Optimal control of mean field equations with monotone coefficients and applications in neuroscience

Alexander Vogler

Technische Universität Berlin (TUB)

June 30 2022, Annecy

E-Mail contact: `vogler@math.tu-berlin.de`

Plan

Plan

FitzHugh-Nagumo network of N neurons

State at time t of each neuron i, described by (Baladron, Fasoli, Faugeras [1]):

$$\begin{cases} dV_t^i = \overbrace{\Big(V_t^i - \frac{(V_t^i)^3}{3} - w_t^i + \alpha_t\Big)dt + \sigma_{\text{ext}} dW_t^i}^{\text{local dynamics}} \\ \qquad \underbrace{-\frac{1}{N}\sum_{j=1}^N (V_t^i - V_{\text{rev}}) y_t^j dt - \frac{1}{N}\sum_{j=1}^N (V_t^i - V_{\text{rev}}) y_t^j dB_t^i}_{= -\widetilde{\mathbb{E}}^{\mu_N}[h(X^i,\widetilde{X}^j)dt + \beta(X^i,\widetilde{X}^j)dB^i] \ ,\mu_N = \text{empirical dist.}} \\ dw_t^i = (V_t^i + a - bw_t^i)dt \ , \\ dy_t^i = (a_r S(V_t^i)(1 - y_t^i) - a_d y_t^i)dt, \qquad \text{for } i = 1, \dots, N, \end{cases}$$

- V^i membrane potential of neuron i
- y^i fraction of open channels (synaptic channels).
- B^i, W^i i.i.d. Brownian motions;
- V_{rev} synaptic reversal potential;
- $S(V)$ = concentration of transmitter released into synaptic cleft;

FitzHugh-Nagumo network of N neurons

State at time t of each neuron i, described by (Baladron, Fasoli, Faugeras [1]):

$$\begin{cases} dV_t^i = \overbrace{\Big(V_t^i - \frac{(V_t^i)^3}{3} - w_t^i + \alpha_t\Big)dt + \sigma_{\text{ext}} dW_t^i}^{\text{local dynamics}} \\ \qquad \underbrace{-\frac{1}{N}\sum_{j=1}^N (V_t^i - V_{\text{rev}}) y_t^j dt - \frac{1}{N}\sum_{j=1}^N (V_t^i - V_{\text{rev}}) y_t^j dB_t^i}_{= -\widetilde{\mathbb{E}}^{\mu_N}[h(X^i,\widetilde{X}^j)dt + \beta(X^i,\widetilde{X}^j)dB^i] \ ,\mu_N = \text{empirical dist.}} \\ dw_t^i = (V_t^i + a - bw_t^i)dt \ , \\ dy_t^i = (a_r S(V_t^i)(1 - y_t^i) - a_d y_t^i)dt, \qquad \text{for } i = 1, \dots, N, \end{cases}$$

- V^i membrane potential of neuron i
- y^i fraction of open channels (synaptic channels).
- B^i, W^i i.i.d. Brownian motions;
- V_{rev} synaptic reversal potential;
- $S(V)$ = concentration of transmitter released into synaptic cleft;

FitzHugh-Nagumo network of N neurons

State at time t of each neuron i, described by (Baladron, Fasoli, Faugeras [1]):

$$\begin{cases} dV_t^i = \overbrace{\Big(V_t^i - \frac{(V_t^i)^3}{3} - w_t^i + \alpha_t\Big)dt + \sigma_{\text{ext}} dW_t^i}^{\text{local dynamics}} \\ \qquad \underbrace{-\frac{1}{N}\sum_{j=1}^N (V_t^i - V_{\text{rev}}) y_t^j dt - \frac{1}{N}\sum_{j=1}^N (V_t^i - V_{\text{rev}}) y_t^j dB_t^i}_{= -\widetilde{\mathbb{E}}^{\mu_N}[h(X^i,\widetilde{X}^j)dt + \beta(X^i,\widetilde{X}^j)dB^i] \ ,\mu_N = \text{empirical dist.}} \\ dw_t^i = (V_t^i + a - bw_t^i)dt \ , \\ dy_t^i = (a_r S(V_t^i)(1 - y_t^i) - a_d y_t^i)dt, \qquad \text{for } i = 1, \dots, N, \end{cases}$$

- V^i membrane potential of neuron i
- y^i fraction of open channels (synaptic channels).
- B^i, W^i i.i.d. Brownian motions;
- V_{rev} synaptic reversal potential;
- $S(V) =$ concentration of transmitter released into synaptic cleft;

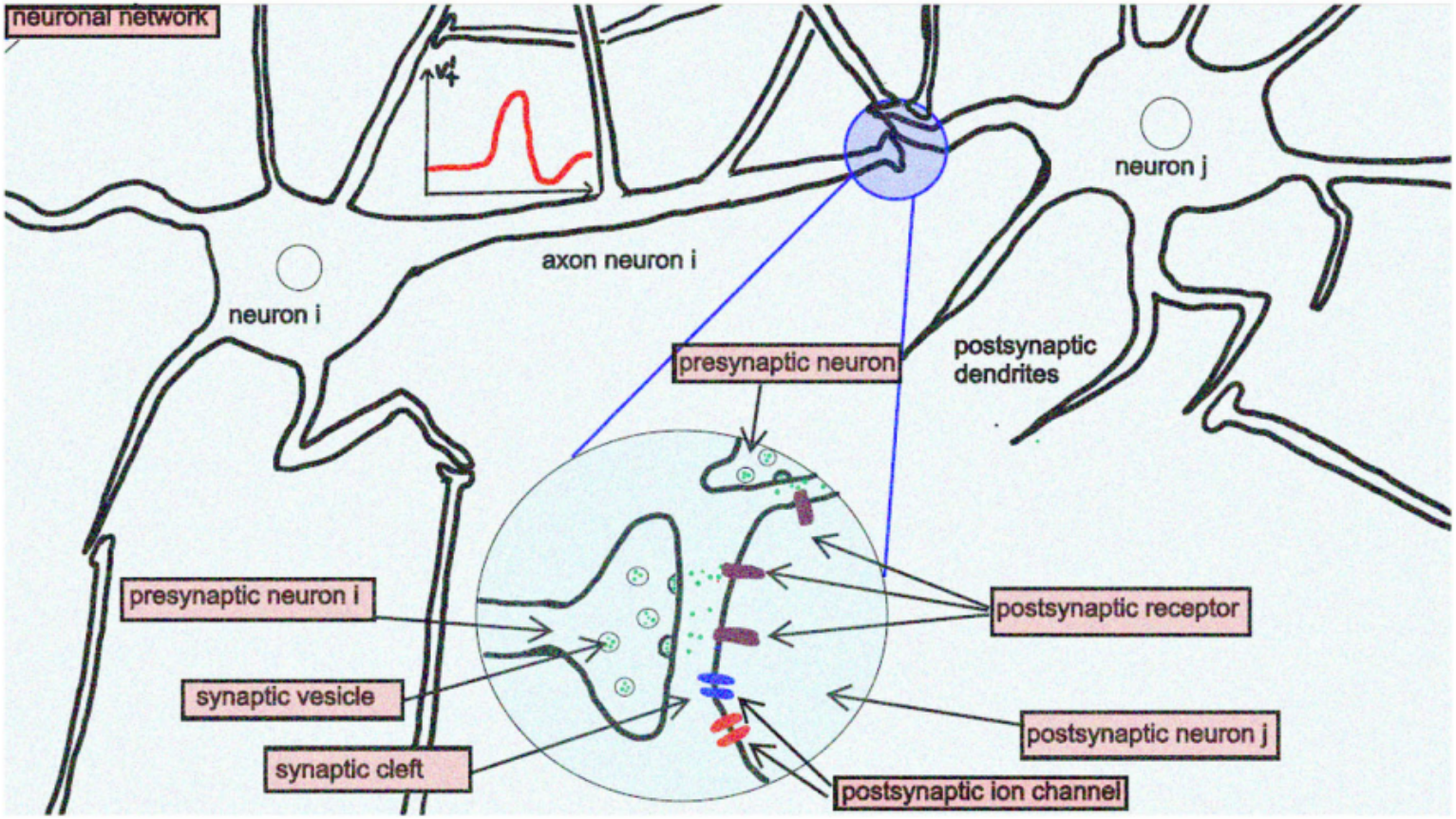

Figure: Synaptic dynamics

Abstract formulation

Propagation of chaos (Bossy, Talay, Faugeras [2]) leads to the controlled mean-field equation of the form

$$dX = b(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(t, X_t, \mathcal{L}(X_t), \alpha_t)dW_t, \quad \text{with convex level set constraint } \pi(X_t) \leq 0, \ \forall t, \tag{1}$$

where

- $X_t = (v_t, w_t, y_t)$;
- W is a Brownian Motion;
- $\pi : \mathbb{R}^3 \to \mathbb{R}, (v, w, y) \mapsto y(y-1)$, *in particular* $\pi(X_t) \leq 0$ *is coherent with the intuition that* y_t *is a fraction of open channels*;
- α_t is a *deterministic control*, e.g. modelling external current.

Goal: Minimize the cost functional

$$J(\alpha) := \mathbb{E}\Big[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt + g(X_T^\alpha, \mathcal{L}(X_T^\alpha))\Big], \tag{2}$$

where $\alpha \in \mathbb{A} := \{\alpha : [0, T] \to A\}$, for $A \subset \mathbb{R}^k$ bounded and convex.

Abstract formulation

Propagation of chaos (Bossy, Talay, Faugeras [2]) leads to the controlled mean-field equation of the form

$$dX = b(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(t, X_t, \mathcal{L}(X_t), \alpha_t)dW_t, \quad \text{with convex level set constraint } \pi(X_t) \leq 0, \ \forall t, \tag{1}$$

where

- $X_t = (v_t, w_t, y_t)$;
- W is a Brownian Motion;
- $\pi : \mathbb{R}^3 \to \mathbb{R}, (v, w, y) \mapsto y(y-1)$, *in particular* $\pi(X_t) \leq 0$ *is coherent with the intuition that* y_t *is a fraction of open channels*;
- α_t is a *deterministic control*, e.g. modelling external current.

Goal: Minimize the cost functional

$$J(\alpha) := \mathbb{E}\Big[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt + g(X_T^\alpha, \mathcal{L}(X_T^\alpha))\Big], \tag{2}$$

where $\alpha \in \mathbb{A} := \{\alpha : [0, T] \to A\}$, for $A \subset \mathbb{R}^k$ bounded and convex.

Abstract formulation

Propagation of chaos (Bossy, Talay, Faugeras [2]) leads to the controlled mean-field equation of the form

$$dX = b(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(t, X_t, \mathcal{L}(X_t), \alpha_t)dW_t, \quad \text{with convex level set constraint } \pi(X_t) \leq 0, \ \forall t, \tag{1}$$

where

- $X_t = (v_t, w_t, y_t)$;
- W is a Brownian Motion;
- $\pi : \mathbb{R}^3 \to \mathbb{R}, (v, w, y) \mapsto y(y-1)$, *in particular* $\pi(X_t) \leq 0$ *is coherent with the intuition that* y_t *is a fraction of open channels*;
- α_t is a *deterministic control*, e.g. modelling external current.

Goal: Minimize the cost functional

$$J(\alpha) := \mathbb{E}\Big[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt + g(X_T^\alpha, \mathcal{L}(X_T^\alpha))\Big], \tag{2}$$

where $\alpha \in \mathbb{A} := \{\alpha : [0, T] \to A\}$, for $A \subset \mathbb{R}^k$ bounded and convex.

