AGH UNIVERSITY OF SCIENCE AND TECHNOLOGY

# Efficient approximation of solutions of SDEs driven by countably dimensional Wiener process and Poisson random measure

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The 9th International Colloquium on BSDEs and Mean Field Systems - Annecy, 28/06/2022

## Main references

[1] P. Przybyłowicz, M. Sobieraj, Ł. Stępień, Efficient approximation of SDEs driven by countably dimensional Wiener process and Poisson random measure, SIAM Journal on Numerical Analysis 60 (2022), 824-855.

[2] A. Kałuża, *Optimal algorithms for solving stochastic initial-value problems with jumps*, PhD thesis, AGH UST, 2020, https://winntbg.bg.agh.edu.pl/rozprawy2/11743/full11743.pdf

[3] S.N. Cohen, R.J. Elliot, *Stochastic calculus and applications, 2nd ed.*, Probability and its applications, New York, Birkhäuser, 2015.

[4] G. Cao, K. He, On a type of stochastic differential equations driven by countably many Brownian motions, *J. Funct. Anal.* **203** (2003), 262–285.

## Lévy measure and Poisson random measure

Let $\mathcal{E} = \mathbb{R}^{d'} \setminus \{0\}$ for some $d' \in \mathbb{N}$. We also assume that a measure $\nu$ defined on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ is a finite Lévy measure ($\lambda := \nu(\mathcal{E}) < +\infty$).

We set $\mathbb{U} := \mathcal{E} \times [0, T]$. Moreover, let $(\mathbb{U}, \mathcal{B}(\mathbb{U}), n)$ be a measure space with non-negative, $\sigma-$ finite measure $n$.

**Definition (point Poisson measure)**

A family of rv's $\{N(B), B \in \mathcal{B}(\mathbb{U})\}$ taking values in $\mathbb{N} \cup \{+\infty\}$ is called a *point (random) Poisson measure* on $\mathbb{U}$ with intensity $n$ iff:

- for every $\omega \in \Omega$, $N(\cdot)(\omega)$ yields a Borel measure on $\mathbb{U}$,
- for each set $B \in \mathcal{B}(\mathbb{U})$, a random variable $N(B)$ follows Poisson law with parameter $n(B)$,
- given any family of disjoint and measurable sets $B_1, \ldots, B_m \subset \mathbb{U}$, the corresponding rv's $N(B_1), \ldots, N(B_m)$ are independent.

## Lévy measure and Poisson random measure - cont'd

Now let

- $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of iid random variables with the distribution $\mu(dy) = \nu(dy)/\lambda$,
- $\{N(t)\}_{t\in[0,T]}$ be a Poisson process with intensity $\lambda$, independent of $\{\xi_n\}$.

Then, the mapping

$$N(E \times (s,t]) := \sum_{N(s)<k\leq N(t)} \mathbf{1}_E(\xi_k) = \sum_{k:s<\tau_k\leq t} \mathbf{1}_E(\xi_k),$$

$$0 \leq s < t \leq T, E \in \mathcal{B}(\mathcal{E}),$$

establishes a random Poisson measure on $\mathcal{E} \times [0, T]$ with intensity $n(dy, dt) = \nu(dy)dt$.

## Preliminaries

$(\Omega, \Sigma, (\Sigma_t)_{t\geq 0}, \mathbb{P})$, $T > 0$, $\mathcal{E} = \mathbb{R}^{d'} \setminus \{0\}$ for some $d' \in \mathbb{N}$,
$W = [W_1, W_2, \ldots]^T$ – cylindrical Wiener process in $\ell^2$,
$N$ – Poisson random measure on $(\mathcal{E} \times [0, T], \mathcal{B}(\mathcal{E} \times [0, T]))$ with intensity measure $\nu(dy)dt$, $\nu$-finite Lévy measure

$$X(t) = \eta + \int_0^t a(s, X(s))ds + \sum_{j=1}^{+\infty} \int_0^t b^{(j)}(s, X(s))dW_j(s)$$
$$+ \int_0^t \int_{\mathcal{E}} c(s, X(s-), y)N(dy, ds), \ \ t \in [0, T]$$

## Preliminaries-cont'd

### Applications

- mathematical finance and insurance
- modelling energy prices
- stochastic control problems/switching systems
- mathematical biology
- and many, many more ...

