Martingale Representations in Progressive Enlargement by Multivariate Point Processes

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9-th International Colloquium on BSDEs and Mean Field Systems
June, 26th - July, 1st, 2022

Overview

$(\Omega, \mathcal{F}, P)$ probability space; $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ a filtration under usual conditions.

- *Goal*
  To represent any $(P, \mathbb{F})$-local martingale through stochastic integration
- *Possible representations*
  - *Strong Predictable Representation*
    Any $(P, \mathbb{F})$-local martingale can be written as a vector stochastic integral with respect a $(P, \mathbb{F})$-local martingale M.

M enjoys the $(P, \mathbb{F})$-*Strong Predictable Representation Property* (SRP)

  - *Weak Predictable Representation*
    any $(P, \mathbb{F})$-local martingale can be written as the sum of a vector stochastic integral with respect to a continuous $(P, \mathbb{F})$-local martingale X^c and an integral with respect to a compensated random measure $\mu - \nu$

When X^c and μ are the countinuous martingale part and the jump measure of a semi-martingale X then
X enjoys the $(P, \mathbb{F})$-*Weak Predictable Representation Property* (WRP)

Multivariate Point Processes

- $(E, \mathcal{E})$, E *Lusin space*; $\mathcal{E}$ its Borel σ-algebra;
- Δ extra point.
- $E_\Delta := E \cup \{\Delta\}$ $\tilde{E} := (0, +\infty) \times E$ $\tilde{E}_\Delta := \tilde{E} \cup \{(+\infty, \Delta)\}$
with $\mathcal{E}_\Delta$, $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}_\Delta$ the Borel σ-algebra of E_Δ, $\tilde{E}$ and $\tilde{E}_\Delta$.
- $\tilde{\Omega} := \Omega \times (0, +\infty) \times E$ $\tilde{\mathcal{P}}(\mathbb{F}) := \mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$,

Definition

A Multivariate Point Process (from now on MPP) is a infinite sequence of $(\tilde{E}_\Delta, \tilde{\mathcal{E}}_\Delta)$-valued r.v's $\{(T_n, X_n)\}_{n\geq 1}$ s.t.

1. *for each n, T_n is a $\mathbb{F}$-stopping time and $T_n \leq T_{n+1}$;*
2. *for each n, X_n is $\mathcal{F}_{T_n}$-measurable;*
3. *if $T_n < \infty$, then $T_n < T_{n+1}$.*

The *Explosion Time* of MPP $\{(T_n, X_n)\}_{n\geq 1}$ is the $(0, +\infty]$-valued r.v. T_∞ defined by

$$T_\infty := \lim_n T_n$$

....▶ Why does Jacod give a so abstract definition?

.... Because it allows to include in the family of MPPs also:

- Explosive jump processes ($P(T_\infty < +\infty) > 0$);
- processes with a finite number of jumps;
- processes with with jump times not necessarily finite.

Example

Occurrence process $\mathbb{1}_{[[\tau,+\infty]]}$ *of a random time* τ*:* $(T_1, X_1) = (\tau, X)$*,* $(T_n, X_n) = (+\infty, \Delta)$ *for any* $n \geq 2$*, where* $X = \mathbb{1}_{[0,+\infty)}(\tau) + \Delta \mathbb{1}_{+\infty}(\tau)$*.*

MPPs and Discrete Random Measures

Any MPP $\{(T_n, X_n)\}_{n\geq 1}$ is completely characterized by a discrete positive random measure from $(\Omega, \mathcal{F})$ to $(\tilde{E}, \tilde{\mathcal{E}})$ defined by

$$\mu(\omega; dt, dx) := \sum_{n\geq 1} \mathbb{1}_{\{T_n<\infty\}}(\omega)\delta_{(T_n(\omega), X_n(\omega))}(dt, dx),$$

Proposition

There exists a positive random measure $\nu(\omega; dt, dx)$ on $(\tilde{E}, \tilde{\mathcal{E}})$ satisfying

$$\nu(\{t\} \times E) \leq 1 \qquad \nu([T_\infty, \infty) \times E) = 0$$

such that for each $B \in \mathcal{E}$

(i) $(\nu(\omega; (0, t] \times B))_{t\geq 0}$ *is predictable ;*

(ii) $(\mu(\omega; (0, t \wedge T_n] \times B) - \nu(\omega; (0, t \wedge T_n] \times B))_{t\geq 0}$ *is a uniformly integrable martingale null at time zero for each $n \geq 1$.*

- ν is the $\mathbb{F}$-*predictable compensator* or $\mathbb{F}$-*dual predictable projection* of μ

Jacod's WRP of MPPs

- $\mathbb{X} := \{\mathcal{X}_t\}_{t\geq 0}$, $\mathcal{X}_t := \sigma\Big(\mu((0,s]\times B) : s \leq t, B \in \mathcal{E}\Big)$ *Natural Filtration* of $\{(T_n, X_n)\}_{n\geq 1}$;
- $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$, $\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{X}_t$ with $\mathcal{F}_0 \subset \mathcal{F}$.