Abstract formulation

Propagation of chaos (Bossy, Talay, Faugeras [2]) leads to the controlled mean-field equation of the form

$$dX = b(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(t, X_t, \mathcal{L}(X_t), \alpha_t)dW_t, \quad \text{with convex level set constraint } \pi(X_t) \leq 0, \ \forall t, \tag{1}$$

where

- $X_t = (v_t, w_t, y_t)$;
- W is a Brownian Motion;
- $\pi : \mathbb{R}^3 \to \mathbb{R}, (v, w, y) \mapsto y(y-1)$, *in particular* $\pi(X_t) \leq 0$ *is coherent with the intuition that* y_t *is a fraction of open channels*;
- α_t is a *deterministic control*, e.g. modelling external current.

Goal: Minimize the cost functional

$$J(\alpha) := \mathbb{E}\Big[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt + g(X_T^\alpha, \mathcal{L}(X_T^\alpha))\Big], \tag{2}$$

where $\alpha \in \mathbb{A} := \{\alpha : [0, T] \to A\}$, for $A \subset \mathbb{R}^k$ bounded and convex.

Abstract formulation

Propagation of chaos (Bossy, Talay, Faugeras [2]) leads to the controlled mean-field equation of the form

$$dX = b(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(t, X_t, \mathcal{L}(X_t), \alpha_t)dW_t, \quad \text{with convex level set constraint } \pi(X_t) \leq 0, \ \forall t, \tag{1}$$

where

- $X_t = (v_t, w_t, y_t)$;
- W is a Brownian Motion;
- $\pi : \mathbb{R}^3 \to \mathbb{R}, (v, w, y) \mapsto y(y-1)$, *in particular* $\pi(X_t) \leq 0$ *is coherent with the intuition that* y_t *is a fraction of open channels*;
- α_t is a *deterministic control*, e.g. modelling external current.

Goal: Minimize the cost functional

$$J(\alpha) := \mathbb{E}\Big[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt + g(X_T^\alpha, \mathcal{L}(X_T^\alpha))\Big], \tag{2}$$

where $\alpha \in \mathbb{A} := \{\alpha : [0, T] \to A\}$, for $A \subset \mathbb{R}^k$ bounded and convex.

Abstract formulation

Propagation of chaos (Bossy, Talay, Faugeras [2]) leads to the controlled mean-field equation of the form

$$dX = b(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(t, X_t, \mathcal{L}(X_t), \alpha_t)dW_t, \quad \text{with convex level set constraint } \pi(X_t) \leq 0, \ \forall t, \tag{1}$$

where

- $X_t = (v_t, w_t, y_t)$;
- W is a Brownian Motion;
- $\pi : \mathbb{R}^3 \to \mathbb{R}, (v, w, y) \mapsto y(y-1)$, *in particular* $\pi(X_t) \leq 0$ *is coherent with the intuition that* y_t *is a fraction of open channels*;
- α_t is a *deterministic control*, e.g. modelling external current.

Goal: Minimize the cost functional

$$J(\alpha) := \mathbb{E}\Big[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt + g(X_T^\alpha, \mathcal{L}(X_T^\alpha))\Big], \tag{2}$$

where $\alpha \in \mathbb{A} := \{\alpha : [0, T] \to A\}$, for $A \subset \mathbb{R}^k$ bounded and convex.

Abstract formulation

Propagation of chaos (Bossy, Talay, Faugeras [2]) leads to the controlled mean-field equation of the form

$$\begin{aligned} dX &= b(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(t, X_t, \mathcal{L}(X_t), \alpha_t)dW_t, \\ &\text{with convex level set constraint } \pi(X_t) \leq 0, \ \forall t, \end{aligned} \tag{1}$$

where

- $X_t = (v_t, w_t, y_t)$;
- W is a Brownian Motion;
- $\pi : \mathbb{R}^3 \to \mathbb{R}, (v, w, y) \mapsto y(y-1)$, *in particular* $\pi(X_t) \leq 0$ *is coherent with the intuition that* y_t *is a fraction of open channels*;
- α_t is a *deterministic control*, e.g. modelling external current.

Goal: Minimize the cost functional

$$J(\alpha) := \mathbb{E}\Big[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt + g(X_T^\alpha, \mathcal{L}(X_T^\alpha))\Big], \tag{2}$$

where $\alpha \in \mathbb{A} := \{\alpha : [0, T] \to A\}$, for $A \subset \mathbb{R}^k$ bounded and convex.

Example

Particular example: We consider the following control problem (SM)

$$\min_{\alpha} J(\alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt\right], \tag{3}$$

where

$$f(t,x,\mu,\alpha) := c_1 |\underbrace{\int_{\mathbb{R}^3} v\mu(dv \times dw \times dy)}_{\text{'local field potential (LFP)'}} - \overline{v}_t|^2 + c_2|x_1 - \hat{v}_t|^2,$$

subject to $X_t^\alpha = (v_t, w_t, y_t)$ satisfying

$$\begin{cases} dv_t = \Big(\overbrace{v_t - \frac{(v_t)^3}{3} - w_t + \alpha_t}^{\text{one-sided Lipschitz}}\Big)dt + \sigma_{ext} dW_t \\ \qquad - \underbrace{\widetilde{\mathbb{E}}\left(J(\widetilde{v}_t - V_{rev})y_t\right)}_{\text{locally Lipschitz}} dt \\ dw_t = c(v_t + a - bw_t)dt \ , \\ dy_t = (a_r S(v_t)(1 - y_t) - a_d y_t)dt. \end{cases} \tag{4}$$

The convex constraint ensures existence of a unique solution to (4).

Example

Particular example: We consider the following control problem (SM)

$$\min_{\alpha} J(\alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt\right], \tag{3}$$

where

$$f(t,x,\mu,\alpha) := c_1 |\underbrace{\int_{\mathbb{R}^3} v\mu(dv \times dw \times dy)}_{\text{'local field potential (LFP)'}} - \overline{v}_t|^2 + c_2|x_1 - \hat{v}_t|^2,$$

subject to $X_t^\alpha = (v_t, w_t, y_t)$ satisfying

$$\begin{cases} dv_t = \Big(\overbrace{v_t - \frac{(v_t)^3}{3} - w_t + \alpha_t}^{\text{one-sided Lipschitz}}\Big)dt + \sigma_{ext}dW_t \\ \qquad - \underbrace{\widetilde{\mathbb{E}}\left(J(\widetilde{v}_t - V_{rev})y_t\right)}_{\text{locally Lipschitz}} dt \\ dw_t = c(v_t + a - bw_t)dt \ , \\ dy_t = (a_r S(v_t)(1-y_t) - a_d y_t)dt. \end{cases} \tag{4}$$

The convex constraint ensures existence of a unique solution to (4).

Example

Particular example: We consider the following control problem (SM)

$$\min_{\alpha} J(\alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt\right], \tag{3}$$

where

$$f(t,x,\mu,\alpha) := c_1 |\underbrace{\int_{\mathbb{R}^3} v\mu(dv \times dw \times dy)}_{\text{'local field potential (LFP)'}} - \overline{v}_t|^2 + c_2|x_1 - \hat{v}_t|^2,$$

subject to $X_t^\alpha = (v_t, w_t, y_t)$ satisfying

$$\begin{cases} dv_t = \Big(\overbrace{v_t - \frac{(v_t)^3}{3} - w_t + \alpha_t}^{\text{one-sided Lipschitz}}\Big)dt + \sigma_{ext}dW_t \\ \qquad - \underbrace{\widetilde{\mathbb{E}}\left(J(\widetilde{v}_t - V_{rev})y_t\right)}_{\text{locally Lipschitz}} dt \\ dw_t = c(v_t + a - bw_t)dt \ , \\ dy_t = (a_r S(v_t)(1-y_t) - a_d y_t)dt. \end{cases} \tag{4}$$

The convex constraint ensures existence of a unique solution to (4).

Example

Particular example: We consider the following control problem (SM)

$$\min_{\alpha} J(\alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt\right], \tag{3}$$

where

$$f(t,x,\mu,\alpha) := c_1 |\underbrace{\int_{\mathbb{R}^3} v\mu(dv \times dw \times dy)}_{\text{'local field potential (LFP)'}} - \overline{v}_t|^2 + c_2|x_1 - \hat{v}_t|^2,$$

subject to $X_t^\alpha = (v_t, w_t, y_t)$ satisfying

$$\begin{cases} dv_t = \Big(\overbrace{v_t - \frac{(v_t)^3}{3} - w_t + \alpha_t}^{\text{one-sided Lipschitz}}\Big)dt + \sigma_{ext} dW_t \\ \qquad - \underbrace{\widetilde{\mathbb{E}}\left(J(\widetilde{v}_t - V_{rev})y_t\right)}_{\text{locally Lipschitz}} dt \\ dw_t = c(v_t + a - bw_t)dt \ , \\ dy_t = (a_r S(v_t)(1 - y_t) - a_d y_t)dt. \end{cases} \tag{4}$$

The convex constraint ensures existence of a unique solution to (4).

Example

Particular example: We consider the following control problem (SM)

$$\min_{\alpha} J(\alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)dt\right], \tag{3}$$

where

$$f(t,x,\mu,\alpha) := c_1 |\underbrace{\int_{\mathbb{R}^3} v\mu(dv \times dw \times dy)}_{\text{'local field potential (LFP)'}} - \overline{v}_t|^2 + c_2|x_1 - \hat{v}_t|^2,$$

subject to $X_t^\alpha = (v_t, w_t, y_t)$ satisfying

$$\begin{cases} dv_t = \Big(\overbrace{v_t - \frac{(v_t)^3}{3} - w_t + \alpha_t}^{\text{one-sided Lipschitz}}\Big)dt + \sigma_{ext} dW_t \\ \qquad - \underbrace{\widetilde{\mathbb{E}}\left(J(\widetilde{v}_t - V_{rev})y_t\right)}_{\text{locally Lipschitz}} dt \\ dw_t = c(v_t + a - bw_t)dt \ , \\ dy_t = (a_r S(v_t)(1 - y_t) - a_d y_t)dt. \end{cases} \tag{4}$$

The convex constraint ensures existence of a unique solution to (4).

Abstract setting

Let $\mathcal{C} := \pi^{-1}((-\infty, 0])$.

Assumptions:

1. $\pi_x(x) \cdot b(t, x, \mu, \alpha) \leq 0$
2. $\mathrm{Im}\,(\sigma(t, x, \mu, \alpha)) \subset \pi_x(x)^{\perp}$
3. $\pi_{xx}(x) \cdot (\sigma\sigma^{\dagger}(t, x, \mu, \alpha)) = 0$,

for all $\mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in A$, $t \in [0, T]$ and $x \in \mathbb{R}^d \setminus \mathcal{C}$.

$\Rightarrow$ we can ensure that

$$\mathbb{P}\left[\pi(X_t^{\alpha,\mu}) \leq 0 \forall t \in [0, T]\right] = 1,$$

for any $\mu \in C([0, T], \mathcal{P}_2^{\mathcal{C}})$, where $X^{\alpha,\mu}$ is the solution to

$$dX = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t.$$

$\rightsquigarrow$ this ensures existence of a unique solution to (4), if the coefficients b, σ are assumed to be Lipschitz continuous in μ on $\mathcal{C}$.

$\rightsquigarrow$ abstract setting for dynamical systems modeling fractions/concentrations

Abstract setting

Let $\mathcal{C} := \pi^{-1}((-\infty, 0])$.

