Martingale Representations in Progressive Enlargement by Multivariate Point Processes

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9-th International Colloquium on BSDEs and Mean Field Systems
June, 26th - July, 1st, 2022
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Overview

(\Omega, \mathcal{F}, P) probability space; \mathbb{F} = (\mathcal{F}_t)_{t \geq 0} a filtration under usual conditions.

- **Goal**
  To represent any \((P, \mathbb{F})\)-local martingale through stochastic integration

- **Possible representations**
  - **Strong Predictable Representation**
    Any \((P, \mathbb{F})\)-local martingale can be written as a vector stochastic integral with respect a \((P, \mathbb{F})\)-local martingale \(M\).

  \(M\) enjoys the \((P, \mathbb{F})\)-Strong Predictable Representation Property (SRP)

  - **Weak Predictable Representation**
    Any \((P, \mathbb{F})\)-local martingale can be written as the sum of a vector stochastic integral with respect to a continuous \((P, \mathbb{F})\)-local martingale \(X^c\) and an integral with respect to a compensated random measure \(\mu - \nu\)

  When \(X^c\) and \(\mu\) are the countinuous martingale part and the jump measure of a semi-martingale \(X\) then \(X\) enjoys the \((P, \mathbb{F})\)-Weak Predictable Representation Property (WRP)
Multivariate Point Processes

- $(E, \mathcal{E})$, $E$ Lusin space; $\mathcal{E}$ its Borel $\sigma$-algebra;
- $\Delta$ extra point.
- $E_\Delta := E \cup \{\Delta\}$  \(\tilde{E} := (0, +\infty) \times E\)  \(\tilde{E}_\Delta := \tilde{E} \cup \{(+\infty, \Delta)\}\)
  with $\mathcal{E}_\Delta$, $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}_\Delta$ the Borel $\sigma$-algebra of $E_\Delta$, $\tilde{E}$ and $\tilde{E}_\Delta$.
- $\tilde{\Omega} := \Omega \times (0, +\infty) \times E$  \(\tilde{\mathcal{P}}(\mathbb{F}) := \mathcal{P}(\mathbb{F}) \otimes \mathcal{E}\),

**Definition**

A Multivariate Point Process (from now on MPP) is a infinite sequence of $(\tilde{E}_\Delta, \tilde{\mathcal{E}}_\Delta)$-valued r.v’s $\{(T_n, X_n)\}_{n \geq 1}$ s.t.

1. for each $n$, $T_n$ is a $\mathbb{F}$-stopping time and $T_n \leq T_{n+1}$;
2. for each $n$, $X_n$ is $\mathcal{F}_{T_n}$-measurable;
3. if $T_n < \infty$, then $T_n < T_{n+1}$.

The Explosion Time of MPP $\{(T_n, X_n)\}_{n \geq 1}$ is the $(0, +\infty]$-valued r.v. $T_\infty$ defined by

$$T_\infty := \lim_{n} T_n$$
Why does Jacod give a so abstract definition?

Because it allows to include in the family of MPPs also:
- Explosive jump processes \( P(T_\infty < +\infty) > 0 \);
- processes with a finite number of jumps;
- processes with jump times not necessarily finite.

Example

Occurrence process \( 1_{[[\tau, +\infty]]} \) of a random time \( \tau \): \( (T_1, X_1) = (\tau, X) \), \( (T_n, X_n) = (+\infty, \Delta) \) for any \( n \geq 2 \), where \( X = 1_{[0, +\infty)}(\tau) + \Delta 1_{+\infty}(\tau) \).
Any MPP \( \{(T_n, X_n)\}_{n \geq 1} \) is completely characterized by a discrete positive random measure from \((\Omega, \mathcal{F})\) to \((\tilde{\mathcal{E}}, \tilde{\mathcal{E}})\) defined by

\[
\mu(\omega; dt, dx) := \sum_{n \geq 1} 1\{T_n < \infty\}(\omega) \delta(T_n(\omega), X_n(\omega))(dt, dx),
\]

**Proposition**

There exists a positive random measure \(\nu(\omega; dt, dx)\) on \((\tilde{\mathcal{E}}, \tilde{\mathcal{E}})\) satisfying

\[
\nu(\{t\} \times E) \leq 1 \quad \nu([T_\infty, \infty) \times E) = 0
\]

such that for each \(B \in \mathcal{E}\)

\[
(i) \quad (\nu(\omega; (0, t] \times B))_{t \geq 0} \text{ is predictable ;}
(ii) \quad (\mu(\omega; (0, t \wedge T_n] \times B) - \nu(\omega; (0, t \wedge T_n] \times B))_{t \geq 0} \text{ is a uniformly integrable martingale null at time zero for each } n \geq 1.
\]