## Preliminaries-cont'd

**Aims**

- rate of convergence for the truncated dimension randomized Euler-Maruyama scheme
- complexity bounds in IBC framework
- efficient implementation (CUDA C, GPUs)

## Preliminaries-cont'd

Recent references on (randomized) algorithms for SDEs (with jumps):

K. Dareiotis, C. Kumar, S. Sabanis (2016), S. Deng , W. Fei , W. Liu , X. Mao (2019), S. Deng , C. Fei , W. Fei , X. Mao (2019), J. Dębowski, P. Przybyłowicz (2016), S. Heinrich (2016, 2019), D. Higham, P.E. Kloeden (2005, 2006, 2007), A. Kałuża, P. Przybyłowicz (2018), R. Kruse, Y. Wu (2019), P. M. Morkisz, P. Przybyłowicz (2014, 2017, 2021), E. Platen, N. Bruti-Liberati (2010), P. Przybyłowicz (2 x 2019), P. Przybyłowicz, M. Szölgyenyi (2021), P. Przybyłowicz, M. Szölgyenyi, F. Xu (2021)

# Preliminaries-cont'd

$\|\cdot\|$ - Hilbert–Schmidt norm

**Assumptions**

*(A) The class $\mathcal{A}(D, L)$, $D, L > 0$, of drift coefficients $a : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$:*

**(A1)** *$a$ is Borel measurable,*

**(A2)** $\|a(t, 0)\| \leq D$ *for all* $t \in [0, T]$,

**(A3)** $\|a(t, x) - a(t, y)\| \leq L\|x - y\|$ *for all* $x, y \in \mathbb{R}^d$, $t \in [0, T]$

**Assumptions**

*(B) Let $\Delta = (\delta(k))_{k=1}^{+\infty} \subset \mathbb{R}$ be a positive, strictly decreasing sequence, converging to zero. The class $\mathcal{B}(C, D, L, \Delta, \varrho_1)$, $C > 0$, $\varrho_1 \in (0, 1]$, of diffusion coefficients $b = (b^{(1)}, b^{(2)}, \ldots) : [0, T] \times \mathbb{R}^d \mapsto \ell^2(\mathbb{R}^d)$:*

**(B1)** $\|b(0, 0)\| \leq D$,

**(B2)** $\|b(t, x) - b(s, x)\| \leq L(1 + \|x\|)|t - s|^{\varrho_1}$ *for all* $x \in \mathbb{R}^d$ *and* $t, s \in [0, T]$,

**(B3)** $\|b(t, x) - b(t, y)\| \leq L\|x - y\|$ *for all* $x, y \in \mathbb{R}^d$, $t \in [0, T]$,

**(B4)** $\sup_{0 \leq t \leq T} \|b(t, x) - P_k b(t, x)\| \leq C(1 + \|x\|)\delta(k)$ *for all* $k \in \mathbb{N}$, $x \in \mathbb{R}$, *where* $P_k : \ell^2(\mathbb{R}^d) \mapsto \ell^2(\mathbb{R}^d)$, $P_k x = (x^{(1)}, x^{(2)}, \ldots, x^{(k)}, 0, 0, \ldots)$ *for all* $x \in \ell^2(\mathbb{R}^d)$. *We denote* $b^k = P_k b$ *and* $P_\infty = Id$.

# Preliminaries-cont'd

**Assumptions**

*(C) Let $p \in [2, +\infty)$, $\varrho_2 \in (0, 1]$, $\nu$ - a finite Lévy measure. The class $\mathcal{C}(p, D, L, \varrho_2, \nu)$ of jump coefficients $c : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^d$:*

**(C1)** *$c$ is Borel measurable,*

**(C2)** $$\left( \int_{\mathcal{E}} \|c(0, 0, y)\|^p \nu(dy) \right)^{1/p} \leq D,$$

**(C3)** $$\left( \int_{\mathcal{E}} \|c(t, x_1, y) - c(t, x_2, y)\|^p \ \nu(dy) \right)^{1/p} \leq L\|x_1 - x_2\| \text{ for all } x_1, x_2 \in \mathbb{R}^d,\ t \in [0, T],$$

**(C4)** $$\left( \int_{\mathcal{E}} \|c(t_1, x, y) - c(t_2, x, y)\|^p \ \nu(dy) \right)^{1/p} \leq L(1 + \|x\|)|t_1 - t_2|^{\varrho_2} \text{ for all } x \in \mathbb{R}^d,$$

$t_1, t_2 \in [0, T]$.

## Preliminaries-cont'd

**Assumptions**

*Initial value*

$$\mathcal{J}(p, D) = \{\eta \in L^p(\Omega) \mid \sigma(\eta) \subset \Sigma_0, \|\eta\|_{L^p(\Omega)} \leq D\}$$

We define

$$\mathcal{A}(D, L) \times \mathcal{B}(C, D, L, \Delta, \varrho_1) \times \mathcal{C}(p, D, L, \varrho_2, \nu) \times \mathcal{J}(p, D)$$

as $\mathcal{F}(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, with $T, d, d', \lambda, p, C, D, L, \varrho_1, \varrho_2, \nu, \Delta$ being the class parameters.
Except for $T, d, d', \lambda, \nu$, the parameters are not known and cannot be used by an algorithm as input parameters.