Assumptions:

1. $\pi_x(x) \cdot b(t, x, \mu, \alpha) \leq 0$
2. $\mathrm{Im}\,(\sigma(t, x, \mu, \alpha)) \subset \pi_x(x)^{\perp}$
3. $\pi_{xx}(x) \cdot (\sigma\sigma^{\dagger}(t, x, \mu, \alpha)) = 0$,

for all $\mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in A$, $t \in [0, T]$ and $x \in \mathbb{R}^d \setminus \mathcal{C}$.

$\Rightarrow$ we can ensure that

$$\mathbb{P}\left[\pi(X_t^{\alpha,\mu}) \leq 0 \forall t \in [0, T]\right] = 1,$$

for any $\mu \in C([0, T], \mathcal{P}_2^{\mathcal{C}})$, where $X^{\alpha,\mu}$ is the solution to

$$dX = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t.$$

$\rightsquigarrow$ this ensures existence of a unique solution to (4), if the coefficients b, σ are assumed to be Lipschitz continuous in μ on $\mathcal{C}$.

$\rightsquigarrow$ abstract setting for dynamical systems modeling fractions/concentrations

Abstract setting

Let $\mathcal{C} := \pi^{-1}((-\infty, 0])$.

Assumptions:

1. $\pi_x(x) \cdot b(t, x, \mu, \alpha) \leq 0$
2. $\mathrm{Im}\,(\sigma(t, x, \mu, \alpha)) \subset \pi_x(x)^{\perp}$
3. $\pi_{xx}(x) \cdot (\sigma\sigma^{\dagger}(t, x, \mu, \alpha)) = 0$,

for all $\mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in A$, $t \in [0, T]$ and $x \in \mathbb{R}^d \setminus \mathcal{C}$.

$\Rightarrow$ we can ensure that

$$\mathbb{P}\left[\pi(X_t^{\alpha,\mu}) \leq 0 \forall t \in [0, T]\right] = 1,$$

for any $\mu \in C([0, T], \mathcal{P}_2^{\mathcal{C}})$, where $X^{\alpha,\mu}$ is the solution to

$$dX = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t.$$

$\rightsquigarrow$ this ensures existence of a unique solution to (4), if the coefficients b, σ are assumed to be Lipschitz continuous in μ on $\mathcal{C}$.

$\rightsquigarrow$ abstract setting for dynamical systems modeling fractions/concentrations

Abstract setting

Let $\mathcal{C} := \pi^{-1}((-\infty, 0])$.

Assumptions:

1. $\pi_x(x) \cdot b(t, x, \mu, \alpha) \leq 0$
2. $\mathrm{Im}\,(\sigma(t, x, \mu, \alpha)) \subset \pi_x(x)^{\perp}$
3. $\pi_{xx}(x) \cdot (\sigma\sigma^{\dagger}(t, x, \mu, \alpha)) = 0$,

for all $\mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in A$, $t \in [0, T]$ and $x \in \mathbb{R}^d \setminus \mathcal{C}$.

$\Rightarrow$ we can ensure that

$$\mathbb{P}\left[\pi(X_t^{\alpha,\mu}) \leq 0 \forall t \in [0, T]\right] = 1,$$

for any $\mu \in C([0, T], \mathcal{P}_2^{\mathcal{C}})$, where $X^{\alpha,\mu}$ is the solution to

$$dX = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t.$$

$\rightsquigarrow$ this ensures existence of a unique solution to (4), if the coefficients b, σ are assumed to be Lipschitz continuous in μ on $\mathcal{C}$.

$\rightsquigarrow$ abstract setting for dynamical systems modeling fractions/concentrations

Abstract setting

Let $\mathcal{C} := \pi^{-1}((-\infty, 0])$.

Assumptions:

1. $\pi_x(x) \cdot b(t, x, \mu, \alpha) \leq 0$
2. $\mathrm{Im}\,(\sigma(t, x, \mu, \alpha)) \subset \pi_x(x)^{\perp}$
3. $\pi_{xx}(x) \cdot (\sigma\sigma^{\dagger}(t, x, \mu, \alpha)) = 0$,

for all $\mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in A,\ t \in [0, T]$ and $x \in \mathbb{R}^d \setminus \mathcal{C}$.

$\Rightarrow$ we can ensure that

$$\mathbb{P}\left[\pi(X_t^{\alpha,\mu}) \leq 0 \forall t \in [0, T]\right] = 1,$$

for any $\mu \in C([0, T], \mathcal{P}_2^{\mathcal{C}})$, where $X^{\alpha,\mu}$ is the solution to

$$dX = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t.$$

$\rightsquigarrow$ this ensures existence of a unique solution to (4), if the coefficients b, σ are assumed to be Lipschitz continuous in μ on $\mathcal{C}$.

$\rightsquigarrow$ abstract setting for dynamical systems modeling fractions/concentrations

Abstract setting

Let $\mathcal{C} := \pi^{-1}((-\infty, 0])$.

Assumptions:

1. $\pi_x(x) \cdot b(t, x, \mu, \alpha) \leq 0$
2. $\mathrm{Im}\,(\sigma(t, x, \mu, \alpha)) \subset \pi_x(x)^{\perp}$
3. $\pi_{xx}(x) \cdot (\sigma\sigma^{\dagger}(t, x, \mu, \alpha)) = 0$,

for all $\mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in A,\ t \in [0, T]$ and $x \in \mathbb{R}^d \setminus \mathcal{C}$.

$\Rightarrow$ we can ensure that

$$\mathbb{P}\left[\pi(X_t^{\alpha,\mu}) \leq 0 \forall t \in [0, T]\right] = 1,$$

for any $\mu \in C([0, T], \mathcal{P}_2^{\mathcal{C}})$, where $X^{\alpha,\mu}$ is the solution to

$$dX = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t.$$

$\rightsquigarrow$ this ensures existence of a unique solution to (4), if the coefficients b, σ are assumed to be Lipschitz continuous in μ on $\mathcal{C}$.

$\rightsquigarrow$ abstract setting for dynamical systems modeling fractions/concentrations

Abstract setting

Let $\mathcal{C} := \pi^{-1}((-\infty, 0])$.

Assumptions:

1. $\pi_x(x) \cdot b(t, x, \mu, \alpha) \leq 0$
2. $\mathrm{Im}\,(\sigma(t, x, \mu, \alpha)) \subset \pi_x(x)^{\perp}$
3. $\pi_{xx}(x) \cdot (\sigma\sigma^{\dagger}(t, x, \mu, \alpha)) = 0$,

for all $\mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in A,\ t \in [0, T]$ and $x \in \mathbb{R}^d \setminus \mathcal{C}$.

$\Rightarrow$ we can ensure that

$$\mathbb{P}\left[\pi(X_t^{\alpha,\mu}) \leq 0 \forall t \in [0, T]\right] = 1,$$

for any $\mu \in C([0, T], \mathcal{P}_2^{\mathcal{C}})$, where $X^{\alpha,\mu}$ is the solution to

$$dX = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t.$$

$\rightsquigarrow$ this ensures existence of a unique solution to (4), if the coefficients b, σ are assumed to be Lipschitz continuous in μ on $\mathcal{C}$.

$\rightsquigarrow$ abstract setting for dynamical systems modeling fractions/concentrations

Abstract setting

Let $\mathcal{C} := \pi^{-1}((-\infty, 0])$.

Assumptions:

1. $\pi_x(x) \cdot b(t, x, \mu, \alpha) \leq 0$
2. $\mathrm{Im}\,(\sigma(t, x, \mu, \alpha)) \subset \pi_x(x)^{\perp}$
3. $\pi_{xx}(x) \cdot (\sigma\sigma^{\dagger}(t, x, \mu, \alpha)) = 0$,

for all $\mu \in \mathcal{P}(\mathbb{R}^d), \alpha \in A,\ t \in [0, T]$ and $x \in \mathbb{R}^d \setminus \mathcal{C}$.

$\Rightarrow$ we can ensure that

$$\mathbb{P}\left[\pi(X_t^{\alpha,\mu}) \leq 0 \forall t \in [0, T]\right] = 1,$$

for any $\mu \in C([0, T], \mathcal{P}_2^{\mathcal{C}})$, where $X^{\alpha,\mu}$ is the solution to

$$dX = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t.$$

$\rightsquigarrow$ this ensures existence of a unique solution to (4), if the coefficients b, σ are assumed to be Lipschitz continuous in μ on $\mathcal{C}$.

$\rightsquigarrow$ abstract setting for dynamical systems modeling fractions/concentrations

Gâteaux derivative of the cost functional

The Gâteaux derivative of the cost functional in direction β is given by

$$\begin{aligned}\partial_\beta J(\alpha) = &\mathbb{E}\left[f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + f_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot \beta_t\right] \\ &+ \mathbb{E}\left[\widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] \\ &+ \mathbb{E}\left[g_x(X_T, \mathcal{L}(X_T)) \cdot Z_T^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[g_\mu(X_T, \mathcal{L}(X_T))(\widetilde{X}_T) \cdot \widetilde{Z}_T^{\alpha,\beta}\right]\right],\end{aligned}$$

where $Z^{\alpha,\beta}$ is the unique solution to

$$\begin{aligned}dZ_t = &\left\{b_x(t, X_t, \mathcal{L}(X_t), \alpha_t)Z_t + b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + B_\mu(t, X_t, \mathcal{L}(X_t, Z_t))\right\}dt \\ &+\left\{\sigma_x(t, X_t, \mathcal{L}(X_t), \alpha_t)Z_t + \sigma_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + \Sigma_\mu(t, X_t, \mathcal{L}(X_t, Z_t))\right\}dW_t \\ Z_0 = &\ 0,\end{aligned}$$

for

$$B_\mu(t, x, \mu) := \int\int_{\mathbb{R}^d \times \mathbb{R}^d} b_\mu(t, x, \mathcal{L}(X_t), \alpha_t)(\widetilde{x})\tilde{y}\mu(d\widetilde{x} \times d\tilde{y}),$$

$$\Sigma_\mu(t, x, \mu) := \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma_\mu(t, x, \mathcal{L}(X_t), \alpha_t)(\widetilde{x})\tilde{y}\mu(d\widetilde{x} \times d\tilde{y}).$$

Gâteaux derivative of the cost functional

The Gâteaux derivative of the cost functional in direction β is given by

$$\begin{aligned}\partial_\beta J(\alpha) = \mathbb{E}\left[f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + f_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot \beta_t\right] \\ + \mathbb{E}\left[\widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] \\ + \mathbb{E}\left[g_x(X_T, \mathcal{L}(X_T)) \cdot Z_T^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[g_\mu(X_T, \mathcal{L}(X_T))(\widetilde{X}_T) \cdot \widetilde{Z}_T^{\alpha,\beta}\right]\right],\end{aligned}$$

where $Z^{\alpha,\beta}$ is the unique solution to

$$\begin{aligned}dZ_t = &\left\{b_x(t, X_t, \mathcal{L}(X_t), \alpha_t)Z_t + b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + B_\mu(t, X_t, \mathcal{L}(X_t, Z_t))\right\}dt \\ &+\left\{\sigma_x(t, X_t, \mathcal{L}(X_t), \alpha_t)Z_t + \sigma_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + \Sigma_\mu(t, X_t, \mathcal{L}(X_t, Z_t))\right\}dW_t \\ Z_0 = &\ 0,\end{aligned}$$

for

$$B_\mu(t, x, \mu) := \int\int_{\mathbb{R}^d \times \mathbb{R}^d} b_\mu(t, x, \mathcal{L}(X_t), \alpha_t)(\widetilde{x})\tilde{y}\mu(d\widetilde{x} \times d\tilde{y}),$$

$$\Sigma_\mu(t, x, \mu) := \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma_\mu(t, x, \mathcal{L}(X_t), \alpha_t)(\widetilde{x})\tilde{y}\mu(d\widetilde{x} \times d\tilde{y}).$$

