Mean-field approach to Bayesian estimation of Markovian signals

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Part I:
Mean-field type approximation of the posterior
The Bayesian approach - general framework

estimating a signal $X \in \mathbb{R}^d$ via noisy data $Y = G(X, \zeta)$

- $\zeta$ - measurement noise
- $Y$ - observation, e.g. $\in \mathbb{R}^p$, $L^2(D)$, ...

given/known in applications

- model for $X$, e.g. in terms of (S)ODE, (S)PDE, ... yields prior $P_X$
- distribution of meas. noise $P(Y \in dy \mid X = x) = e^{-\ell(y,x)} dy$

yields posterior distribution (Bayes theorem)

$$P_{X \mid Y = y}(dx) = \frac{e^{-\ell(y,x)} P_X(dx)}{\int e^{-\ell(y,\tilde{x})} P_X(d\tilde{x})} \propto e^{-\ell(y,x)} P_X(dx)$$
Monte-Carlo approximation of $P_{X|Y=y}$

given $X_1, \ldots, X_M \in \mathbb{R}^d$ (particles), e.g. indep. with common distr. $P_X$, then

$$\pi^M := \frac{1}{M} \sum_{i=1}^{M} \delta_{X_i} \approx P_X$$

since the SLLN implies

$$\int f \, d\pi^M = \frac{1}{M} \sum_{i=1}^{M} f(X_i) \to E(f(X)) = \int f \, dP_X$$

introducing weights $w_i = \exp(-l(y, X_i))$ yields

$$\tilde{\pi}^M := \sum_{i=1}^{M} \frac{w_i}{\sum_{j=1}^{M} w_j} \delta_{X_i} \approx P_{X|Y=y}$$

since

$$\int f \, d\tilde{\pi}^M = \frac{\sum_{i=1}^{M} w_i f(X_i)}{\sum_{i=1}^{M} w_i} \to \frac{\int f \, e^{-\ell(y, \cdot)} \, dP_X}{\int e^{-\ell(y, \cdot)} \, dP_X}$$

**drawback** $w_i$ might degenerate for most particles, requires resampling in iterations (causing additional approximation errors)
The posterior as a push-forward measure

**Ansatz** realize $P_{X|Y=y}$ as image measure $P_{X|Y=y} = \Phi \# P_X$

$$\int f(x) P_{X|Y=y}(dx) = \int f(\Phi(x)) P_X(dx)$$

if $P_X(dx) = \pi(x) \, dx$, $\Phi$ may be obtained as solutions of the Monge-Ampère equation

$$\det (D\Phi(x)) = Z \exp (\ell(y, \Phi(x))) \frac{\pi(x)}{\pi(\Phi(x))},$$

with normalizing constant

$$Z = \int \exp (-\ell(y, x)) \pi(x) \, dx$$

- fully nonlinear
- non-unique: $\Phi \circ \Psi$ (resp. $\Psi \circ \Phi$) again solution for any $\Psi$ differentiable and measure-preserving $P_X$ (resp. $P_{X|Y=y}$)
- requires additional constraints for uniqueness
Linearized Monge-Ampère equation

\[ \ell \ll 1 \text{ implies } \Phi(x) \approx x + K(x) \text{ with } K \ll 1 \]

neglecting higher order terms in \( \ell \), \( K \) and Jacobian \( DK \) gives

\[
\det(D\Phi(x)) \approx 1 + \text{trace}(DK(x)) = 1 + \nabla \cdot K(x)
\]
\[
\exp(-\ell(y, \Phi(x))) \approx (1 - \ell(y, \Phi(x))) \approx (1 - \ell(y, x))
\]
\[
\eta(\Phi(x)) \approx \eta(x) + \nabla \eta(x) \cdot K(x)
\]

inserting approximations gives

\[
\nabla \cdot (\pi K) = \pi \nabla \cdot K + \nabla \pi \cdot K = -\pi (\ell(y, \cdot) - \int \ell(y, \cdot) \pi \, dx) \quad (1)
\]

MC-approximation: \( X_1, \ldots, X_M \) i.i.d. with distr. \( P_X \)

\[
\frac{1}{M} \sum_{m=1}^{M} \delta_{X_m} \approx P_X \quad \Rightarrow \quad \frac{1}{M} \sum_{m=1}^{M} \delta_{X_m + K(X_m)} \approx P_{X|Y=y}
\]

and \( K(x) = K(x, \pi) \) is called \textit{Kalman gain}.
Gradient type solutions - Poisson equation

\[ \nabla \cdot (\pi K(\cdot, \pi)) = -\pi(\ell - \pi[\ell]), \quad \pi[\ell] = \int \ell(y, x) \pi(x) \, dx \]

unique up to \( \pi \)-measure preserving transform. \( \mathcal{J}(x, \pi) \), i.e. \( \nabla \cdot (\pi \mathcal{J}) = 0 \)