## Truncated dimension approximation

$M \in \mathbb{N}$, $X^M = (X^M(t))_{t\in[0,T]}$

$$X^M(t) = \eta + \int_0^t a(s, X^M(s))\,\mathrm{d}s + \int_0^t b^M(s, X^M(s))\,\mathrm{d}W(s)$$
$$+ \int_0^t \int_{\mathcal{E}} c(s, X^M(s-), y) N(\mathrm{d}y, \mathrm{d}s), \quad t \in [0, T],$$

$$\int_0^t b^M(s, X^M(s))\,\mathrm{d}W(s) = \Big[\sum_{j=1}^M \int_0^t b_k^{(j)}(s, X^M(s))\,\mathrm{d}W_j(s)\Big]_{k=1,2,\ldots,d},$$

# Truncated dimension approximation-cont'd

**Proposition 1. ([1])**

There exist $K_1, K_2 \in (0, +\infty)$, $M_0 \in \mathbb{N}$ such that for any $M \in \mathbb{N}$ it holds

$$\sup_{(a,b,c,\eta)\in\mathcal{F}(p,C,D,L,\Delta,\varrho_1,\varrho_2,\nu)} \sup_{0\le t\le T} \|X(a,b,c,\eta)(t) - X^M(a,b,c,\eta)(t)\|_{L^p(\Omega)} \le K_1\delta(M),$$

and for any $M \ge M_0$ we have

$$\sup_{(a,b,c,\eta)\in\mathcal{F}(p,C,D,L,\Delta,\varrho_1,\varrho_2,\nu)} \sup_{0\le t\le T} \|X(a,b,c,\eta)(t) - X^M(a,b,c,\eta)(t)\|_{L^2(\Omega)} \ge \frac{1}{2}K_2 T^{1/2}\delta(M).$$

Hence,

$$\sup_{(a,b,c,\eta)\in\mathcal{F}(p,C,D,L,\Delta,\varrho_1,\varrho_2,\nu)} \sup_{0\le t\le T} \|X(a,b,c,\eta)(t) - X^M(a,b,c,\eta)(t)\|_{L^p(\Omega)} = \Theta(\delta(M)),$$

as $M \to +\infty$.

## Truncated dimension randomized Euler scheme

$M, n \in \mathbb{N}$, $t_j = jT/n$, $j = 0, 1, \ldots, n$,
$\Delta W_j = [\Delta W_{j,1}, \Delta W_{j,2}, \ldots]^T$, $\Delta W_{j,k} = W_k(t_{j+1}) - W_k(t_j)$ for $k \in \mathbb{N}$,
$N(t) := N(\mathcal{E} \times (0, t])$ - Poisson process with the intensity $\lambda = \nu(\mathcal{E})$,
$(\xi_k)_{k=1}^{+\infty}$ - iid sequence of $\mathcal{E}$-valued rvs. with the law $\nu/\lambda$,
$(\theta_j)_{j=1}^{n-1}$ - indep. rvs, $\theta_j \sim U[t_j, t_{j+1}]$, $j = 0, 1, \ldots, n-1$,
$\sigma(\theta_0, \theta_1, \ldots, \theta_{n-1})$ and $\Sigma_\infty$ are independent,

$$\begin{cases} X_{M,n}^{RE}(0) = \eta \\ X_{M,n}^{RE}(t_{j+1}) = X_{M,n}^{RE}(t_j) + a(\theta_j, X_{M,n}^{RE}(t_j))\frac{T}{n} + b^M(t_j, X_{M,n}^{RE}(t_j))\Delta W_j \\ \qquad\qquad + \sum_{k=N(t_j)+1}^{N(t_{j+1})} c(t_j, X_{M,n}^{RE}(t_j), \xi_k), \quad j = 0, 1, \ldots, n-1. \end{cases}$$

$$\bar{X}_{M,n}^{RE}(a, b, c, \eta) = X_{M,n}^{RE}(T).$$

## Truncated dimension randomized Euler scheme-cont'd

**Theorem 1. ([1])**

There exists a positive constant $K$, depending only on the parameters of the class $\mathcal{F}(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, such that for every $M, n \in \mathbb{N}$ and $(a, b, c, \eta) \in \mathcal{F}(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$ it holds

$$\|X(a, b, c, \eta)(T) - \bar{X}^{RE}_{M,n}(a, b, c, \eta)\|_{L^p(\Omega)} \leq K\Big(n^{-\min\{\varrho_1, \varrho_2, 1/p\}} + \delta(M)\Big).$$