Gâteaux derivative of the cost functional

The Gâteaux derivative of the cost functional in direction β is given by

$$\begin{aligned}\partial_\beta J(\alpha) = \mathbb{E}\left[f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + f_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot \beta_t\right] \\ + \mathbb{E}\left[\widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] \\ + \mathbb{E}\left[g_x(X_T, \mathcal{L}(X_T)) \cdot Z_T^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[g_\mu(X_T, \mathcal{L}(X_T))(\widetilde{X}_T) \cdot \widetilde{Z}_T^{\alpha,\beta}\right]\right],\end{aligned}$$

where $Z^{\alpha,\beta}$ is the unique solution to

$$\begin{aligned}dZ_t &= \left\{b_x(t, X_t, \mathcal{L}(X_t), \alpha_t)Z_t + b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + B_\mu(t, X_t, \mathcal{L}(X_t, Z_t))\right\}dt \\ &\quad + \left\{\sigma_x(t, X_t, \mathcal{L}(X_t), \alpha_t)Z_t + \sigma_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + \Sigma_\mu(t, X_t, \mathcal{L}(X_t, Z_t))\right\}dW_t \\ Z_0 &= 0,\end{aligned}$$

for

$$\begin{aligned}B_\mu(t, x, \mu) &:= \int\int_{\mathbb{R}^d \times \mathbb{R}^d} b_\mu(t, x, \mathcal{L}(X_t), \alpha_t)(\widetilde{x})\tilde{y}\mu(d\widetilde{x} \times d\tilde{y}), \\ \Sigma_\mu(t, x, \mu) &:= \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma_\mu(t, x, \mathcal{L}(X_t), \alpha_t)(\widetilde{x})\tilde{y}\mu(d\widetilde{x} \times d\tilde{y}).\end{aligned}$$

Adjoint equation and duality

In order to derive a gradient representation for the cost functional, we consider the adjoint equation for $Z^{\alpha,\beta}$:

Let (P, Q) be the solution to:

$$\begin{cases} dP_t = -\Big(b_x(t, X_t, \mathcal{L}(X_t), \alpha_t) P_t + \sigma_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Q_t \\ \qquad + f_x(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t) + \widetilde{\mathbb{E}}\Big[b_\mu(t, \widetilde{X}_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \widetilde{P}_t \Big] \\ \qquad + \widetilde{\mathbb{E}}\Big[f_\mu(t, \widetilde{X}_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \Big] \Big) dt - Q_t dW_t \\ P_T = g_x(X_T^\alpha, \mathcal{L}(X_T^\alpha)) + \widetilde{\mathbb{E}}\Big[g_\mu(\widetilde{X}_t, \mathcal{L}(X_t))(\widetilde{X}_t) \Big], \end{cases} \tag{5}$$

i.e. (P, Q) solves the equation (5) and is **adapted** to the given filtration $(\mathcal{F}_t)_{t\in[0,T]}$, which is generated by $(X_t)_{t\in[0,T]}$.

Adjoint equation and duality

In order to derive a gradient representation for the cost functional, we consider the adjoint equation for $Z^{\alpha,\beta}$:

Let (P, Q) be the solution to:

$$\begin{cases} dP_t = -\Big(b_x(t, X_t, \mathcal{L}(X_t), \alpha_t) P_t + \sigma_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Q_t \\ \qquad + f_x(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t) + \widetilde{\mathbb{E}}\Big[b_\mu(t, \widetilde{X}_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \widetilde{P}_t \Big] \\ \qquad + \widetilde{\mathbb{E}}\Big[f_\mu(t, \widetilde{X}_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \Big] \Big) dt - Q_t dW_t \\ P_T = g_x(X_T^\alpha, \mathcal{L}(X_T^\alpha)) + \widetilde{\mathbb{E}}\Big[g_\mu(\widetilde{X}_t, \mathcal{L}(X_t))(\widetilde{X}_t) \Big], \end{cases} \tag{5}$$

i.e. (P, Q) solves the equation (5) and is **adapted** to the given filtration $(\mathcal{F}_t)_{t\in[0,T]}$, which is generated by $(X_t)_{t\in[0,T]}$.

Adjoint equation and duality

In order to derive a gradient representation for the cost functional, we consider the adjoint equation for $Z^{\alpha,\beta}$:

Let (P, Q) be the solution to:

$$\begin{cases} dP_t = -\Big(b_x(t, X_t, \mathcal{L}(X_t), \alpha_t) P_t + \sigma_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Q_t \\ \qquad + f_x(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t) + \widetilde{\mathbb{E}}\Big[b_\mu(t, \widetilde{X}_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \widetilde{P}_t \Big] \\ \qquad + \widetilde{\mathbb{E}}\Big[f_\mu(t, \widetilde{X}_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \Big] \Big) dt - Q_t dW_t \\ P_T = g_x(X_T^\alpha, \mathcal{L}(X_T^\alpha)) + \widetilde{\mathbb{E}}\Big[g_\mu(\widetilde{X}_t, \mathcal{L}(X_t))(\widetilde{X}_t) \Big], \end{cases} \tag{5}$$

i.e. (P, Q) solves the equation (5) and is **adapted** to the given filtration $(\mathcal{F}_t)_{t \in [0,T]}$, which is generated by $(X_t)_{t \in [0,T]}$.

Duality

Then the following duality holds true:

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right] \tag{6}$$

$\Rightarrow$ **Gradient representation of the cost functional:**

$$\begin{aligned}\partial_\beta J(\alpha) &= \int_0^T \mathbb{E}\left[P_t \cdot b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)\right]\beta_t + \mathbb{E}\left[Q_t \cdot \sigma_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\right]\beta_t dt \\ &\quad + \mathbb{E}\left[f_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\right]\beta_t dt \\ &= \int_0^T \mathbb{E}\left[H_\alpha(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \alpha_t)\right]\beta_t dt \\ &=: \int_0^T \nabla J(\alpha)(t)\beta_t dt,\end{aligned} \tag{7}$$

where $H(t, x, \mu, p, q, \alpha) := b(t, x, \mu, \alpha) \cdot p + \sigma(t, x, \mu, \alpha) \cdot q + f(t, x, \mu, \alpha)$ denotes the Hamiltonian.

Duality

Then the following duality holds true:

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right] \tag{6}$$

$\Rightarrow$ **Gradient representation of the cost functional:**

$$\begin{aligned}\partial_\beta J(\alpha) &= \int_0^T \mathbb{E}\left[P_t \cdot b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)\right]\beta_t + \mathbb{E}\left[Q_t \cdot \sigma_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\right]\beta_t dt \\ &\quad + \mathbb{E}\left[f_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\right]\beta_t dt \\ &= \int_0^T \mathbb{E}\left[H_\alpha(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \alpha_t)\right]\beta_t dt \\ &=: \int_0^T \nabla J(\alpha)(t)\beta_t dt,\end{aligned} \tag{7}$$

where $H(t, x, \mu, p, q, \alpha) := b(t, x, \mu, \alpha) \cdot p + \sigma(t, x, \mu, \alpha) \cdot q + f(t, x, \mu, \alpha)$ denotes the Hamiltonian.

Duality

Then the following duality holds true:

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right] \tag{6}$$

$\Rightarrow$ **Gradient representation of the cost functional:**

$$\begin{aligned}\partial_\beta J(\alpha) &= \int_0^T \mathbb{E}\left[P_t \cdot b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)\right]\beta_t + \mathbb{E}\left[Q_t \cdot \sigma_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\right]\beta_t dt \\ &\quad + \mathbb{E}\left[f_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\right]\beta_t dt \\ &= \int_0^T \mathbb{E}\left[H_\alpha(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \alpha_t)\right]\beta_t dt \\ &=: \int_0^T \nabla J(\alpha)(t)\beta_t dt,\end{aligned} \tag{7}$$

where $H(t, x, \mu, p, q, \alpha) := b(t, x, \mu, \alpha) \cdot p + \sigma(t, x, \mu, \alpha) \cdot q + f(t, x, \mu, \alpha)$ denotes the Hamiltonian.

Maximum principle

In particular, if the Hamiltonian H is convex in α we can obtain the following maximum principle:

Let $\overline{\alpha} \in \mathbb{A}$ be a minimizer of J as in (2) and (P, Q) the solution to the corresponding adjoint equation (5), then it holds for Lebesgue-almost every $t \in [0, T]$

$$\mathbb{E}\left[H(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \overline{\alpha}_t)\right] \leq \mathbb{E}\left[H(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \alpha)\right],$$

for every $\alpha \in A$.

Since we consider deterministic controls, the maximum principle is formulated in terms of the expectation and does not hold $dt \otimes \mathbb{P}$-almost everywhere.

Maximum principle

In particular, if the Hamiltonian H is convex in α we can obtain the following maximum principle:

Let $\overline{\alpha} \in \mathbb{A}$ be a minimizer of J as in (2) and (P, Q) the solution to the corresponding adjoint equation (5), then it holds for Lebesgue-almost every $t \in [0, T]$

$$\mathbb{E}\left[H(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \overline{\alpha}_t)\right] \leq \mathbb{E}\left[H(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \alpha)\right],$$

for every $\alpha \in A$.

Since we consider deterministic controls, the maximum principle is formulated in terms of the expectation and does not hold $dt \otimes \mathbb{P}$-almost everywhere.

Maximum principle

In particular, if the Hamiltonian H is convex in α we can obtain the following maximum principle:

Let $\overline{\alpha} \in \mathbb{A}$ be a minimizer of J as in (2) and (P, Q) the solution to the corresponding adjoint equation (5), then it holds for Lebesgue-almost every $t \in [0, T]$

$$\mathbb{E}\left[H(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \overline{\alpha}_t)\right] \leq \mathbb{E}\left[H(t, X_t, \mathcal{L}(X_t), P_t, Q_t, \alpha)\right],$$

for every $\alpha \in A$.

Since we consider deterministic controls, the maximum principle is formulated in terms of the expectation and does not hold $dt \otimes \mathbb{P}$-almost everywhere.

Duality

Problem: It is difficult to approximate the **adapted** solution to the adjoint equation (5) numerically

However: The duality

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right].$$

holds true for any 'non-adapted solution' to (5), as long as σ does not depend on $X, \mathcal{L}(X)$ and α $\rightsquigarrow$ In this case the duality can be proven pathwise.

$\rightsquigarrow$Is it necessary to compute the **adapted** solution of (5) to determine the gradient?

In the case of additive noise, any 'non-adapted solution' leads to the same gradient as the adapted solution, however in the general case this is no longer true.

Duality

Problem: It is difficult to approximate the **adapted** solution to the adjoint equation (5) numerically

However: The duality

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right].$$

holds true for any 'non-adapted solution' to (5), as long as σ does not depend on $X, \mathcal{L}(X)$ and α $\rightsquigarrow$ In this case the duality can be proven pathwise.

$\rightsquigarrow$Is it necessary to compute the **adapted** solution of (5) to determine the gradient?