**Ansatz** \( K(\cdot, \pi) = \nabla \phi \) leads to weighted Poisson eq

\[ \nabla \cdot (\pi \nabla \phi) = -\pi(\ell - \pi[\ell]) \quad (2) \]

(Young, Mehta, Meyn, et al (2013)), with formal solution

\[ \phi = \int_{0}^{\infty} e^{u(\Delta + \nabla \pi \cdot \nabla)} (\ell - \pi[\ell]) \, du \]

solving (2) diffusion map approximation

\[ \phi = e^{\varepsilon(\Delta + \nabla \pi \cdot \nabla)} \phi + \int_{0}^{\varepsilon} e^{u(\Delta + \nabla \pi \cdot \nabla)} (\ell - \pi[\ell]) \, du \]

\[ \approx e^{\varepsilon(\Delta + \nabla \pi \cdot \nabla)} \phi + \varepsilon(\ell - \pi[\ell]) \approx T_{\varepsilon} \phi_{\varepsilon} + \varepsilon(\ell - \pi[\ell]) \]

yields MCMC approximation ([Pathiraja, S., FoDS '21])

\[ \phi_{\varepsilon} = \varepsilon \sum_{k=0}^{\infty} T_{\varepsilon}^{k} (\ell - \pi_{\varepsilon}[\ell]), \quad T_{\varepsilon}(x, A) = \frac{P_{2\varepsilon}^{BM}(1_{A} \tilde{\pi})}{P_{2\varepsilon}^{BM}(\tilde{\pi})}, \quad \tilde{\pi} = \frac{\pi}{\sqrt{P_{2\varepsilon}^{BM} \pi}} \]
Part II:
Bayesian inference of time-structured data
State estimation of sde’s

\begin{align*}
(S) \quad dX_t &= f(X_t)dt + Q(X_t)^{\frac{1}{2}}dW_t \\
(O) \quad dY_t &= g(X_t)dt + R^{\frac{1}{2}}dV_t
\end{align*}

Bayesian Ansatz in principle known:

- law of \( X_{0:t} \)
- conditional law of \( Y | X \) - Brownian motion with drift, i.e.
  \[ Y_{0:t} | X_{0:t} \sim \exp \left( \int_0^t g(X_s)R^{-1}dY_s - \frac{1}{2} \int_0^t \|R^{-\frac{1}{2}}g(X_s)\|^2 ds \right) P_0(dY_{0:t}) \]
  where \( P_0 = \) Wiener meas. (Girsanov-Theorem)

yields - according to Bayes rule -

\[ X_{0:t} | Y_{0:t} \sim \frac{\exp \left( \int_0^t g(X_s)R^{-1}dY_s - \frac{1}{2} \int_0^t \|R^{-\frac{1}{2}}g(X_s)\|^2 ds \right) P(dX_{0:t})}{\int \exp \left( \int_0^t g(X_s)R^{-1}dY_s - \frac{1}{2} \int_0^t \|R^{-\frac{1}{2}}g(X_s)\|^2 ds \right) P(dX_{0:t})} \]
Kushner-Stratonovich equation (SPDE)

\[ \pi_t(dx) := \pi^{Y_{0:t}}(dx) = P(X_t \in dx \mid Y_{0:t}) \]

= cond. distr. of \( X_t \) given \( Y_s, s \in [0, t] \)

solves the measure-valued spde

\[
d\pi_t(\varphi) = \pi_t[L\varphi] + \text{cov}_{\pi_t}(g, \varphi) R^{-1} dl_t
\]

where

\[ L u(x) = \frac{1}{2} \text{tr} \left( Q(x) u''(x) \right) + f(x) \nabla u(x) \]