Idea of the proof:

$$X - \bar{X}^{RE}_{M,n} = (X - X^M) + (X^M - \bar{X}^{RE}_{M,n})$$

## Lower error bounds and complexity

$$\mathcal{G}_i(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu) = \mathcal{A}(D, L) \times \mathcal{B}(C, D, L, \Delta, \varrho_1) \times \mathcal{C}_i(p, D, L, \varrho_2, \nu) \times \mathcal{J}(p, D)$$

for $i = 1, 2$, where

$$\mathcal{C}_1(p, D, L, \varrho_2, \nu) = \{c \in \mathcal{C}(p, D, L, \varrho_2, \nu) \mid c(t, x, y) = c(t, x, 0) \\ \text{for all } (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d'}\},$$

$$\mathcal{C}_2(p, D, L, \varrho_2, \nu) = \{c \in \mathcal{C}(p, D, L, \varrho_2, \nu) \mid \exists_{\tilde{c}:[0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^{d\times d'}} : c(t, x, y) = \tilde{c}(t, x)y \\ \text{for all } (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d'}\}.$$

$$\text{(D)} \qquad \kappa_p := \left( \int_{\mathcal{E}} \|y\|^p \nu(\,\mathrm{d}y) \right)^{1/p} < +\infty.$$

## Lower error bounds and complexity-cont'd

$$\int_0^t \int_{\mathcal{E}} c(s, X(s-), y) N(\,\mathrm{d}y,\, \mathrm{d}s) = \begin{cases} \int_0^t c(s, X(s-), 0)\,\mathrm{d}N(s), & \text{if } c \in \mathcal{C}_1, \\ \int_0^t \tilde{c}(s, X(s-))\,\mathrm{d}L(s), & \text{if } c \in \mathcal{C}_2, \end{cases}$$

$$L(t) = \int_0^t \int_{\mathcal{E}} y \; N(dy, ds)$$

# Lower error bounds and complexity-cont'd

$\Phi_{M,n}$, $M \in \mathbb{N}$ (truncation dimension parameter), $n \in \mathbb{N}$ (discretization parameter)

$$\begin{aligned}\mathcal{N}_{M,n}(a, b, c, \eta, W, Z) = \Big[ & a(\theta_0, y_0), a(\theta_1, y_1), \ldots, a(\theta_{k_1-1}, y_{k_1-1}), \\ & b^M(t_0, z_0), b^M(t_1, z_1), \ldots, b^M(t_{k_1-1}, z_{k_1-1}), \\ & \bar{c}(u_0, v_0), \bar{c}(u_1, v_1), \ldots, \bar{c}(u_{k_1-1}, v_{k_1-1}), \\ & W^M(s_0), W^M(s_1), \ldots, W^M(s_{k_2-1}), \\ & Z(q_0), Z(q_1), \ldots, Z(q_{k_3-1}), \eta \Big],\end{aligned}$$

where $\bar{c}(t, x) = c(t, x, 0)$ (if $Z = N$) or $\bar{c}(t, x) = \tilde{c}(t, x)$ (if $Z = L$)

$$\max_{1 \le i \le 3} k_i = O(n), \tag{1}$$

$[\theta_0, \theta_1, \ldots, \theta_{k_1-1}]^T$ is a $[0, T]^{k_1}$-valued random vector on $(\Omega, \Sigma, \mathbb{P})$, $\sigma(\theta_0, \theta_1, \ldots, \theta_{k_1-1})$ and $\Sigma_\infty$ are independent $\sigma$-fields,
$t_0, t_1, \ldots, t_{k_1-1}$, $s_0, s_1, \ldots, s_{k_2-1}$, $u_0, u_1, \ldots, u_{k_1-1}, q_0, q_1, \ldots, q_{k_3-1} \in [0, T]$ - fixed,

# Lower error bounds and complexity-cont'd

$\psi_j$, $j = 0, 1, \ldots, k_1 - 1$ (Borel measurable functions), such that

$$(y_0, z_0, v_0) = \psi_0\Big(W^M(s_0), \ldots, W^M(s_{k_2-1}), Z(q_0), \ldots, Z(q_{k_3-1}), \eta\Big),$$

and for $j = 1, 2, \ldots, k_1 - 1$

$$\begin{aligned}(y_j, z_j, v_j) \quad = \quad \psi_j\Big(& a(\theta_0, y_0), a(\theta_1, y_1), \ldots, a(\theta_{j-1}, y_{j-1}),\\ & b^M(t_0, z_0), b^M(t_1, z_1), \ldots, b^M(t_{j-1}, z_{j-1}),\\ & \bar{c}(u_0, v_0), \bar{c}(u_1, v_1), \ldots, \bar{c}(u_{j-1}, v_{j-1}),\\ & W(s_0), W(s_1), \ldots, W(s_{k_2-1}),\\ & Z(q_0), Z(q_1), \ldots, Z(q_{k_3-1}), \eta\Big).\end{aligned}$$