In the case of additive noise, any 'non-adapted solution' leads to the same gradient as the adapted solution, however in the general case this is no longer true.

Duality

Problem: It is difficult to approximate the **adapted** solution to the adjoint equation (5) numerically

However: The duality

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right].$$

holds true for any 'non-adapted solution' to (5), as long as σ does not depend on $X, \mathcal{L}(X)$ and α $\rightsquigarrow$ In this case the duality can be proven pathwise.

$\rightsquigarrow$Is it necessary to compute the **adapted** solution of (5) to determine the gradient?

In the case of additive noise, any 'non-adapted solution' leads to the same gradient as the adapted solution, however in the general case this is no longer true.

Duality

Problem: It is difficult to approximate the **adapted** solution to the adjoint equation (5) numerically

However: The duality

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right].$$

holds true for any 'non-adapted solution' to (5), as long as σ does not depend on $X, \mathcal{L}(X)$ and $\alpha \rightsquigarrow$ In this case the duality can be proven pathwise.

$\rightsquigarrow$Is it necessary to compute the **adapted** solution of (5) to determine the gradient?

In the case of additive noise, any 'non-adapted solution' leads to the same gradient as the adapted solution, however in the general case this is no longer true.

Duality

Problem: It is difficult to approximate the **adapted** solution to the adjoint equation (5) numerically

However: The duality

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right].$$

holds true for any 'non-adapted solution' to (5), as long as σ does not depend on $X, \mathcal{L}(X)$ and $\alpha \rightsquigarrow$ In this case the duality can be proven pathwise.

$\rightsquigarrow$Is it necessary to compute the **adapted** solution of (5) to determine the gradient?

In the case of additive noise, any 'non-adapted solution' leads to the same gradient as the adapted solution, however in the general case this is no longer true.

Duality

Problem: It is difficult to approximate the **adapted** solution to the adjoint equation (5) numerically

However: The duality

$$\mathbb{E}\left[P_T \cdot Z_T^{\alpha,\beta} + \int_0^T f_x(t, X_t, \mathcal{L}(X_t), \alpha_t) \cdot Z_t^{\alpha,\beta} + \widetilde{\mathbb{E}}\left[f_\mu(t, X_t, \mathcal{L}(X_t), \alpha_t)(\widetilde{X}_t) \cdot \widetilde{Z}_t^{\alpha,\beta}\right]\right] dt$$
$$= \mathbb{E}\left[\int_0^T P_t \cdot b_\alpha(t, X_t, \mathcal{L}(X_t), \alpha_t)\beta_t + Q_t \cdot \sigma_\alpha(t, X_t, (X_t), \alpha_t)\beta_t dt\right].$$

holds true for any 'non-adapted solution' to (5), as long as σ does not depend on $X, \mathcal{L}(X)$ and $\alpha \rightsquigarrow$ In this case the duality can be proven pathwise.

$\rightsquigarrow$Is it necessary to compute the **adapted** solution of (5) to determine the gradient?

In the case of additive noise, any 'non-adapted solution' leads to the same gradient as the adapted solution, however in the general case this is no longer true.

Adapted vs. 'non-adapted solutions'

Example: Minimize $J(\alpha) = \mathbb{E}\left[\int_0^T |X_t^\alpha|^2 dt\right]$ subject to

$$dX_t^\alpha = \{\alpha_t + \mu X_t^\alpha\}\, dt + \sigma X_t^\alpha dW_t, \quad X_0 = 1. \tag{8}$$

The BSDE reduces to

$$dP_t = -\{\mu P_t + \sigma Q_t + 2X_t^\alpha\}\, dt - Q_t dW_t, \quad P_T = 0. \tag{9}$$

The (non-unique) 'Non-adapted solution' $(\hat{P}, \hat{Q})$, where $\hat{Q} = 0$ and

$$\hat{P}_t = 2e^{-\mu t} \int_t^T e^{\mu s} X_s^\alpha ds$$

leads to the following approximation of the gradient for $\alpha \equiv 0$:

$$\begin{aligned} \widehat{\nabla J(\alpha)}(t) &:= \mathbb{E}\left[b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)\hat{P}_t\right] \\ &= 2\mathbb{E}\left[e^{-\mu t} \int_t^T e^{\mu s} X_s^\alpha ds\right] \\ &\overset{\alpha \equiv 0}{=} \frac{1}{\mu} e^{-\mu t}(e^{2\mu T} - e^{2\mu t}) \end{aligned}$$

Adapted vs. 'non-adapted solutions'

Example: Minimize $J(\alpha) = \mathbb{E}\left[\int_0^T |X_t^\alpha|^2 dt\right]$ subject to

$$dX_t^\alpha = \{\alpha_t + \mu X_t^\alpha\}\, dt + \sigma X_t^\alpha dW_t, \quad X_0 = 1. \tag{8}$$

The BSDE reduces to

$$dP_t = -\{\mu P_t + \sigma Q_t + 2X_t^\alpha\}\, dt - Q_t dW_t, \quad P_T = 0. \tag{9}$$

The (non-unique) 'Non-adapted solution' $(\hat{P}, \hat{Q})$, where $\hat{Q} = 0$ and

$$\hat{P}_t = 2e^{-\mu t} \int_t^T e^{\mu s} X_s^\alpha ds$$

leads to the following approximation of the gradient for $\alpha \equiv 0$:

$$\begin{aligned}\widehat{\nabla J(\alpha)}(t) &:= \mathbb{E}\left[b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)\hat{P}_t\right] \\ &= 2\mathbb{E}\left[e^{-\mu t} \int_t^T e^{\mu s} X_s^\alpha ds\right] \\ &\overset{\alpha \equiv 0}{=} \frac{1}{\mu} e^{-\mu t}(e^{2\mu T} - e^{2\mu t})\end{aligned}$$

Adapted vs. 'non-adapted solutions'

Example: Minimize $J(\alpha) = \mathbb{E}\left[\int_0^T |X_t^\alpha|^2 dt\right]$ subject to

$$dX_t^\alpha = \{\alpha_t + \mu X_t^\alpha\}\, dt + \sigma X_t^\alpha dW_t, \quad X_0 = 1. \tag{8}$$

The BSDE reduces to

$$dP_t = -\{\mu P_t + \sigma Q_t + 2X_t^\alpha\}\, dt - Q_t dW_t, \quad P_T = 0. \tag{9}$$

The (non-unique) 'Non-adapted solution' $(\hat{P}, \hat{Q})$, where $\hat{Q} = 0$ and

$$\hat{P}_t = 2e^{-\mu t} \int_t^T e^{\mu s} X_s^\alpha ds$$

leads to the following approximation of the gradient for $\alpha \equiv 0$:

$$\begin{aligned}\widehat{\nabla J(\alpha)}(t) &:= \mathbb{E}\left[b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)\hat{P}_t\right] \\ &= 2\mathbb{E}\left[e^{-\mu t} \int_t^T e^{\mu s} X_s^\alpha ds\right] \\ &\overset{\alpha\equiv 0}{=} \frac{1}{\mu} e^{-\mu t}(e^{2\mu T} - e^{2\mu t})\end{aligned}$$

Adapted vs. 'non-adapted solutions'

The **unique** adapted solution to the linear BSDE (5) is given by

$$P_t = 2\mathbb{E}\left[e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\int_t^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s} X_s^\alpha ds|\mathcal{F}_t\right]$$

$$Q_t = -\sigma P_t + e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\eta_t,$$

where η results from the martingale representation theorem for the martingale

$$M_t := -2\mathbb{E}\left[\int_0^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s} X_s^\alpha ds|\mathcal{F}_t\right].$$

Thus

$$\begin{aligned}\nabla J(\alpha)(t) &= \mathbb{E}\left[b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)P_t\right]\\ &= 2\mathbb{E}\left[e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\int_t^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s} X_s^\alpha ds\right]\\ &= \frac{2}{\sigma^2+2\mu}e^{-\mu t}(e^{2\mu T}e^{\sigma^2 T} - e^{2\mu t}e^{\sigma^2 t})\end{aligned}$$

Coincides with $\widehat{\nabla J(\alpha)}$ only if $\sigma = 0$.

Adapted vs. 'non-adapted solutions'

The **unique** adapted solution to the linear BSDE (5) is given by

$$P_t = 2\mathbb{E}\left[e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\int_t^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s}X_s^\alpha ds|\mathcal{F}_t\right]$$

$$Q_t = -\sigma P_t + e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\eta_t,$$

where η results from the martingale representation theorem for the martingale

$$M_t := -2\mathbb{E}\left[\int_0^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s}X_s^\alpha ds|\mathcal{F}_t\right].$$

Thus

$$\begin{aligned}\nabla J(\alpha)(t) &= \mathbb{E}\left[b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)P_t\right]\\ &= 2\mathbb{E}\left[e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\int_t^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s}X_s^\alpha ds\right]\\ &= \frac{2}{\sigma^2+2\mu}e^{-\mu t}(e^{2\mu T}e^{\sigma^2 T} - e^{2\mu t}e^{\sigma^2 t})\end{aligned}$$

Coincides with $\widehat{\nabla J(\alpha)}$ only if $\sigma = 0$.

Adapted vs. 'non-adapted solutions'

The **unique** adapted solution to the linear BSDE (5) is given by

$$P_t = 2\mathbb{E}\left[e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\int_t^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s}X_s^\alpha ds|\mathcal{F}_t\right]$$

$$Q_t = -\sigma P_t + e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\eta_t,$$

where η results from the martingale representation theorem for the martingale

$$M_t := -2\mathbb{E}\left[\int_0^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s}X_s^\alpha ds|\mathcal{F}_t\right].$$

Thus

$$\begin{aligned}\nabla J(\alpha)(t) &= \mathbb{E}\left[b_\alpha(t, X_t^\alpha, \mathcal{L}(X_t^\alpha), \alpha_t)P_t\right]\\ &= 2\mathbb{E}\left[e^{-(\mu-\frac{\sigma^2}{2})t-\sigma W_t}\int_t^T e^{(\mu-\frac{\sigma^2}{2})s+\sigma W_s}X_s^\alpha ds\right]\\ &= \frac{2}{\sigma^2+2\mu}e^{-\mu t}(e^{2\mu T}e^{\sigma^2 T} - e^{2\mu t}e^{\sigma^2 t})\end{aligned}$$

Coincides with $\widehat{\nabla J(\alpha)}$ only if $\sigma = 0$.

Plan

Approximation of the adjoint equation

How to approximate the adapted solution to (5)?

In general we consider the MFFBSDE of the following type:

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t))dt + \sigma(t, X_t, \mathcal{L}(X_t))dW_t \\ dP_t = [f(t, X_t, P_t) + h(t, X_t, \mathcal{L}(X_t, P_t))]\, dt - Q_t dW_t \\ X_0 = \xi \\ P_T = g(X_T). \end{cases} \tag{10}$$

Backward Euler-scheme:

$$P^\pi_{t_k} = \mathbb{E}\left(P^\pi_{t_{k+1}} | \mathcal{F}_{t_k}\right) - (t_{k+1} - t_k)\left\{f(t_k, X^\pi_{t_k}, P^\pi_{t_k}) + h(t_{k+1}, X^\pi_{t_k}, \mathcal{L}(X^\pi_{t_{k+1}}, P^\pi_{t_{k+1}}))\right\}$$

$$Q^\pi_{t_k} = (t_{k+1} - t_k)^{-1}\mathbb{E}\left(P^\pi_{t_k}(W_{t_{k+1}} - W_{t_k}) | \mathcal{F}_{t_k}\right),$$

$$P^\pi_{t_N} = g(X^\pi_{t_N}), \quad Q^\pi_{t_N} = 0,$$

for a given grid $\pi : 0 = t_0 < t_1 < ... < t_N = T$.