\(\nu\) is the \(\mathbb{F}\)-predictable compensator or \(\mathbb{F}\)-dual predictable projection of \(\mu\)
Jacod’s WRP of MPPs

- $\mathbb{X} := \{\mathcal{X}_t\}_{t \geq 0}$, $\mathcal{X}_t := \sigma\left(\mu((0, s] \times B) : s \leq t, B \in \mathcal{E}\right)$ Natural Filtration of $\{(T_n, X_n)\}_{n \geq 1}$;
- $\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}$, $\mathcal{F}_t := \mathcal{F}_0 \vee \mathcal{X}_t$ with $\mathcal{F}_0 \subset \mathcal{F}$.

**Theorem**

$Z = (Z_t)_{t \geq 0}$ $\mathbb{F}$-adapted, càdlàg process. The following statements are equivalent

(i) there exists $\{S_n\}_{n \geq 1}$ of $\mathbb{F}$-stopping times, $S_n \nearrow T_\infty$ s.t., for any $n \geq 1$, $Z_{t \wedge S_n}$ is a U.I. martingale;

(ii) there exists a finite $\tilde{\mathbb{P}}(\mathbb{F})$-measurable function $W$ s.t. on $\{t < T_\infty\}$

\[
\int_0^t \int_E |W(s, x)| \nu(ds, dx) < \infty \text{ a.s.}
\]

\[
Z_t = Z_0 + \int_0^t \int_E W(s, x)(\mu(ds, dx) - \nu(ds, dx)) \text{ a.s.}
\]
any MPP satisfies the WRP up to $T_\infty$ (just WRP if $T_\infty = +\infty$ a.s.) with respect to its initially enlarged natural filtration.

...and, when $T_\infty = +\infty$?

**Corollary**

When $P(T_\infty < +\infty) = 0$ the semimartingale $(X_t)_{t \geq 0}$ defined by

$$X_t := \sum_{n \geq 1} X_n 1\{T_n \leq t\}$$

satisfies the $\mathbb{F}$-WRP.
Put together $d$ MPPs: The Merging Process

- $\{(T_n^i, X_n^i)\}_{n \geq 1}$ MPP in $(\tilde{E}_\Delta^i, \tilde{E}_\Delta^i)$, $E^i$ a Lusin space, $i = 1, \ldots, d$;
- $\mathcal{F}_t^i = \mathcal{F}_0^i \lor X_t^i$, $\mathcal{F}_0^i \subset \mathcal{F}$ and $X^i = (X_t^i)_{t \geq 0}$ the natural filtration of $\{(T_n^i, X_n^i)\}_{n \geq 1}$ $i = 1, \ldots, d$;
- $E := E_0^1 \times E_0^2 \ldots \times E_0^d$, with $E_0^i := E^i \cup \{0\}$;
- $\mathcal{G} := \{G_t\}_{t \geq 0}$ with $G_t := \cap_{s > t} \lor_{i=1}^d \mathcal{F}_s^i$.

▶ A natural candidate for the $\mathcal{G}$-WRP: the **Merging Process**

The $d$-dimensional MPP $\{(T_n, V_n)\}_{n \geq 1}$ taking values in $(\tilde{E}_\Delta, \tilde{E}_\Delta)$ with explosion time $T_\infty = \min(T_\infty^1, \ldots, T_\infty^d)$, where

- the sequence of its jump times is obtained rearranging pointwise in nondecreasing way the set $\{T_n^i, n \geq 1, i = 1, \ldots, d\}$;
- the mark at any jump time is the vector whose $i$-th component coincides with $X_k^i$ on the set $T_n = T_k^i$, $i = 1, \ldots, d$. 
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Overview

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From WRP to SRP

Looking for a basis: a sufficient condition

\[ T_1(\omega) := \begin{cases} \inf \{ T_i^1(\omega) : T_1^i(\omega) < +\infty, \; i = 1, \ldots, d \} & \text{if } \{ \ldots \} \neq \emptyset \\ +\infty & \text{otherwise} \end{cases} \]

\[ \vdots \]

\[ T_n(\omega) := \begin{cases} \inf \{ T_k^i(\omega) : T_{n-1}(\omega) < T_k^i(\omega) < +\infty, \; i = 1, \ldots, d, \; k \geq 1 \} & \text{if } \{ \ldots \} \neq \emptyset \\ +\infty & \text{otherwise} \end{cases} \]

\[ \vdots \]

\[ V_n := \begin{cases} (V_1^n, \ldots, V_d^n) & \text{if } T_n < +\infty \\ \Delta & \text{otherwise} \end{cases} \]

\[ V_n^i(\omega) := \begin{cases} X_k^i(\omega) & \text{if there exists } k \geq 1 \text{ such that } T_k^i(\omega) = T_n(\omega); \\ 0 & \text{otherwise}. \end{cases} \]