Theorem

$Z = (Z_t)_{t\geq 0}$ $\mathbb{F}$-*adapted, càdlàg process. The following statements are equivalent*

(i) *there exists* $\{S_n\}_{n\geq 1}$ *of* $\mathbb{F}$-*stopping times,* $S_n \nearrow T_\infty$ *s.t., for any* $n \geq 1$, $Z_{t\wedge S_n}$ *is a U.I. martingale;*

(ii) *there exists a finite* $\tilde{\mathcal{P}}(\mathbb{F})$-*measurable function* W *s.t. on* $\{t < T_\infty\}$

$$\int_0^t \int_E |W(s,x)|\nu(ds,dx) < \infty \quad a.s.$$

$$Z_t = Z_0 + \int_0^t \int_E W(s,x)(\mu(ds,dx) - \nu(ds,dx)) \quad a.s.$$

....beyond the formulas:

any MPP satisfies the WRP up to T_∞ (just WRP if $T_\infty = +\infty$ *a.s.*) with respect to its initially enlarged natural filtration.

...and, when $T_\infty = +\infty$?

Corollary

When $P(T_\infty < +\infty) = 0$ *the semimartingale* $(X_t)_{t\geq 0}$ *defined by*

$$X_t := \sum_{n\geq 1} X_n \mathbb{1}_{\{T_n \leq t\}}$$

satisfies the $\mathbb{F}$*-WRP.*

Put together d MPPs: The Merging Process

- $\{(T^i_n, X^i_n)\}_{n\geq 1}$ MPP in $(\tilde{E}^i_\Delta, \tilde{\mathcal{E}}^i_\Delta)$, E^i a Lusin space, $i = 1, \dots, d$;
- $\mathcal{F}^i_t = \mathcal{F}^i_0 \vee \mathcal{X}^i_t$, $\mathcal{F}^i_0 \subset \mathcal{F}$ and $\mathbb{X}^i = (\mathcal{X}^i_t)_{t\geq 0}$ the natural filtration of $\{(T^i_n, X^i_n)\}_{n\geq 1}$ $i = 1, \dots, d$;
- $E := E^1_0 \times E^2_0 \dots \times E^d_0$, with $E^i_0 := E^i \cup \{0\}$;
- $\mathbb{G} := \{\mathcal{G}_t\}_{t\geq 0}$ with
$$\mathcal{G}_t := \cap_{s>t} \vee_{i=1}^d \mathcal{F}^i_s.$$

▶ A natural candidate for the $\mathbb{G}$-WRP: the *Merging Process*

The d-dimensional MPP $\{(T_n, V_n)\}_{n\geq 1}$ taking values in $(\tilde{E}_\Delta, \tilde{\mathcal{E}}_\Delta)$ with explosion time $T_\infty = \min\left(T^1_\infty, \dots, T^d_\infty\right)$, where

- the sequence of its jump times is obtained rearranging pointwise in nondecreasing way the set $\{T^i_n, n \geq 1, i = 1, \dots, d\}$;
- the mark at any jump time is the vector whose i-th component coincides with X^i_k on the set $T_n = T^i_k$, $i = 1, \dots, d$.

$$T_1(\omega) := \begin{cases} \inf\{\, T_1^i(\omega) : \;\; T_1^i(\omega) < +\infty, \quad i = 1, ..., d\} & \text{if } \{\ldots\} \neq \emptyset \\ +\infty & \text{otherwise} \end{cases}$$

$$\vdots$$

$$T_n(\omega) := \begin{cases} \inf\{\, T_k^i(\omega) : \;\; T_{n-1}(\omega) < T_k^i(\omega) < +\infty, \quad i = 1, ..., d, \;\; k \geq 1\} & \text{if } \{\ldots\} \\ +\infty & \text{otherwi} \end{cases}$$

$$\vdots$$

$$V_n := \begin{cases} (V_n^1, \cdots, V_n^d) & \text{if } T_n < +\infty \\ \Delta & \text{otherwise} \end{cases}$$

$$V_n^i(\omega) := \begin{cases} X_k^i(\omega) & \text{if there exists } k \geq 1 \text{ such that } T_k^i(\omega) = T_n(\omega); \\ 0 & \text{otherwise.} \end{cases}$$