Approximation of the adjoint equation

How to approximate the adapted solution to (5)?

In general we consider the MFFBSDE of the following type:

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t))dt + \sigma(t, X_t, \mathcal{L}(X_t))dW_t \\ dP_t = [f(t, X_t, P_t) + h(t, X_t, \mathcal{L}(X_t, P_t))]\, dt - Q_t dW_t \\ X_0 = \xi \\ P_T = g(X_T). \end{cases} \tag{10}$$

Backward Euler-scheme:

$$P^\pi_{t_k} = \mathbb{E}\left(P^\pi_{t_{k+1}} | \mathcal{F}_{t_k}\right) - (t_{k+1} - t_k)\left\{f(t_k, X^\pi_{t_k}, P^\pi_{t_k}) + h(t_{k+1}, X^\pi_{t_k}, \mathcal{L}(X^\pi_{t_{k+1}}, P^\pi_{t_{k+1}}))\right\}$$

$$Q^\pi_{t_k} = (t_{k+1} - t_k)^{-1}\mathbb{E}\left(P^\pi_{t_k}(W_{t_{k+1}} - W_{t_k}) | \mathcal{F}_{t_k}\right),$$

$$P^\pi_{t_N} = g(X^\pi_{t_N}), \quad Q^\pi_{t_N} = 0,$$

for a given grid $\pi : 0 = t_0 < t_1 < ... < t_N = T$.

Approximation of the adjoint equation

How to approximate the adapted solution to (5)?

In general we consider the MFFBSDE of the following type:

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t))dt + \sigma(t, X_t, \mathcal{L}(X_t))dW_t \\ dP_t = [f(t, X_t, P_t) + h(t, X_t, \mathcal{L}(X_t, P_t))]\, dt - Q_t dW_t \\ X_0 = \xi \\ P_T = g(X_T). \end{cases} \tag{10}$$

Backward Euler-scheme:

$$P^\pi_{t_k} = \mathbb{E}\left(P^\pi_{t_{k+1}} | \mathcal{F}_{t_k}\right) - (t_{k+1} - t_k)\left\{f(t_k, X^\pi_{t_k}, P^\pi_{t_k}) + h(t_{k+1}, X^\pi_{t_k}, \mathcal{L}(X^\pi_{t_{k+1}}, P^\pi_{t_{k+1}}))\right\}$$

$$Q^\pi_{t_k} = (t_{k+1} - t_k)^{-1}\mathbb{E}\left(P^\pi_{t_k}(W_{t_{k+1}} - W_{t_k}) | \mathcal{F}_{t_k}\right),$$

$$P^\pi_{t_N} = g(X^\pi_{t_N}), \quad Q^\pi_{t_N} = 0,$$

for a given grid $\pi : 0 = t_0 < t_1 < ... < t_N = T$.

Approximation of the adjoint equation

There exists $u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ (decoupling filed) (Carmona, Delarue [3]), such that

$$P_t = u(t, X_t, \mathcal{L}(X_t)).$$

Thus

$$P^\pi_{t_k+1} = u(t_{k+1}, X^\pi_{t_{k+1}}, \mathcal{L}(X^\pi_{t_{k+1}})) =: \hat{u}(t_{k+1}, X^\pi_{t_{k+1}})$$

This leads to the following representation of the conditional expectation in terms of a function $\tilde{u}$ by

$$\mathbb{E}\left(P^\pi_{t_k+1} | \mathcal{F}_{t_k}\right) = \tilde{u}(t_{k+1}, X^\pi_{t_k}).$$

Approximate $\tilde{u}(t_{k+1}, \cdot)$ with gaussian radial basis functions, by solving the following minimization problem for fixed nodes $x_1, ..., x_L$:

$$\min_\alpha \mathbb{E}\left(|P^\pi_{t_k+1} - \sum_{i=1}^L \alpha_i(t_{k+1}) e^{\frac{1}{2\delta}\|X^\pi_{t_k} - x_i\|^2}|^2\right),$$

for $\alpha = (\alpha_1(t_{k+1}), ..., \alpha_L(t_{k+1}))^\dagger$, where $\delta > 0$ and $L \in \mathbb{N}$ are fixed.

Approximation of the adjoint equation

There exists $u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ (decoupling filed) (Carmona, Delarue [3]), such that

$$P_t = u(t, X_t, \mathcal{L}(X_t)).$$

Thus

$$P^\pi_{t_k+1} = u(t_{k+1}, X^\pi_{t_{k+1}}, \mathcal{L}(X^\pi_{t_{k+1}})) =: \hat{u}(t_{k+1}, X^\pi_{t_{k+1}})$$

This leads to the following representation of the conditional expectation in terms of a function $\tilde{u}$ by

$$\mathbb{E}\left(P^\pi_{t_k+1} | \mathcal{F}_{t_k}\right) = \tilde{u}(t_{k+1}, X^\pi_{t_k}).$$

Approximate $\tilde{u}(t_{k+1}, \cdot)$ with gaussian radial basis functions, by solving the following minimization problem for fixed nodes $x_1, ..., x_L$:

$$\min_\alpha \mathbb{E}\left(|P^\pi_{t_k+1} - \sum_{i=1}^L \alpha_i(t_{k+1}) e^{\frac{1}{2\delta}\|X^\pi_{t_k} - x_i\|^2}|^2\right),$$

for $\alpha = (\alpha_1(t_{k+1}), ..., \alpha_L(t_{k+1}))^\dagger$, where $\delta > 0$ and $L \in \mathbb{N}$ are fixed.

Approximation of the adjoint equation

There exists $u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ (decoupling filed) (Carmona, Delarue [3]), such that

$$P_t = u(t, X_t, \mathcal{L}(X_t)).$$

Thus

$$P^\pi_{t_k+1} = u(t_{k+1}, X^\pi_{t_{k+1}}, \mathcal{L}(X^\pi_{t_{k+1}})) =: \hat{u}(t_{k+1}, X^\pi_{t_{k+1}})$$

This leads to the following representation of the conditional expectation in terms of a function $\tilde{u}$ by

$$\mathbb{E}\left(P^\pi_{t_k+1} | \mathcal{F}_{t_k}\right) = \tilde{u}(t_{k+1}, X^\pi_{t_k}).$$

Approximate $\tilde{u}(t_{k+1}, \cdot)$ with gaussian radial basis functions, by solving the following minimization problem for fixed nodes $x_1, ..., x_L$:

$$\min_\alpha \mathbb{E}\left(|P^\pi_{t_k+1} - \sum_{i=1}^L \alpha_i(t_{k+1}) e^{\frac{1}{2\delta}\|X^\pi_{t_k} - x_i\|^2}|^2\right),$$

for $\alpha = (\alpha_1(t_{k+1}), ..., \alpha_L(t_{k+1}))^\dagger$, where $\delta > 0$ and $L \in \mathbb{N}$ are fixed.

Approximation of the adjoint equation

There exists $u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ (decoupling filed) (Carmona, Delarue [3]), such that

$$P_t = u(t, X_t, \mathcal{L}(X_t)).$$

Thus

$$P^\pi_{t_k+1} = u(t_{k+1}, X^\pi_{t_{k+1}}, \mathcal{L}(X^\pi_{t_{k+1}})) =: \hat{u}(t_{k+1}, X^\pi_{t_{k+1}})$$

This leads to the following representation of the conditional expectation in terms of a function $\tilde{u}$ by

$$\mathbb{E}\left(P^\pi_{t_k+1} | \mathcal{F}_{t_k}\right) = \tilde{u}(t_{k+1}, X^\pi_{t_k}).$$

Approximate $\tilde{u}(t_{k+1}, \cdot)$ with gaussian radial basis functions, by solving the following minimization problem for fixed nodes $x_1, ..., x_L$:

$$\min_\alpha \mathbb{E}\left(|P^\pi_{t_k+1} - \sum_{i=1}^L \alpha_i(t_{k+1}) e^{\frac{1}{2\delta}\|X^\pi_{t_k} - x_i\|^2}|^2\right),$$

for $\alpha = (\alpha_1(t_{k+1}), ..., \alpha_L(t_{k+1}))^\dagger$, where $\delta > 0$ and $L \in \mathbb{N}$ are fixed.

Gradient decent algorithm

To approximate the optimal control, we fix an initial control α_0 and proceed according to the following gradient decent algorithm:

1. solve the state equation to determine X^{α_n};
2. approximate the gradient of the cost functional $\nabla J(\alpha_n)(t) \approx \frac{1}{N}\sum_{k=1}^{N} H_\alpha(t, X_t^{\alpha_n,k}, \frac{1}{N}\sum_{k=1}^{N}\delta_{X_t^{\alpha_n,k}}, P_t^k, Q_t^k, \alpha_n(t)), t \in [0, T]$, where $(P^1, Q^1), ..., (P^N, Q^N)$ are N samples of the solution to the adjoint equation;
3. update control according to $\alpha_{n+1} := \alpha_n - s\nabla J(\alpha_n)$;

Problem:

In many situations, the membrane potential v becomes highly sensitive to small perturbations of the control at specific times.

This sensitivity can lead to high fluctuations of the sample mean for the corresponding adjoint equation

Gradient decent algorithm

To approximate the optimal control, we fix an initial control α_0 and proceed according to the following gradient decent algorithm:

1. solve the state equation to determine X^{α_n};
2. approximate the gradient of the cost functional $\nabla J(\alpha_n)(t) \approx \frac{1}{N}\sum_{k=1}^{N} H_\alpha(t, X_t^{\alpha_n,k}, \frac{1}{N}\sum_{k=1}^{N}\delta_{X_t^{\alpha_n,k}}, P_t^k, Q_t^k, \alpha_n(t)), t \in [0, T]$, where $(P^1, Q^1), ..., (P^N, Q^N)$ are N samples of the solution to the adjoint equation;
3. update control according to $\alpha_{n+1} := \alpha_n - s\nabla J(\alpha_n)$;

Problem:

In many situations, the membrane potential v becomes highly sensitive to small perturbations of the control at specific times.

This sensitivity can lead to high fluctuations of the sample mean for the corresponding adjoint equation

Gradient decent algorithm

To approximate the optimal control, we fix an initial control α_0 and proceed according to the following gradient decent algorithm:

1. solve the state equation to determine X^{α_n};
2. approximate the gradient of the cost functional $\nabla J(\alpha_n)(t) \approx \frac{1}{N}\sum_{k=1}^{N} H_\alpha(t, X_t^{\alpha_n,k}, \frac{1}{N}\sum_{k=1}^{N}\delta_{X_t^{\alpha_n,k}}, P_t^k, Q_t^k, \alpha_n(t)), t \in [0, T]$, where $(P^1, Q^1), ..., (P^N, Q^N)$ are N samples of the solution to the adjoint equation;
3. update control according to $\alpha_{n+1} := \alpha_n - s\nabla J(\alpha_n)$;

Problem:

In many situations, the membrane potential v becomes highly sensitive to small perturbations of the control at specific times.

