On the existence of value for a general stochastic differential game with ergodic payoff

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Outline

1. Formulation of stochastic differential games (SDGs)

2. The existence of the viscosity solution of ergodic HJBI equation

3. The existence of the value of our SDGs

4. Application in pollution accumulation problems with long-run average welfare

Formulation of the SDGs

Let b, σ, f be measurable mappings defined over appropriate spaces. We are interested in a type of two-player SDGs where

Dynamics: For $t \geq 0$, $x \in \mathbb{R}^n$,

$$X_t^{x,u,v} = x + \int_0^t b(X_s^{x,u,v}, u_s, v_s)ds + \int_0^t \sigma(X_s^{x,u,v}, u_s, v_s)dB_s. \tag{1.1}$$

Time-average payoff:

$$J(T,x,u,v) = \frac{1}{T}E[\int_0^T f(X_s^{x,u,v}, u_s, v_s)ds]. \tag{1.2}$$

Zero-sum game with ergodic payoff:

$$
\begin{aligned}
&\text{Player 1 minimizes } \liminf_{T \to \infty} J(T,x,u,v) \text{ via } u = (u_s);\\
&\text{Player 2 maximizes } \limsup_{T \to \infty} J(T,x,u,v) \text{ via } v = (v_s).
\end{aligned}
\tag{1.3}
$$

Recall finite time horizon SDGs

1) For finite time horizon case, i.e., for fixed $T > 0$,

Player 1 minimizes $T \cdot J(T, x, u, v)$ via $u = (u_s)$;

Player 2 maximizes $T \cdot J(T, x, u, v)$ via $v = (v_s)$.

Such SDGs have been intensively studied by using dynamic programming principle (DPP):

• Pioneering work: Fleming, Souganidis (1989, Indiana Univ. Math. J.)

    <u>The basic idea of their work</u>: The upper and lower value functions satisfy the DPP, thus they are the unique viscosity solution of the associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations, respectively. Under the Isaacs condition, the upper and lower value functions coincide (i.e., the value exists).

Recall finite time horizon SDGs

- Extension works:

    SDGs with recursive payoffs: Buckdahn, Li (2008, SICON); Li, Wei (2015, AMO);

    with jumps: Buckdahn, Hu, Li (2011, SPA); Biswas (2012, SICON);

    with asymmetric information: Cardaliaguet, Rainer (2009, AMO);

    without Isaacs condition: Buckdahn, Li, Quincampoix (2014, AOP); Li, Li (2017, ESAIM: COCV; 2019, Stochastics)

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2) For infinite horizon case (i.e., $T \to \infty$), such SDGs (1.1)-(1.3) have been studied when

- σ is *non-degenerate* and independent of controls: Borkar, Ghosh (1992, JOTA), Arapostathis, Borkar, Kumar (2013, Annals of the ISDG);
- σ is *non-degenerate* and b, σ, f are periodic w.r.t. x: Alvarez, Bardi (2007, Annals of the ISDG).

<u>Remark</u>: All these works require the *non-degenerate* condition, i.e., the least eigenvalue of $\sigma\sigma^*$ uniformly bounded away from zero. This condition ensures the ergodic property of the diffusion process and plays an important role in their works.

Objective of the talk

Problem: how to study such SDGs (1.1)-(1.3) without non-degenerate assumption (one of the main challenges in differential game theory[1])?

Under the non ergodic settings (i.e., the value may depend upon the initial state), there have been some results:

• Optimal control problems: Quincampoix, Renault (2011, SICON), Buckdahn, Goreac, Quimcampoix (2014, AMO), Li, Zhao (2019, SPA), Buckdahn, Li, Quincampoix, Renault (2020, SICON);

• SDGs: Buckdahn, Li, Zhao (2021, JDE).

Remark: In the above works, certain *non-expansivity conditions* are given in order to study the existence of asymptotic values.

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[1]See the discussions in Buckdahn, Cardaliaguet, Quincampoix (2011, Dynamic Games and Appl.)

Our objective: We want to study the existence of the value of the game with ergodic payoff <u>without non-degenerate</u> assumption but still in <u>ergodic</u> settings.

<u>Our approach</u>: The associated HJBI equation has a viscosity solution $\Rightarrow$ the estimates for upper and lower ergodic value functions in terms of this viscosity solution $\Rightarrow$ value exists under the Isaacs condition.

<u>Remark</u>: Compared with the traditional way introduced by Fleming and Souganidis, this approach in some sense is a reverse process and it was first introduced by Świech (1996, JMAA) to get sub and super optimality of DPP for discounted SDGs (with some fixed discounted factor).