\{V_n\}_{n \geq 1} \text{ takes values in } E \cup \{\Delta\} \text{ and}

\[ V_n^i = \sum_{k \geq 1} X_k^i \mathbb{1}_{\{T_k^i=T_n\}} \mathbb{1}_{\{T_n<+\infty\}} + \Delta \mathbb{1}_{\{T_n=+\infty\}}. \]
**Remark**

If $V_n = x$, with $x = (x_1, \ldots, x_d) \in E_0^1 \times \ldots \times E_0^d$, then

- $x$ cannot be $\{0, \ldots, 0\}$;
- if $x_i = 0$ for some (but not all!) $i = 1, \ldots, d$ then the corresponding $i$-th MPPs don't jump at $T_n$;
- if $x_i \neq 0$ for all $i = 1, \ldots, d$, then all the MPPs are jumping together.

- $X = (X_t)_{t \geq 0}$ the natural filtration of $\{(T_n, V_n)\}_{n \geq 1}$;
- $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the right-continuous filtration defined by $\mathcal{F}_t := \mathcal{F}_0 \vee X_t$, where $\mathcal{F}_0 := \vee_{i=1}^d \mathcal{F}_0^i$.

**Theorem**

The $\{(T_n, V_n)\}_{n \geq 1}$ satisfies the $\mathbb{F}$-WRP up to $T_\infty := \min\left(T_1^\infty, \ldots, T_d^\infty\right)$.
In the frame of Progressive Enlargement

- \{(T_n, X_n^i)\}_{n \geq 1}, \quad F^i = \mathcal{F}_0^i \vee X^i, \quad i = 1, \ldots, d,
- \mathcal{F}_0^i \subset \mathcal{F} \quad \text{and} \quad X^i = (X^i_t)_{t \geq 0} \quad \text{the natural filtration of the i-th MPP};
- \mathcal{G} := F^1 \vee \ldots \vee F^d

**Remark**

\[ F \text{ does not coincide in general with } \mathcal{G}. \]

*If \( T_\infty < +\infty \) then \( F \) doesn’t contain the natural filtration of any MPP of the family either non explosive or with explosion time greater that \( T_\infty \).*

**Assumption A1**

*The explosion time \( T^i_\infty \) is infinite a.s., for any \( i \in \{1, \ldots, d\} \).*

**Lemma**

Assume A1. Then \( F = \mathcal{G} \).


Propagation of WRP under enlargement by MPPs

\[\text{Theorem}\]

Assume $A1$. Then the MPP \(\{(T_n, V_n)\}_{n \geq 1}\) satisfies the $G$-WRP.

\[\begin{align*}
\textbf{Summarizing } \\
\text{If } P(T_\infty < +\infty) = 0 \text{ then } \\
\begin{align*}
&\textbullet \ G := F^1 \lor \ldots \lor F^d \text{ is the progressive enlargement of } F^1 \text{ by } F^2 \lor \ldots \lor F^d; \\
&\textbullet \ F^2 \lor \ldots \lor F^d \text{ coincides with the natural filtration of the merging of } \\
&\{(T^2_n, X^2_n)\}_{n \geq 1}, \ldots , \{(T^d_n, X^d_n)\}_{n \geq 1} \text{ initially enlarged by } \mathcal{F}^2_0 \lor \ldots \lor \mathcal{F}^d_0; \\
&\textbullet \text{ the stability of the WRP of } \{(T^1_n, X^1_n)\}_{n \geq 1} \text{ under progressive enlargement by } \\
&\{(T^2_n, X^2_n)\}_{n \geq 1}, \ldots , \{(T^d_n, X^d_n)\}_{n \geq 1} \text{ holds.}
\end{align*}\]
From WRP to SRP

- \( \{(T_n^i, X_n^i)\}_{n \geq 1} \) MPP in \((\tilde{E}_\Delta^i, \tilde{\mathcal{E}}_\Delta^i), E^i\) a Lusin space, \(i = 1, \ldots, d\);
- \( T_\infty = \min\left( T_1^\infty, \ldots, T_d^\infty \right) \) where \( T_i^\infty = \lim_{n \to +\infty} T_n^i, i = 1, \ldots, d \);
- \( E := E_1^0 \times E_2^0 \ldots \times E_d^0 \), with \( E_0^i := E^i \cup \{0\} \);

**Assumption A2**

\( E^i \) is discrete, \(i \in \{1, \ldots, d\}\).