This sensitivity can lead to high fluctuations of the sample mean for the corresponding adjoint equation

Gradient decent algorithm

To approximate the optimal control, we fix an initial control α_0 and proceed according to the following gradient decent algorithm:

1. solve the state equation to determine X^{α_n};
2. approximate the gradient of the cost functional $\nabla J(\alpha_n)(t) \approx \frac{1}{N}\sum_{k=1}^{N} H_\alpha(t, X_t^{\alpha_n,k}, \frac{1}{N}\sum_{k=1}^{N}\delta_{X_t^{\alpha_n,k}}, P_t^k, Q_t^k, \alpha_n(t)), t \in [0, T]$, where $(P^1, Q^1), ..., (P^N, Q^N)$ are N samples of the solution to the adjoint equation;
3. update control according to $\alpha_{n+1} := \alpha_n - s\nabla J(\alpha_n)$;

Problem:

In many situations, the membrane potential v becomes highly sensitive to small perturbations of the control at specific times.

This sensitivity can lead to high fluctuations of the sample mean for the corresponding adjoint equation

Gradient decent algorithm

To approximate the optimal control, we fix an initial control α_0 and proceed according to the following gradient decent algorithm:

1. solve the state equation to determine X^{α_n};
2. approximate the gradient of the cost functional $\nabla J(\alpha_n)(t) \approx \frac{1}{N}\sum_{k=1}^{N} H_\alpha(t, X_t^{\alpha_n,k}, \frac{1}{N}\sum_{k=1}^{N}\delta_{X_t^{\alpha_n,k}}, P_t^k, Q_t^k, \alpha_n(t)), t \in [0, T]$, where $(P^1, Q^1), ..., (P^N, Q^N)$ are N samples of the solution to the adjoint equation;
3. update control according to $\alpha_{n+1} := \alpha_n - s\nabla J(\alpha_n)$;

Problem:

In many situations, the membrane potential v becomes highly sensitive to small perturbations of the control at specific times.

This sensitivity can lead to high fluctuations of the sample mean for the corresponding adjoint equation

Gradient decent algorithm

To approximate the optimal control, we fix an initial control α_0 and proceed according to the following gradient decent algorithm:

1. solve the state equation to determine X^{α_n};
2. approximate the gradient of the cost functional $\nabla J(\alpha_n)(t) \approx \frac{1}{N}\sum_{k=1}^{N} H_\alpha(t, X_t^{\alpha_n,k}, \frac{1}{N}\sum_{k=1}^{N}\delta_{X_t^{\alpha_n,k}}, P_t^k, Q_t^k, \alpha_n(t)), t \in [0, T]$, where $(P^1, Q^1), ..., (P^N, Q^N)$ are N samples of the solution to the adjoint equation;
3. update control according to $\alpha_{n+1} := \alpha_n - s\nabla J(\alpha_n)$;

Problem:

In many situations, the membrane potential v becomes highly sensitive to small perturbations of the control at specific times.

This sensitivity can lead to high fluctuations of the sample mean for the corresponding adjoint equation

Gradient decent algorithm

To approximate the optimal control, we fix an initial control α_0 and proceed according to the following gradient decent algorithm:

1. solve the state equation to determine X^{α_n};
2. approximate the gradient of the cost functional $\nabla J(\alpha_n)(t) \approx \frac{1}{N}\sum_{k=1}^{N} H_\alpha(t, X_t^{\alpha_n,k}, \frac{1}{N}\sum_{k=1}^{N}\delta_{X_t^{\alpha_n,k}}, P_t^k, Q_t^k, \alpha_n(t)), t \in [0, T]$, where $(P^1, Q^1), ..., (P^N, Q^N)$ are N samples of the solution to the adjoint equation;
3. update control according to $\alpha_{n+1} := \alpha_n - s\nabla J(\alpha_n)$;

Problem:

In many situations, the membrane potential v becomes highly sensitive to small perturbations of the control at specific times.

This sensitivity can lead to high fluctuations of the sample mean for the corresponding adjoint equation

High sensitivity of the membrane potential

Example Consider the control problem (SM) for $J = \sigma_{ext} = c_1 = 0$ and $a = 0.7, b = 0.8, c = 0.08$, where for the approximation of the optimal control, the initial control α_0 is chosen close to the bifurcation value for the supercritical Hopf-bifurcation point of equation (4).

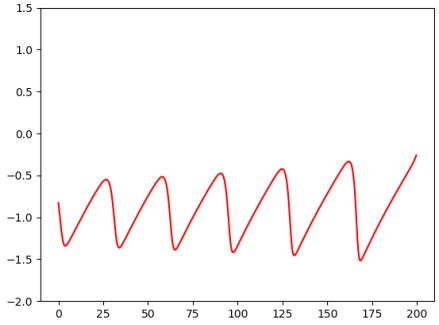

Figure: Membrane potential of the solution to the state equation at $\alpha_0 = 0.315$

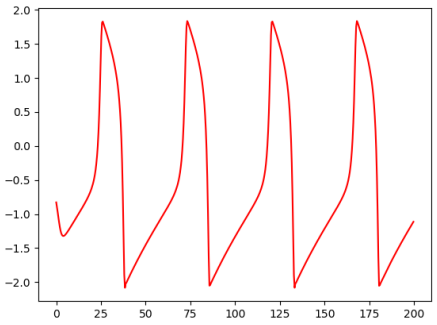

Figure: Reference profile $\tilde{v}$ generated by solving the state equation for $\alpha \equiv 0.33$

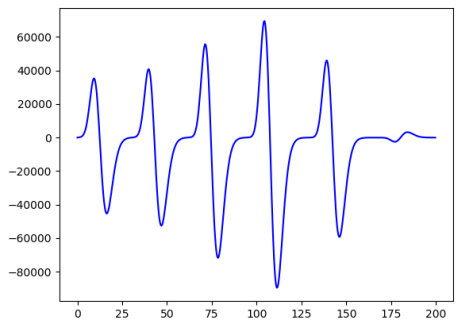

Figure: Solution $P_t^{\alpha_0}$ to the corresponding adjoint equation

High sensitivity of the membrane potential

Example Consider the control problem (SM) for $J = \sigma_{ext} = c_1 = 0$ and $a = 0.7, b = 0.8, c = 0.08$, where for the approximation of the optimal control, the initial control α_0 is chosen close to the bifurcation value for the supercritical Hopf-bifurcation point of equation (4).

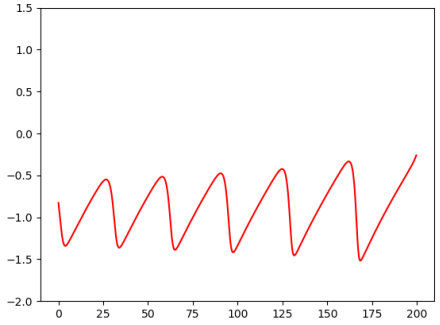

Figure: Membrane potential of the solution to the state equation at $\alpha_0 = 0.315$

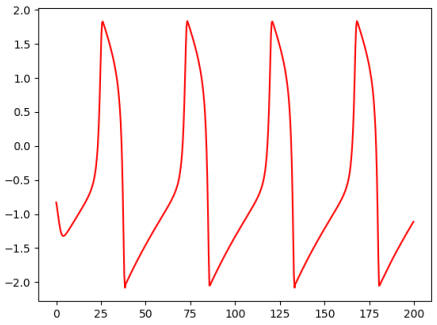

Figure: Reference profile $\tilde{v}$ generated by solving the state equation for $\alpha \equiv 0.33$

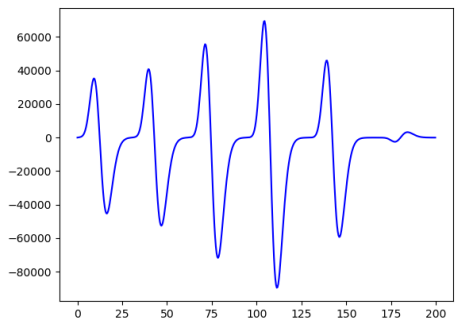

Figure: Solution $P_t^{\alpha_0}$ to the corresponding adjoint equation

Fluctuations of the adjoint samples

Example Consider the control problem (SM) for $J = c_1 = 1, c_2 = 0, \sigma_{ext} = 0.08$ and $a = 0.7, b = 0.8, c = 0.08$.

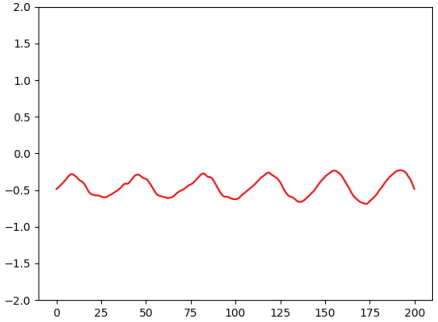

Figure: $\mathbb{E}[V_t]$, where $X_t = (V_t, w_t, y_t)$ is the solution to (4) for $\alpha_0 \equiv 0$

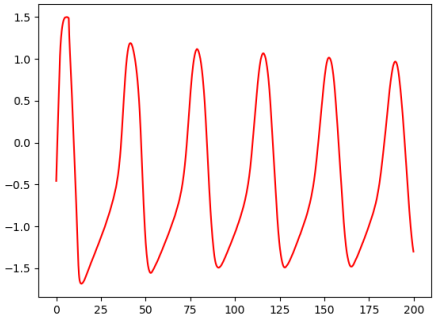

Figure: Reference profile $v_t = \mathbb{E}[V_t]$, where $X_t = (V_t, w_t, y_t)$ is the solution to (4) for $\alpha(t) = 0.8$, if $t \leq 7$

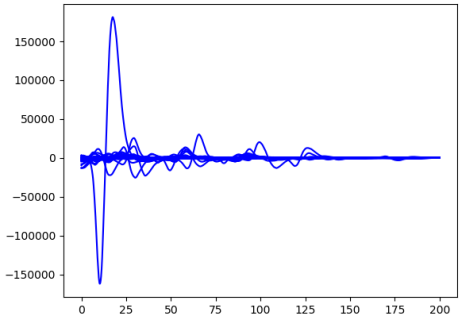

Figure: Samples of the solution to the adjoint equation for α_0

Fluctuations of the adjoint samples

Example Consider the control problem (SM) for $J = c_1 = 1, c_2 = 0, \sigma_{ext} = 0.08$ and $a = 0.7, b = 0.8, c = 0.08$.

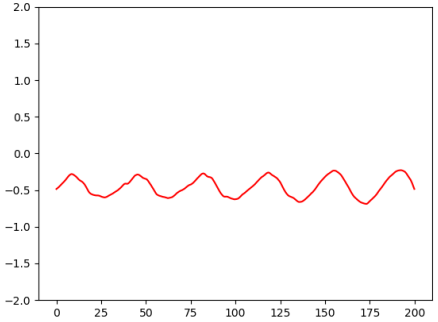

Figure: $\mathbb{E}[V_t]$, where $X_t = (V_t, w_t, y_t)$ is the solution to (4) for $\alpha_0 \equiv 0$

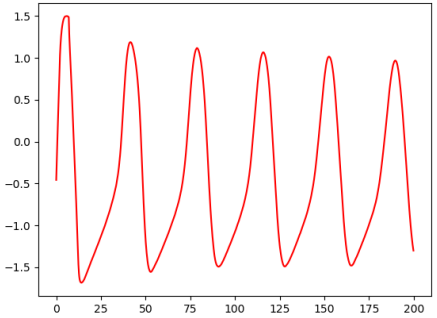

Figure: Reference profile $v_t = \mathbb{E}[V_t]$, where $X_t = (V_t, w_t, y_t)$ is the solution to (4) for $\alpha(t) = 0.8$, if $t \leq 7$

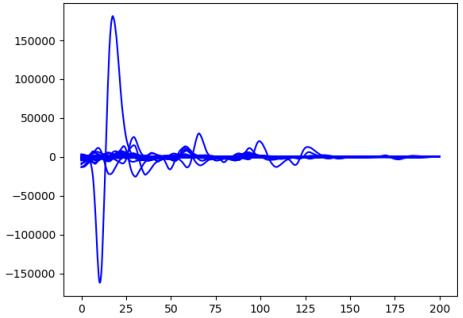

Figure: Samples of the solution to the adjoint equation for α_0

Parameter setting for the optimal control problem

Parameter setting of the control problem: $a = 0.7, b = 0.8, c = 0.08$, $\sigma_{ext} = 0.08, J = 0.46$.