$\{V_n\}_{n\geq 1}$ takes values in $E \cup \{\Delta\}$ and

$$V_n^i = \sum_{k\geq 1} X_k^i \mathbb{1}_{\{T_k^i = T_n\}} \mathbb{1}_{\{T_n < +\infty\}} + \Delta\, \mathbb{1}_{\{T_n = +\infty\}}.$$

WRP of the Merging Process

Remark

If $V_n = x$, with $x = (x_1, \ldots, x_d) \in E_0^1 \times \ldots \times E_0^d$, then

- *x cannot be $\{0, \ldots, 0\}$;*
- *if $x_i = 0$ for some (but not all!) $i = 1, \ldots, d$ then the corresponding i-th MPPs don't jump at T_n;*
- *if $x_i \neq 0$ for all $i = 1, \ldots, d$, then all the MPPs are jumping together.*

- $\mathbb{X} = (\mathcal{X}_t)_{t \geq 0}$ the natural filtration of $\{(T_n, V_n)\}_{n \geq 1}$;
- $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the right-continuous filtration defined by $\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{X}_t$, where $\mathcal{F}_0 := \vee_{i=1}^d \mathcal{F}_0^i$.

Theorem

The $\{(T_n, V_n)\}_{n \geq 1}$ satisfies the $\mathbb{F}$-WRP up to $T_\infty := \min\left(T_\infty^1, \ldots, T_\infty^d\right)$.

In the frame of Progressive Enlargement

- $\{(T^i_n, X^i_n)\}_{n\geq 1}$, $\mathbb{F}^i = \mathcal{F}^i_0 \vee \mathbb{X}^i$, $i = 1, ..., d$, $\mathcal{F}^i_0 \subset \mathcal{F}$ and $\mathbb{X}^i = (\mathcal{X}^i_t)_{t\geq 0}$ the natural filtration of the i-th MPP;
- $\mathbb{G} := \mathbb{F}^1 \vee \ldots \vee \mathbb{F}^d$

Remark

$\mathbb{F}$ does not coincide in general with $\mathbb{G}$.
If $T_\infty < +\infty$ then $\mathbb{F}$ doesn't contain the natural filtration of any MPP of the family either non explosive or with explosion time greater that T_∞.

Assumption A1

The explosion time T^i_∞ is infinite a.s., for any $i \in \{1, ..., d\}$.

Lemma

Assume A1. Then $\mathbb{F} = \mathbb{G}$.

Propagation of WRP under enlargement by MPPs

Theorem

Assume A1. Then the MPP $\{(T_n, V_n)\}_{n\geq 1}$ satisfies the $\mathbb{G}$-WRP.

▶ Summarizing

If $P(T_\infty < +\infty) = 0$ then

- $\mathbb{G} := \mathbb{F}^1 \vee \ldots \vee \mathbb{F}^d$ is the progressive enlargement of $\mathbb{F}^1$ by $\mathbb{F}^2 \vee \ldots \vee \mathbb{F}^d$;
- $\mathbb{F}^2 \vee \ldots \vee \mathbb{F}^d$ coincides with the natural filtration of the merging of $\{(T^2_n, X^2_n)\}_{n\geq 1}, \ldots, \{(T^d_n, X^d_n)\}_{n\geq 1}$ initially enlarged by $\mathcal{F}^2_0 \vee \ldots . \vee \mathcal{F}^d_0$;
- the stability of the WRP of $\{(T^1_n, X^1_n)\}_{n\geq 1}$ under progressive enlargement by $\{(T^2_n, X^2_n)\}_{n\geq 1}, \ldots, \{(T^d_n, X^d_n)\}_{n\geq 1}$ holds.