Informational cost = the total number of scalar (finite dimensional) evaluations of $(a, b, c, \eta)$, $W$, and $Z$.
If $(Z, \mathcal{G}) = (N, \mathcal{G}_1)$, then it is equal to

$$2dk_1 + M(dk_1 + k_2) + k_3 + d$$

and

$$d(1 + d')k_1 + M(dk_1 + k_2) + k_3 d' + d$$

if $(Z, \mathcal{G}) = (L, \mathcal{G}_2)$. By (1) in both cases the cost is $O(Mn)$.

## Lower error bounds and complexity-cont'd

An algorithm that approximates $X(T)$

$$\bar{X}_{M,n}(a,b,c,\eta,W,Z) = \phi_{M,n}\big(\mathcal{N}_{M,n}(a,b,c,\eta,W,Z)\big),$$

for some Borel measurable function $\phi_{M,n}$.

The error of $\bar{X}_{M,n} \in \Phi_{M,n}$ for a fixed $(a,b,c,\eta) \in \mathcal{G}$

$$e^{(p)}\Big(\bar{X}_{M,n},(a,b,c,\eta)\Big) = \Big(\mathbb{E}\|X(a,b,c,\eta)(T) - \bar{X}_{M,n}(a,b,c,\eta,W,Z)\|^p\Big)^{1/p},$$

where $(Z,\mathcal{G}) \in \{(N,\mathcal{G}_1),(L,\mathcal{G}_2)\}$.

The <u>worst-case error</u> of $\bar{X}_{M,n} \in \Phi_{M,n}$ in $\mathcal{G}$

$$e^{(p)}(\bar{X}_{M,n},\mathcal{G}) = \sup_{(a,b,c,\eta)\in\mathcal{G}} e^{(p)}\Big(\bar{X}_{M,n},(a,b,c,\eta)\Big).$$

## Lower error bounds and complexity-cont'd

For $\varepsilon \in (0, +\infty)$ we define the $\varepsilon$-complexity in $\mathcal{G} \in \{\mathcal{G}_1, \mathcal{G}_2\}$ as follows

$$\mathrm{comp}(\varepsilon, \mathcal{G}) = \inf\Big\{ nM \mid M, n \in \mathbb{N} \text{ are such that}$$
$$\exists_{\phi_{M,n}, \mathcal{N}_{M,n}} \text{ with } \max_{1 \le i \le 3} k_i = O(n), \ \max_{i=1,2} k_i = \Omega(n) \text{ and } e^{(2)}(\bar{X}_{M,n}, \mathcal{G}) \le \varepsilon,$$
$$\text{where } \bar{X}_{M,n} = \phi_{M,n} \circ \mathcal{N}_{M,n} \Big\}.$$

Only the case $p = 2$ covered, for $p > 2$ gap between upper and lower bounds can be observed.

## Lower error bounds and complexity-cont'd

**Corollary 1. ([1])**

**(i)** There exists a positive constant $K$, depending only on the parameters of the class $\mathcal{G}_1(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, such that for every $M, n \in \mathbb{N}$ and $(a, b, c, \eta) \in \mathcal{G}_1(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$ it holds

$$\|X(a, b, c, \eta)(T) - \bar{X}^{RE}_{M,n}(a, b, c, \eta, W, N)\|_{L^p(\Omega)} \leq K\left(n^{-\min\{\varrho_1, \varrho_2, 1/p\}} + \delta(M)\right).$$

**(ii)** Let $\kappa_p < +\infty$. There exists a positive constant $K$, depending only on the parameters of the class $\mathcal{G}_2(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, such that for every $M, n \in \mathbb{N}$ and $(a, b, c, \eta) \in \mathcal{G}_2(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$ it holds

$$\|X(a, b, c, \eta)(T) - \bar{X}^{RE}_{M,n}(a, b, c, \eta, W, L)\|_{L^p(\Omega)} \leq K\left(n^{-\min\{\varrho_1, \varrho_2, 1/p\}} + \delta(M)\right).$$

For the both classes the (informational) cost of $\bar{X}^{RE}_{M,n}$ is $\Theta(Mn)$.