\text{gen. of (S) hence } \pi_t[L\varphi] dt \text{ is of Fokker-Planck type}

\[ dl_t = dY_t - \pi_t[g] dt \text{ measures deviation of data from forecast} \]

if \( \pi_t \ll dx \), its density \( \pi(t, x) \) is a solution of the spde

\[
d_t \pi(t, x) = \frac{1}{2} \sum_{i,j} \partial^2_{x_i x_j} (Q_{ij}(x) \pi(t, x)) - \sum_i \partial_{x_i} (f_i(x) \pi(t, x))
\]

\[ + \left( g^T(x) - \int g^T(y) \pi(t, y) dy \right) \pi(t, x) R^{-1} \frac{dl_t}{dt} \]
Part III:
The feedback particle filter
Feedback Particle Filter

approximate $\pi_t$ in terms of $\pi_t^M = \frac{1}{M} \sum_{i=1}^{M} \delta_{X^i_t}$ with

$$dX^i_t = f(X^i_t)dt + Q^{\frac{1}{2}}(X^i_t)dW^i_t + K \left( X^i_t, \pi_t^M \right) \circ \left( dY_t - \pi_t^M[g] dt \right)$$

and $K(\cdot, \pi)$ solves

$$\nabla \cdot (\pi K(\cdot, \pi)) = -\pi (g - \pi[g]) R^{-1}, \quad \pi[g] = \int g(x) \pi(x) \, dx$$

can expect for large $M$ convergence of $X^i_t$ to ind. solutions of

$$d\tilde{X}_t = f(\tilde{X}_t) dt + Q^{\frac{1}{2}}(\tilde{X}_t) dW_t + K(\tilde{X}_t, \pi_t) \circ (dY_t - \pi_t[g] dt)$$

Math. problems

(1) Kalman gain $K$ well-defined and sufficiently regular

(2) well-posedness of the mf-sde

(3) well-posedness and convergence of $X^i_t$ to (ind. copies of ) $\tilde{X}_t$ (mf-limit)

Rem fair to say: (2) & (3) completely open
Well-posedness of the FPF - first steps

Ansatz for (1)

solution to the weighted Poisson equation

$$\nabla \cdot (\pi_t \nabla \phi_t) = (g - \pi_t[g]) \pi_t$$

can formally be represented as

$$\phi_t = \int_0^\infty e^{u \left( \Delta + \frac{\nabla \pi_t}{\pi_t} \cdot \nabla \right)} (g - \pi_t[g]) \, du$$

exp decay of $e^{u \left( \Delta + \frac{\nabla \pi_t}{\pi_t} \cdot \nabla \right)} (g - \pi_t[g])$ (in $u$) implied by Poincaré inequality

$$\int (\varphi - \pi_t[\varphi])^2 \, d\pi_t \leq \kappa_t \int |\nabla \varphi|^2 \, d\pi_t$$

our Ansatz 'Poincaré via log-concavity' $\pi_t \propto e^{-G_t}$, $G_t$ convex, e.g. $N(m, Q)$
Thm [Pathiraja, Reich, S., SICON ’21] Let $\pi_t(x)$ be the posterior density. Suppose that

- $f = -\nabla F$ gradient type, $F'' \geq c_F \text{Id}$, $c_F > 0$, $g(x) = Gx$ linear
- $V = -\Delta F + |\nabla F|^2 + |Gx|^2$ uniformly strictly convex, $V'' \geq c_V \text{Id}$
- initial density $\pi_0 \propto \exp(-G_0)$, $G_0'' \geq c_G \text{Id}$ with $c_G > c_F$

then

$$\int (\varphi - \pi_t[\varphi])^2 \, d\pi_t \leq \kappa_t \int |\nabla \varphi|^2 \, d\pi_t$$

with constant less than

$$\kappa_t = \left( c_F + \min\left( c_G - c_F, \sqrt{c_V / 2} \right) \right)^{-1}.$$

**main feature** $\pi_t$ log-concave for all $t$ (very restrictive (!), e.g. excludes multi target tracking)
Part IV:

The Ensemble Kalman-Bucy filter:  
the popular relative of the FPF
Linear signals: Kalman-Bucy filter (KBF)

\[ (S) \quad dX_t = (AX_t + b) \, dt + Q^{\frac{1}{2}} \, dW_t \]

\[ (O) \quad dY_t = GX_t \, dt + R^{\frac{1}{2}} \, dV_t \]

Main feature

\[ \pi_0 = N(m_0, P_0) \implies \pi_t = N(m_t, P_t) \]

with

\[ dm_t = (Am_t + b) \, dt + P_t \, G^T R^{-1} (dY_t - Gm_t \, dt) \]

\[ \frac{d}{dt} P_t = AP_t + P_t A^T + Q - P_t G^T R^{-1} G P_t \]

Rem dim of (S) might be large, hence computation of \( P_t \) costly, requires low-dimensional approx.
Mean-field representation of the KBF

(S) \[ dX_t = (AX_t + b) \, dt + Q^{\frac{1}{2}} \, dW_t \]