From WRP to SRP

- $\{(T^i_n, X^i_n)\}_{n\geq 1}$ MPP in $(\tilde{E}^i_\Delta, \tilde{\mathcal{E}}^i_\Delta)$, E^i a Lusin space, $i = 1, \ldots, d$;
- $T_\infty = \min\left(T^1_\infty, \ldots, T^d_\infty\right)$ where $T^i_\infty = \lim_{n\to+\infty} T^i_n$, $i = 1, \ldots, d$;
- $E := E^1_0 \times E^2_0 \ldots \times E^d_0$, with $E^i_0 := E^i \cup \{0\}$;

Assumption A2

E^i is discrete, $i \in \{1, ..., d\}$.

$$\Downarrow$$

The space $E \setminus \{0, \ldots, 0\}$ is countable: $E \setminus \{0, \ldots, 0\} = \{x_1, x_2, \ldots\}$.

- Assume A1. Then T_∞ is infinite and $\{(T_n, V_n)\}_{n\geq 1}$ enjoys the $\mathbb{G}$-WRP: for any $\mathbb{G}$-local martingale Z

$$Z_t = Z_0 + \int_0^t \int_E W(s,x)\big(\mu(ds,dx) - \nu(ds,dx)\big), \quad a.s.$$

where μ is the random measure associated to (T_n, V_n) and ν its $\mathbb{G}$-dual predictable projection.

- Assume A1, A2. Then

$$Z_t = Z_0 + \int_0^t \int_{E\setminus\{0,\dots,0\}} \sum_{h\geq 1} W(s,x)\mathbb{1}_{\{x=x_h\}}\big(\mu(ds,dx) - \nu(ds,dx)\big) =$$

$$= Z_0 + \sum_{h\geq 1}\int_0^t W(s,x_h)\big(\mu(ds,\{x_h\}) - \nu(ds,\{x_h\})\big), \text{ a.s.}$$

$$M_t^h := \mu((0,t],\{x_h\}) - \nu((0,t],\{x_h\}), \quad h \geq 1$$
$$W_t(x_h) := W(t, x_h).$$

- For any $h \geq 1$ the process $\left(M_t^h\right)_{t \geq 0}$ is a $\mathbb{G}$-local martingale null at 0;
- for any $h \geq 1$ the process $(W_t(x_h))_{t \geq 0}$ is a $\mathbb{G}$-predictable process;
- $Z_t = Z_0 + \sum_{h \geq 1} \int_0^t W_s(x_h) dM_s^h, \quad a.s.$

Theorem

Set $M := (M^1, \ldots, M^h, \ldots)$. Then M enjoys the $\mathbb{G}$-SRP.

A sufficient condition for the orthogonality

Lemma

Let X be a $\mathbb{G}$-adapted pure jump process of locally integrable variation and let X^p be its $\mathbb{G}$-dual predictable projection. Then, for any fixed stopping time S, $\Delta X_S = 0$, a.s., implies $\Delta X^p_S = 0$, a.s.

Let X and Y be two general $\mathbb{G}$-adapted locally integrable pure jump processes.

Assumption A3:Mutual Avoiding Predictable Jump Times

$P(\Delta X_\sigma \neq 0) > 0$ implies $\Delta Y_\sigma = 0$, a.s., for any finite $\mathbb{G}$-predictable stopping time σ.

Proposition

Let X and Y be $\mathbb{G}$-adapted pure jump processes of locally integrable variation which verify A3. Then for any finite $\mathbb{G}$-predictable stopping time σ

$$\Delta X^p_\sigma \Delta Y^p_\sigma = \Delta X_\sigma \Delta Y^p_\sigma = \Delta X^p_\sigma \Delta Y_\sigma = 0, \quad a.s.$$

$$M^h = N^h - N^{h,p},\ h \geq 1,$$

where

- $N_t^h := \mu((0,t], \{x_h\}), \quad h \geq 1;$
- $N^{h,p}$ the $\mathbb{G}$-dual predictable projection of N^h, $\quad h \geq 1$.

Theorem

Let Assumption A3 be in force for all pair N^h, N^k with $h \neq k$. Then M is a $\mathbb{G}$-basis.

Proof.

- $\mathbb{G}$ is strongly represented by M;
- $[M^h, M^k] = [N^h, N^k] - [N^h, N^{k,p}] - [N^{h,p}, N^k] + [N^{h,p}, N^{k,p}]$;
- N^h, $N^{h,p}$, $h \geq 1$ are bounded variation processes then the quadratic covariations coincide with the sum of common jumps;
- N^h and N^k with $h \neq k$ do not jump together and A3 yields $[M^h, M^k] = 0$.

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Thank you!