## Lower error bounds and complexity-cont'd

**Theorem 2. ([1])**

**(i)** There exist positive constants $\hat{C}, n_0, M_0$, depending only on the parameters of the class $\mathcal{G}_1(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, such that for all $n \geq n_0, M \geq M_0$ and for every method $\bar{X}_{M,n} \in \Phi_{M,n}$ it holds

$$e^{(p)}(\bar{X}_{M,n}, \mathcal{G}_1) \geq \hat{C}(n^{-\min\{\varrho_1, \varrho_2, 1/2\}} + \delta(M)).$$

**(ii)** Let $\kappa_p < +\infty$. There exist positive constants $\hat{C}, n_0, M_0$, depending only on the parameters of the class $\mathcal{G}_2(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, such that for all $n \geq n_0, M \geq M_0$ and for every method $\bar{X}_{M,n} \in \Phi_{M,n}$ it holds

$$e^{(p)}(\bar{X}_{M,n}, \mathcal{G}_2) \geq \hat{C}(n^{-\min\{\varrho_1, \varrho_2, 1/2\}} + \delta(M)).$$

# Lower error bounds and complexity-cont'd

**Lemma 1. ([1])**

Let $Z \in \{N, L\}$. There exists $M_0 \in \mathbb{N}$, depending only on the parameters of the class $\mathcal{F}(p, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, such that for all $M \geq M_0$ and any $\sigma(\mathcal{H}_M \cup \Sigma_\infty^Z)$-measurable random vector $Y : \Omega \mapsto \mathbb{R}^d$ it holds

$$\sup_{b \in \mathcal{B}_0(C,D,L,\Delta,\varrho_1)} \mathbb{E}\|\mathcal{I}(b) - Y\|^2 \geq \sup_{b \in \mathcal{B}_0(C,D,L,\Delta,\varrho_1)} \mathbb{E}\|\mathcal{I}(b) - \mathcal{I}(b^M)\|^2 \geq C^2 T(\delta(M))^2,$$

where

$$\mathcal{B}_0(C, D, L, \Delta, \varrho_1) = \{b \in \mathcal{B}(C, D, L, \Delta, \varrho_1) \mid b(t, x) = b(t, 0) \ \text{ for } t \in [0, T], x \in \mathbb{R}^d\}$$

and

$$\mathcal{I}(b) = \int_0^T b(t, 0)\, \mathrm{d}W(t).$$

## Lower error bounds and complexity-cont'd

**Theorem 3. ([1])**

Let $\gamma = \min\{\varrho_1, \varrho_2, 1/2\}$.

**(i)** There exist positive constants $C_1, C_2, C_3, C_4, \varepsilon_0$, depending only on the parameters of the class $\mathcal{G}_1(2, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds

$$C_1(1/\varepsilon)^{1/\gamma}\delta^{-1}(\varepsilon/C_2) \leq \mathrm{comp}(\varepsilon, \mathcal{G}_1) \leq C_3(1/\varepsilon)^{1/\gamma}\delta^{-1}(\varepsilon/C_4).$$

**(ii)** Let $\kappa_2 < +\infty$. There exist positive constants $C_1, C_2, C_3, C_4, \varepsilon_0$, depending only on the parameters of the class $\mathcal{G}_2(2, C, D, L, \Delta, \varrho_1, \varrho_2, \nu)$, such that for all $\varepsilon \in (0, \varepsilon_0)$ the following holds

$$C_1(1/\varepsilon)^{1/\gamma}\delta^{-1}(\varepsilon/C_2) \leq \mathrm{comp}(\varepsilon, \mathcal{G}_2) \leq C_3(1/\varepsilon)^{1/\gamma}\delta^{-1}(\varepsilon/C_4).$$

## Lower error bounds and complexity-cont'd

**Example 1.**

If $\gamma = 1/2$, $\delta(M) = \Theta(M^{-\alpha+1/2})$, $\alpha \in [1, +\infty)$, then

$$\mathrm{comp}(\varepsilon, \mathcal{G}_i) = \Theta\left(\left(\frac{1}{\varepsilon}\right)^{\frac{4\alpha}{2\alpha-1}}\right), \ i = 1, 2.$$

Hence, if $\alpha = 1$ then the minimal cost is $O(\varepsilon^{-4})$, while in the case of finite dimensional $W$ the minimal cost is $O(\varepsilon^{-2})$.

## Numerical experiments

For details on CUDA C implementation on GPU architecture, see [1], [2].