(O) \[ dY_t = GX_t \, dt + R^{\frac{1}{2}} \, dV_t \]

lin. Monge-Ampère eq reduces for \( \pi = N(m, P) \) to

\[ \langle P^{-1}(x - m), K \rangle - \nabla \cdot K = \langle R^{-1}G(x - m), dY_t \rangle - \frac{1}{2} \langle R^{-1}Gx, Gx \rangle + \text{const.} \]

with explicit solution

\[ K(x, \pi) = PG^T R^{-1} \left( dY_t - \frac{1}{2} G(x + m) \, dt \right) \]

and McKean-Vlasov sde

\[ d\bar{X}_t = (A\bar{X}_t + b) \, dt + Q^{\frac{1}{2}} dW_t + P_t G^T R^{-1} \left( dY_t - \frac{1}{2} G(\bar{X}_t + m_t) \, dt \right) \]
**Ensemble Kalman-Bucy filter (EnKBF)**

**Main idea** replace

$$m_t \sim \tilde{x}_t^M := \frac{1}{M} \sum_{i=1}^{M} X_t^i$$

$$P_t \sim P_t^M := \frac{1}{M-1} \sum_{i=1}^{M} (X_t^i - \tilde{x}_t^M)(X_t^i - \tilde{x}_t^M)^T$$

where

$$dX_t^i = f(X_t^i)dt + Q(X_t^i)^{\frac{1}{2}}dW_t^i + P_t^M G^T R^{-1} \left( dY_t - \frac{1}{2} G(X_t^i + \tilde{x}_t^M)dt \right)$$

for $M$ ind. Brownian motions $W_t^i$, or its deterministic analogon

$$dX_t^i = f(X_t^i)dt + \frac{1}{2} Q(X_t^i) \left( P_t^M \right)^{-1} (X_t^i - \tilde{x}_t^M)dt$$

$$+ P_t^M G^T R^{-1} \left( dY_t - \frac{1}{2} G(X_t^i + \tilde{x}_t^M)dt \right)$$

avoids degeneracy problem of classical weighted particle filters, in part. avoids resampling
Well-posedness of strong solutions

coefficients of

\[
dX_t^i = f(X_t^i)dt + Q(X_t^i)\frac{1}{2}dW_t^i + P_t^M G^T R^{-1} \left( dY_t - \frac{1}{2} G(X_t^i + \bar{x}_t^M)dt \right)
\]

have cubic growth, but \( \text{tr}(P_t^M) \) has semi-martingale decomposition

\[
d\text{tr}(P_t^M) = \frac{2}{M - 1} \sum_{i=1}^{M} \left\langle f(X_t^i) - f(\bar{x}_t), X_t^i - \bar{x}_t \right\rangle dt
\]

\[
+ \text{tr} \left( \frac{1}{M} \sum_{i=1}^{M} Q(X_t^i) - P_t^M G^T R^{-1} GP_t^M \right) dt + d\text{tr}(N_t)
\]

with local martingale

\[
dN_t := \frac{1}{M - 1} \sum_{i=1}^{M} \left( Q(X_t^i)\frac{1}{2} d \left( W_t^i - \bar{w}_t \right) \right) (X_t^i - \bar{x}_t)^T
\]

\[
+ (X_t^i - \bar{x}_t) \left( Q(X_t^i)\frac{1}{2} d \left( W_t^i - \bar{w}_t \right) \right)^T dt
\]

stochastic Gronwall lemma now implies for all \( p < 1 \)

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \xi} \text{tr}(P_t^M)^p \right] \leq C_p e^{2p\|f\|_{\text{Lip}} T} \mathbb{E} \left[ \left( \int_0^{T \wedge \xi} \frac{1}{M} \sum_{i=1}^{M} \text{tr}(Q(X_t^i)) dt \right)^p \right]
\]

up to some life-time \( \xi \)
Well-posedness of strong solutions, ctd.