- Initial states are uniformly distributed on the limit cycle of (4) for $\sigma_{ext} = 0, \alpha = 0$
- Slow gating variable by decreased closing rate a_d of the synaptic gates
- $\rightsquigarrow$ activity of a large number of neurons in the network can lead to further activity at later times without external input

Example

- Controlling the local field potential $\mathbb{E}[V_t]$ of an uncontrolled network of coupled FitzHugh-Nagumo neurons into a reference profile which is chosen to be the local field potential of an excited network
- Controlling the local field potential $\mathbb{E}[V_t]$ of a uncontrolled network of uncoupled FitzHugh-Nagumo neurons into the same reference profile

Parameter setting for the optimal control problem

Parameter setting of the control problem: $a = 0.7, b = 0.8, c = 0.08$, $\sigma_{ext} = 0.08, J = 0.46$.

- Initial states are uniformly distributed on the limit cycle of (4) for $\sigma_{ext} = 0, \alpha = 0$
- Slow gating variable by decreased closing rate a_d of the synaptic gates
- $\rightsquigarrow$ activity of a large number of neurons in the network can lead to further activity at later times without external input

Example

- Controlling the local field potential $\mathbb{E}[V_t]$ of an uncontrolled network of coupled FitzHugh-Nagumo neurons into a reference profile which is chosen to be the local field potential of an excited network
- Controlling the local field potential $\mathbb{E}[V_t]$ of a uncontrolled network of uncoupled FitzHugh-Nagumo neurons into the same reference profile

Parameter setting for the optimal control problem

Parameter setting of the control problem: $a = 0.7, b = 0.8, c = 0.08$, $\sigma_{ext} = 0.08, J = 0.46$.

- Initial states are uniformly distributed on the limit cycle of (4) for $\sigma_{ext} = 0, \alpha = 0$
- Slow gating variable by decreased closing rate a_d of the synaptic gates
- $\rightsquigarrow$ activity of a large number of neurons in the network can lead to further activity at later times without external input

Example

- Controlling the local field potential $\mathbb{E}[V_t]$ of an uncontrolled network of coupled FitzHugh-Nagumo neurons into a reference profile which is chosen to be the local field potential of an excited network
- Controlling the local field potential $\mathbb{E}[V_t]$ of a uncontrolled network of uncoupled FitzHugh-Nagumo neurons into the same reference profile

Parameter setting for the optimal control problem

Parameter setting of the control problem: $a = 0.7, b = 0.8, c = 0.08$, $\sigma_{ext} = 0.08, J = 0.46$.

- Initial states are uniformly distributed on the limit cycle of (4) for $\sigma_{ext} = 0, \alpha = 0$
- Slow gating variable by decreased closing rate a_d of the synaptic gates
- $\rightsquigarrow$ activity of a large number of neurons in the network can lead to further activity at later times without external input

Example

- Controlling the local field potential $\mathbb{E}[V_t]$ of an uncontrolled network of coupled FitzHugh-Nagumo neurons into a reference profile which is chosen to be the local field potential of an excited network
- Controlling the local field potential $\mathbb{E}[V_t]$ of a uncontrolled network of uncoupled FitzHugh-Nagumo neurons into the same reference profile

Parameter setting for the optimal control problem

Parameter setting of the control problem: $a = 0.7, b = 0.8, c = 0.08$, $\sigma_{ext} = 0.08, J = 0.46$.

- Initial states are uniformly distributed on the limit cycle of (4) for $\sigma_{ext} = 0, \alpha = 0$
- Slow gating variable by decreased closing rate a_d of the synaptic gates
- $\rightsquigarrow$ activity of a large number of neurons in the network can lead to further activity at later times without external input

Example

- Controlling the local field potential $\mathbb{E}[V_t]$ of an uncontrolled network of coupled FitzHugh-Nagumo neurons into a reference profile which is chosen to be the local field potential of an excited network
- Controlling the local field potential $\mathbb{E}[V_t]$ of a uncontrolled network of uncoupled FitzHugh-Nagumo neurons into the same reference profile

Parameter setting for the optimal control problem

Parameter setting of the control problem: $a = 0.7, b = 0.8, c = 0.08$, $\sigma_{ext} = 0.08, J = 0.46$.

- Initial states are uniformly distributed on the limit cycle of (4) for $\sigma_{ext} = 0, \alpha = 0$
- Slow gating variable by decreased closing rate a_d of the synaptic gates
- $\rightsquigarrow$ activity of a large number of neurons in the network can lead to further activity at later times without external input

Example

- Controlling the local field potential $\mathbb{E}[V_t]$ of an uncontrolled network of coupled FitzHugh-Nagumo neurons into a reference profile which is chosen to be the local field potential of an excited network
- Controlling the local field potential $\mathbb{E}[V_t]$ of a uncontrolled network of uncoupled FitzHugh-Nagumo neurons into the same reference profile

Example

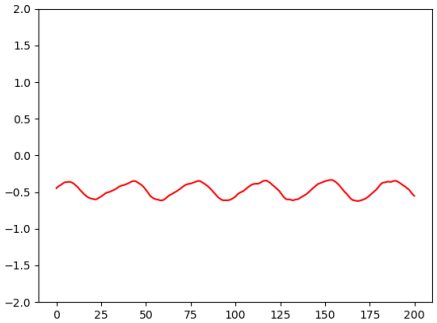

Figure: Uncontrolled local field potential $J = 0.46$

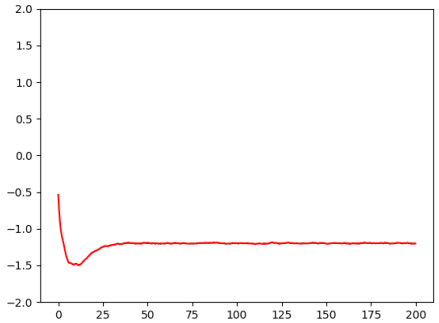

Figure: Uncontrolled local field potential $J = 0$

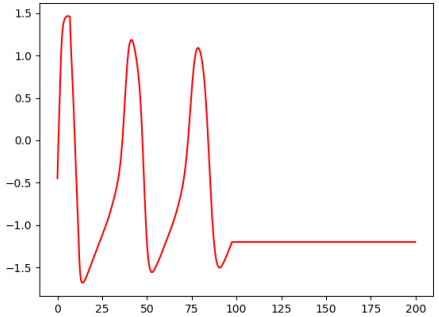

Figure: Reference profile

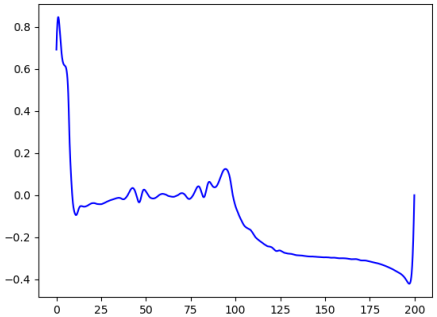

Figure: Optimal control $J = 0.46$

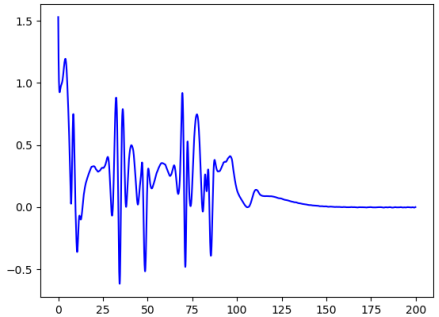

Figure: Optimal control $J = 0$

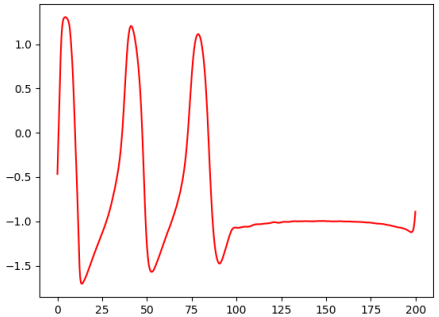

Figure: Local field potential with optimal control

Plan

Feedback control

Consider controls of the form

$$\alpha_t = \phi(t, X_t) \approx \sum_{i=1}^{L} \tilde{\alpha}_i(t) e^{-\frac{1}{2\delta}\|X_t - x_i(t)\|^2} = h(\tilde{\alpha}_t, X_t),$$

where $\tilde{\alpha}_t = (\tilde{\alpha}_1(t), ..., \tilde{\alpha}_L(t))$, $h : \mathbb{R}^L \times \mathbb{R}^d \to \mathbb{R}^d$, $h(\alpha, x) := \sum_{i=1}^{L} \tilde{\alpha}_i e^{-\frac{1}{2\delta}\|x - x_i(t)\|^2}$.

By defining

$$\begin{aligned}
\tilde{b}(t, X_t, \mathcal{L}(X_t), \tilde{\alpha}_t) &:= b(t, X_t, \mathcal{L}(X_t), h(\tilde{\alpha}_t, X_t)) \\
\tilde{\sigma}(t, X_t, \mathcal{L}(X_t), \tilde{\alpha}_t) &:= \sigma(t, X_t, \mathcal{L}(X_t), h(\tilde{\alpha}_t, X_t)) \\
\tilde{f}(t, X_t, \mathcal{L}(X_t), \tilde{\alpha}_t) &:= f(t, X_t, \mathcal{L}(X_t), h(\tilde{\alpha}_t, X_t))
\end{aligned}$$

we can solve the new optimal control problem with coefficients $\tilde{b}, \tilde{\sigma}, \tilde{f}$ for deterministic controls to get an approximation of the optimal feedback function ϕ.

References

J. Baladron, D. Fasoli, O. Faugeras, and J. Touboul. Mean-field description and propagation of chaos in networks of hodgkin-huxley and fitzhugh-nagumo neurons. *The Journal of Mathematical Neuroscience*, 2(1):10, 2012.

M. Bossy, D. Talay and O. Faugeras. Clarification and Complement to "Mean-Field Description and Propagation of Chaos in Networks of Hodgkin–Huxley and FitzHugh–Nagumo Neurons. The Journal of Mathematical Neuroscience (JMN), 2015.

R. Carmona and F. Delarue. Probabilistic Theory of Mean Field Games with Applications I-II. Springer, 2018.

G. Dos Reis, W. Salkeld, and J. Tugaut. Freidlin–wentzell ldp in path space for mckean–vlasov equations and the functional iterated logarithm law. *The Annals of Applied Probability*, 29(3):1487–1540, 2019.

S. Peng A General Stochastic Maximum Principle for Optimal Control Problems. *SIAM Journal on Control and Optimization*, 28:966-979, 1990.

A. Hocquet and A. Vogler. Optimal control of mean field equations with monotone coefficients and applications in neuroscience. *Appl. Math. Optim.*, 84:1925–1968, 2021.

Thank you for your attention.