(1) Ornstein–Uhlenbeck process with jumps

$$X(t) = \eta + \int_0^t (\mu - AX(s))\,\mathrm{d}s + \sum_{j=1}^{+\infty} \int_0^t \frac{\sigma_j}{j^\alpha} \mathrm{d}W_j(s) + \int_0^t s\mathrm{d}N(s)$$

(2) Merton model driven by compound Poisson process

$$X(t) = \eta + \int_0^t \mu X(s)\mathrm{d}s + \sum_{j=1}^{+\infty} \int_0^t \frac{\sigma_j}{j^\alpha} X(s)\mathrm{d}W_j(s) + \int_0^t X(s-)\mathrm{d}L(s)$$

## Numerical experiments-cont'd

**Example 1.-cont'd**

For $\bar{X}^{RE}_{M,n}$ the optimal (up to constants) choice of $(M, n)$ is

$$M(\varepsilon) = O\Big((1/\varepsilon)^{\frac{2}{2\alpha-1}}\Big), n(\varepsilon) = O\Big((1/\varepsilon)^2\Big).$$

Hence, we take $n = O\Big(M^{2\alpha-1}\Big)$. The error of $\bar{X}^{RE}_{M,n}$ is $O\Big((cost(\bar{X}^{RE}_{M,n}))^{\frac{1}{4\alpha}-\frac{1}{2}}\Big)$ and the slope for the $\log(error)$ vs $\log(cost)$ plot should be close to $\frac{1}{4\alpha} - \frac{1}{2}$.

# Numerical experiments-cont'd

| Simulation parameters | | |
|---|---|---|
| Model | Ornstein-Uhlenbeck with jumps | Infinite Merton model |
| T | 1,53 | |
| x0 (eta) | 1 | |
| mu | 0,08 | |
| sig | 0,4 | |
| INTENSITY | 1,21 | |
| ALPHA | N/A | 1,0 |
| ALPHA_ORN | 1,2 | N/A |

**Figure:** Specified simulation parameters for the introduced models.

# Numerical experiments-cont'd

```
#ifdef ORNSTEIN
__device__ double func_a(double t, double x) {
    return (mu - A * x);
}

__device__ double func_b(int k, double t, double x) {
    return (sig / pow((double)k, ALPHA_ORN));
    //return (1.0 / pow((double) k, (2 * varrho + 1))) * (pow(t, varrho)) * x;
}

__device__ double func_c(double t, double x, double xi) {
    return t;
}

__device__ double sigma(int j) {
    return sig;
}

__device__ double exact_solution(double t, double* w, int wiener_dim, Jump* jumps_head) {
    return 0.0;
}
#endif // ORNSTEIN
```