\[ dX_t^i = f(X_t^i)dt + Q(X_t^i)\frac{1}{2}dW_t^i + P_t^M G^T R^{-1} \left( dY_t - \frac{1}{2} G(X_t^i + \bar{x}_t^M)dt \right) \]

similarly,

\[ d\bar{x}_t^M = (\frac{1}{M} \sum_{i=1}^{M} f(X_t^i) - P_t^M G^T R^{-1} G\bar{x}_t^M)dt \]

\[ + P_t^M G^T R^{-1} dY_t + d\bar{N}_t \]

with local martingale

\[ d\bar{N}_t := \frac{1}{M} \sum_{i=1}^{M} Q(X_t^i)\frac{1}{2}dW_t^i \]

another application of the stochastic Gronwall lemma implies

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \xi} \| \bar{x}_t^M \|^p \right] < \infty \]

so that \( P[\xi \leq T] = 0 \) for all \( T \)
Numerical Illustration

Lorenz-63 system with additive noise

\[(S) \quad dX_t = f(X_t)dt + \sqrt{2}dW_t\]

\[(O) \quad dY_t = X_t dt + \sqrt{\varepsilon}dV_t\]

\[f(x) = \begin{pmatrix}
10(x_2 - x_1) \\
(28 - x_3)x_1 - x_2 \\
x_1x_2 - \frac{8}{3}x_3
\end{pmatrix}\]

EnKBF with ensemble size \( M = 4 \)

accuracy: [de Wiljes, Reich, S., SIADS '18]
Numerical Illustration, $M = 2, 3, P^M_t$ sing.

Lorenz-63 system with additive noise

\[(S) \quad dX_t = f(X_t)dt + \sqrt{2}dW_t\]

\[(O) \quad dY_t = X_t dt + \sqrt{\epsilon}dV_t\]

\[f(x) = \begin{pmatrix} 10(x_2 - x_1) \\ (28 - x_3)x_1 - x_2 \\ x_1x_2 - \frac{8}{3}x_3 \end{pmatrix}\]
Part V:

The MF-limit of the EnKBF
Mean field limit for $M \to \infty$

**Thm** [Lange, S., FoDS '21]

- $Q(x) \equiv Q$ (additive signal noise)
- $X_0^i, i = 1, \ldots, M$, i.i.d. with distribution $\bar{\pi}_0$ having finite second moments
- $W_t^i$ ind. Brownian motions

Then

$$X_t^i = \bar{X}_t^i + r_t^i$$  (3)

- $\bar{X}_t^i$ are sol. to

$$d\bar{X}_t^i = f(\bar{X}_t^i)dt + Q_{1/2}^i dW_t^i + \text{Cov}(\bar{X}_t^i)R^{-1}\left(dt - \frac{1}{2}G(\bar{X}_t^i + E(\bar{X}_t^i))dt\right)$$

- $r_t^i$ satisfies a.s. w.r.t. to the distr. of $Y$

$$\sup_{t \in [0,T]} \frac{1}{M} \sum_{i=1}^M \left\|r_t^i\right\|^2 \to 0, \quad M \to \infty,$$  (4)

in probability for all $T \geq 0$

**Rem** extension to correlated meas. noise in [Ertel, S., arXiv2205.14253]
The spde driving $P\bar{X}_t$
with the notation $\bar{m}_t = E(\bar{X}_t)$, $\bar{P}_t = \text{Cov}_P(\bar{X}_t)$

Itô's formula implies that for
\[
d\bar{X}_t = f(\bar{X}_t)dt + Q(\bar{X}_t)^{\frac{1}{2}}dW_t + \bar{P}_tG^TR^{-1}\left(dY_t - \frac{1}{2}G(\bar{X}_t + \bar{m}_t)dt\right)
\]

the distr. $\bar{\pi}_t = \text{Law}(\bar{X}_t)$ satisfies the nonl. Fokker-Planck spde

\[
d\int \varphi d\bar{\pi}_t = \int L\varphi d\bar{\pi}_t dt + \frac{1}{2} \int \text{tr} \left(\bar{P}_tG^TR^{-1}G\bar{P}_t\varphi''\right) d\bar{\pi}_t dt
\]
\[
+ \int \nabla \varphi d\bar{\pi}_t \bar{P}_tG^TR^{-1}(dY_t - G\bar{m}_t dt)
\]
\[
- \frac{1}{2} \int \nabla \varphi \bar{P}_tG^TR^{-1}G(x - \bar{m}_t) d\bar{\pi}_t dt
\]
\[
=: (I) + (II) + (III) + (IV)
\]

where $L\varphi(x) = \frac{1}{2}\text{tr}(Q(x)\varphi'') (x) + f(x) \cdot \nabla \varphi(x)$

Rem: does not coincide with the KS-eq, unless $\bar{\pi}_t = \mathcal{N}(\bar{m}_t, \bar{P}_t)$

in this case partial integration in (II) yields $(II) = -(IV)$, and (III) reduces to

\[
\int \varphi G(x - \bar{m}_t)^T d\bar{\pi}_t R^{-1} (dY_t - G\bar{m}_t dt)
\]