)+\gamma_{3}(t) \hat{Z}(t)+L_{3}(t)\right] d t+\hat{Z}(t) d B(t), \\
\hat{X}(0)= & x_{0}, \hat{Y}(T)=\langle\kappa, \hat{X}(T)\rangle+\varsigma . \tag{8}
\end{align*}\right.
$$

where $\alpha_{1}(t)=\tilde{b}_{x}^{\epsilon}(t)=\int_{0}^{1} b_{x}\left(t, \Theta(t)+\theta\left(\Theta^{\epsilon}(t)-\Theta(t)\right), u^{\epsilon}(t)\right) d \theta$ for $0 \leq \varepsilon<T$, other terms are similar.

Other assumptions, roughly speaking,

- There exist unique solutions to the adjoint equations.
- The algebra equation of $\Delta(t)$ has a unique solution.


## Conclusion:

- For any well-posedness conditions which can guarantee the above assumptions hold, our SMP does hold.
- Our approach to derive SMP is essentially independent of different well-posedness conditions.
- M. Hu, S. Ji and X. Xue, A note on the global stochastic maximum principle for fully coupled forward-backward stochastic systems, arXiv.1812.10469, (2018).


## General case

In more details, our approach holds under the following two kinds of assumptions in Hu, Ji and Xue (2018).

## The first kind of assumptions is:

- There exists a unique $L^{p}$-solution $(p \geq 8)$ to $\operatorname{FBSDE}(7)$;
- There exists a unique bounded solution to the first-order adjoint equation;
- There exists a unique solution to the algebra equation;
- There exists a unique $L^{p}$-solution $(p \geq 8)$ to FBSDE (8).


## General case

The second kind of assumptions is:

- There exists a unique $L^{p}$-solution ( $p \geq 8$ ) to FBSDE (7);
- There exists a unique solution to the first-order adjoint equation;
- There exists a unique solution to the algebra equation;
- The solution to linear FBSDE (8) has $L^{p}$-estimates $(p \geq 8)$;
- $\sigma$ is linear in $z$, and $\left\|\sigma_{z}\right\|$ is small enough.

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10 M. Hu, S. Ji and X. Xue, A note on the global stochastic maximum principle for fully coupled forward-backward stochastic systems, arXiv.1812.10469, (2018).

## Thank you!