**Figure:** Source code representation of the input functions for Ornstein–Uhlenbeck process

# Numerical experiments-cont'd

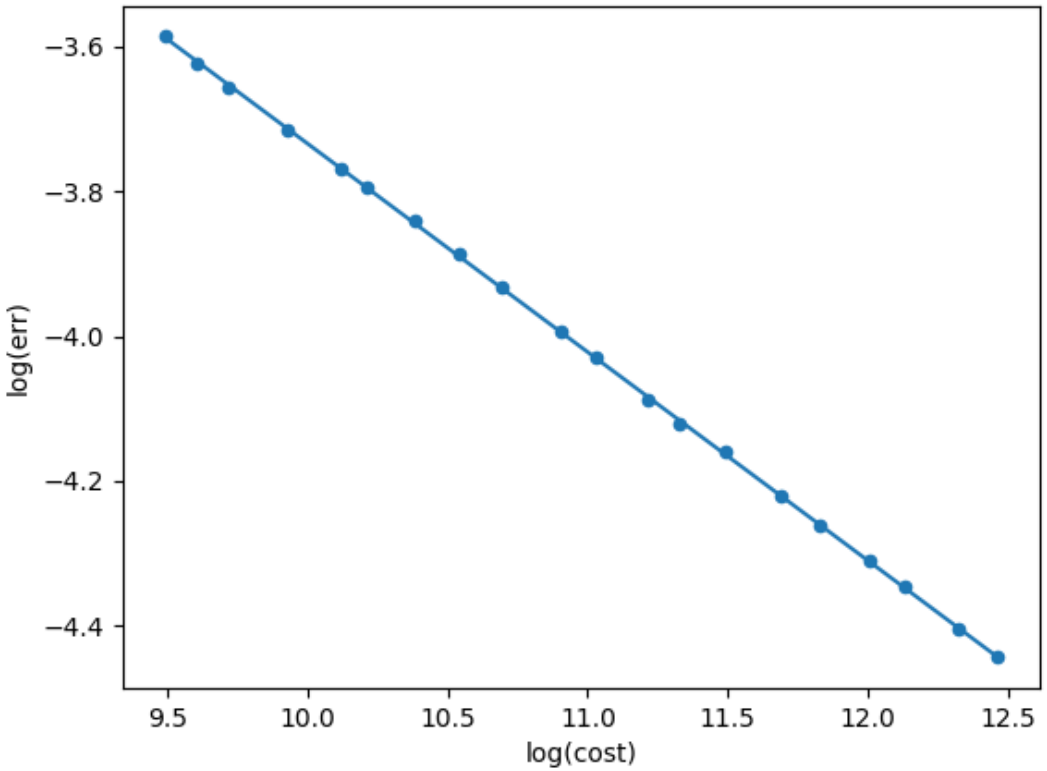


**Figure:** The log(*error*) vs log(*cost*) plot for (1).

$\alpha = 1.2$, theoretical conv. rate $= 0.292$, empirical conv. rate $= 0.288$

# Numerical experiments-cont'd

```
#ifdef INFWIENER
__device__ double func_a(double t, double x) {
    return mu * x;
}

__device__ double sigma(int j) {
    return sig;
}

__device__ double func_b(int k, double t, double x) {
    return (sigma(k) / pow((double) k, ALPHA)) * x;
    //return (1.0 / pow((double) k, (2 * varrho + 1))) * (pow(t, varrho)) * x;
}

__device__ double func_c(double t, double x, double xi) {
    return 0.0;
}

__device__ double exact_solution(double t, double* w, int wiener_dim, Jump* jumps_head) {
    double sum_j = 0.0;
    double sum_w = 0.0;
    for (int j = 1; j <= wiener_dim; j++) {
        sum_j += pow(sigma(j), 2.0) / pow((double)j, 2 * ALPHA);
        sum_w += (sigma(j) / pow((double)j, ALPHA)) * w[j-1];
    }
    return x0 * exp(mu * t - 0.5 * sum_j * t + sum_w);
}
#endif // INFWIENER
```

**Figure:** Source code representation of the input functions for Merton model

# Numerical experiments-cont'd

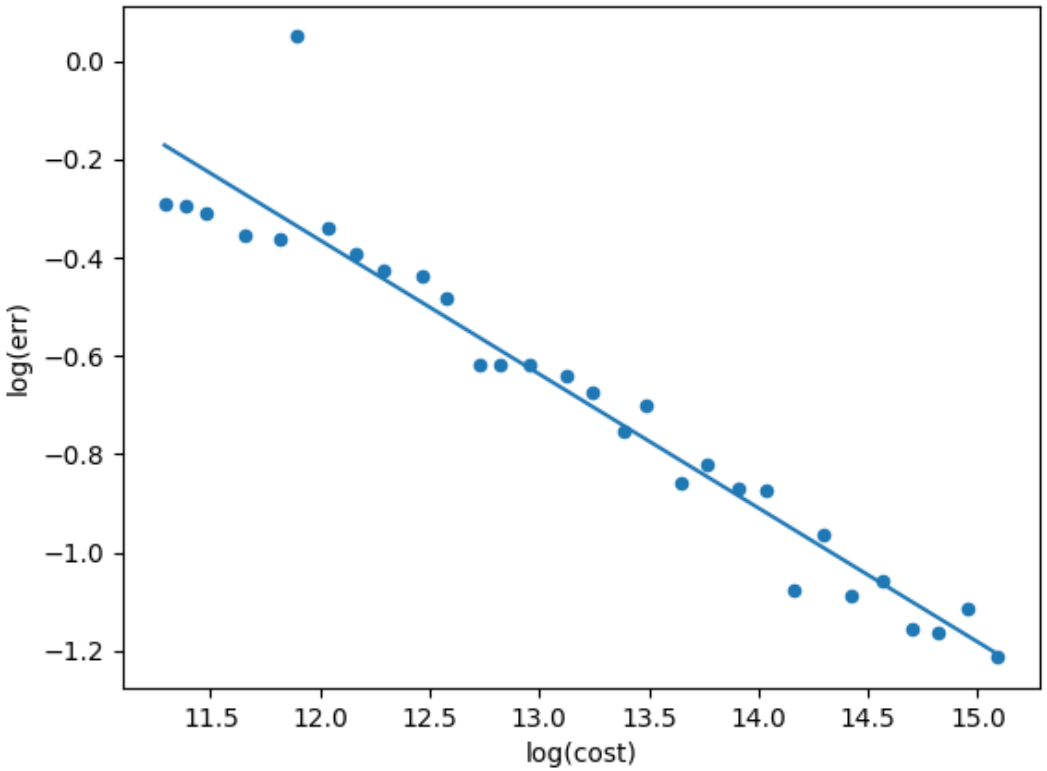


**Figure:** The log(*error*) vs log(*cost*) plot for (2).

$\alpha = 1.0$, theoretical conv. rate $= 0.25$, empirical conv. rate $= 0.272$

## Future work

- (Randomized) schemes of higher order of convergence
- Exact rate of convergence together with asymptotic constants
- Construction and analysis of suitable MLMC algorithms
- Algorithm convergence analysis under perturbed initial information
- Development of cuSTOCH library (joint AGH-NVIDIA project)

Thank you for your attention!