The space \( E \setminus \{0, \ldots, 0\} \) is countable: \( E \setminus \{0, \ldots, 0\} = \{x_1, x_2, \ldots\} \).
**Assume A1.** Then $T_\infty$ is infinite and $\{(T_n, V_n)\}_{n \geq 1}$ enjoys the $\mathbb{G}$-WRP: for any $\mathbb{G}$-local martingale $Z$

$$Z_t = Z_0 + \int_0^t \int_E W(s, x) (\mu(ds, dx) - \nu(ds, dx)), \quad a.s.$$ 

where $\mu$ is the random measure associated to $(T_n, V_n)$ and $\nu$ its $\mathbb{G}$-dual predictable projection.

**Assume A1, A2.** Then

$$Z_t = Z_0 + \int_0^t \int_{E \setminus \{0, \ldots, 0\}} \sum_{h \geq 1} W(s, x) \mathbb{1}_{\{x = x_h\}} (\mu(ds, dx) - \nu(ds, dx)) =$$

$$= Z_0 + \sum_{h \geq 1} \int_0^t W(s, x_h) (\mu(ds, \{x_h\}) - \nu(ds, \{x_h\})), \quad a.s.$$
\[ \begin{align*}
M_t^h & := \mu((0, t], \{x_h\}) - \nu((0, t], \{x_h\}), \quad h \geq 1 \\
W_t(x_h) & := W(t, x_h).
\end{align*} \]

- For any \( h \geq 1 \) the process \( (M_t^h)_{t \geq 0} \) is a \( \mathcal{G} \)-local martingale null at 0;
- for any \( h \geq 1 \) the process \( (W_t(x_h))_{t \geq 0} \) is a \( \mathcal{G} \)-predictable process;
- \[ Z_t = Z_0 + \sum_{h \geq 1} \int_0^t W_s(x_h) dM_s^h, \quad a.s. \]

**Theorem**

Set \( M := (M^1, \ldots, M^h, \ldots) \). Then \( M \) enjoys the \( \mathcal{G} \)-SRP.
A sufficient condition for the orthogonality

**Lemma**

Let $X$ be a $\mathcal{G}$-adapted pure jump process of locally integrable variation and let $X^p$ be its $\mathcal{G}$-dual predictable projection. Then, for any fixed stopping time $S$, $\Delta X_S = 0$, a.s., implies $\Delta X^p_S = 0$, a.s.

Let $X$ and $Y$ be two general $\mathcal{G}$-adapted locally integrable pure jump processes.

**Assumption A3:** Mutual Avoiding Predictable Jump Times

$P(\Delta X_\sigma \neq 0) > 0$ implies $\Delta Y_\sigma = 0$, a.s., for any finite $\mathcal{G}$-predictable stopping time $\sigma$.

**Proposition**

Let $X$ and $Y$ be $\mathcal{G}$-adapted pure jump processes of locally integrable variation which verify A3. Then for any finite $\mathcal{G}$-predictable stopping time $\sigma$

$$\Delta X^p_\sigma \Delta Y^p_\sigma = \Delta X_\sigma \Delta Y^p_\sigma = \Delta X^p_\sigma \Delta Y_\sigma = 0, \text{ a.s.}$$
$M^h = N^h - N^{h,p}, \ h \geq 1$,

where

- $N^h_t := \mu((0, t], \{x_h\}), \ h \geq 1$;
- $N^{h,p}$ the $\mathcal{G}$-dual predictable projection of $N^h, \ h \geq 1$.

**Theorem**

Let Assumption A3 be in force for all pair $N^h, N^k$ with $h \neq k$. Then $M$ is a $\mathcal{G}$-basis.

**Proof.**

- $\mathcal{G}$ is strongly represented by $M$;
- $[M^h, M^k] = [N^h, N^k] - [N^h, N^{k,p}] - [N^{h,p}, N^k] + [N^{h,p}, N^{k,p}]$;
- $N^h, N^{h,p}, \ h \geq 1$ are bounded variation processes then the quadratic covariations coincide with the sum of common jumps;
- $N^h$ and $N^k$ with $h \neq k$ do not jump together and A3 yields $[M^h, M^k] = 0$. 

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Thank you!