Preliminaries

<u>Probability Space</u>:

$+$ $(\Omega, \mathcal{F}, P)$: complete probability space;

$+$ B: d-dimensional B.M. over $(\Omega, \mathcal{F}, P)$;

$+$ $\mathbb{F}^B$: filtration generated by B, and augmented by all P-null sets.

<u>Admissible control spaces</u>: for two compact metric spaces U and V,

$$\mathcal{U} = \{(u_t)_{0 \leq t < \infty} : U\text{-valued } \mathbb{F}^B\text{-adapted process}\};$$

$$\mathcal{V} = \{(v_t)_{0 \leq t < \infty} : V\text{-valued } \mathbb{F}^B\text{-adapted process}\}.$$

Preliminaries

<u>Main assumptions</u>: The functions

$$b : \mathbb{R}^n \times U \times V \to \mathbb{R}^n, \ \sigma : \mathbb{R}^n \times U \times V \to \mathbb{R}^{n \times d}, \ f : \mathbb{R}^n \times U \times V \to \mathbb{R},$$

satisfy the following conditions

(H1) For every fixed $x \in \mathbb{R}^n$, b, σ, f are continuous in $(u, v) \in U \times V$;

(H2) For $l = b, \sigma, f$, there exists a constant C_l such that, for all $x, y \in \mathbb{R}^n$, $u \in U$, $v \in V$,

$$|l(x, u, v) - l(y, u, v)| \le C_l |x - y|.$$

Under the assumptions (H1) and (H2), the dynamics (1.1) has a unique solution $X^{x,u,v}$ for each $(u, v) \in \mathcal{U} \times \mathcal{V}$.

Preliminaries

(H3) There exists a constant $K > C_\sigma^2$ such that, for all $(u, v) \in U \times V$, $x, y \in \mathbb{R}^n$,

$$2(x - y)\big(b(x, u, v) - b(y, u, v)\big) \leq -K|x - y|^2.$$

<u>Remark</u> Assumptions (H2) and (H3) imply the classical dissipativity condition, i.e., for all $(u, v) \in U \times V$, $x, y \in \mathbb{R}^n$,

$$2(x - y)\big(b(x, u, v) - b(y, u, v)\big) + \|\sigma(x, u, v) - \sigma(y, u, v)\|^2$$
$$\leq -(K - C_\sigma^2)|x - y|^2.$$

$+$ This condition will ensure the existence and the uniqueness of the invariant measure of the state process $X^{x,u,v}$ when either b, σ are independent of the controls u, v or u, v are feedback controls.

$+$ Note that in our case the state process $X^{x,u,v}$ is time-inhomogenous. Thus, the classical theory of ergodicity may fail to be applied to our model.

Preliminaries

Lemma 1.1

Under the assumptions (H1)-(H3), there exist constants C, $c > 0$ such that for all $t > 0$, $\delta > 0$, $(u,v) \in \mathcal{U} \times \mathcal{V}$, $x, y \in \mathbb{R}^n$, we have the following estimates,

$$
\begin{aligned}
&E|X_t^{x,u,v}|^2 \leq C(1 + |x|^2 e^{-ct}); \\
&E|X_t^{x,u,v} - X_t^{y,u,v}|^2 \leq e^{-ct}|x-y|^2; \\
&E[\sup_{t \leq s \leq t+\delta} |X_s^{x,u,v} - X_t^{x,u,v}|^2] \leq C(\delta^2 + \delta).
\end{aligned}
\tag{1.4}
$$

<u>Games of the type</u>: "strategy against control".

• An admissible strategy for Player 1 is a mapping $\alpha : \mathcal{V} \to \mathcal{U}$ satisfying the non-anticipative property. The set of all admissible strategies for Player 1 is denoted by $\mathcal{A}$.

• An admissible strategy $\beta \in \mathcal{B}$ for Player 2 is defined similarly.

Preliminaries

Let us now introduce the following upper and lower ergodic value functions:

$$\rho^+(x) = \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} \limsup_{T \to \infty} J(T, x, u, \beta(u)), \ x \in \mathbb{R}^n,$$

$$\rho^-(x) = \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} \liminf_{T \to \infty} J(T, x, \alpha(v), v), \ x \in \mathbb{R}^n.$$

Definition 1.1

If for all x,

$$\rho^+(x) = \rho^-(x) = \rho \ (\text{a constant}),$$

we say that our SDGs with ergodic payoff has a value ρ (we also say that the long time average cost game is ergodic).

The associated HJBI equation

In order to prove the existence of the value of the game, we consider the following related ergodic HJBI equations,

$$\rho = \inf_{u \in U} \sup_{v \in V} H(x, Dw(x), D^2w(x), u, v), \ x \in \mathbb{R}^n, \tag{2.1}$$

$$\rho = \sup_{v \in V} \inf_{u \in U} H(x, Dw(x), D^2w(x), u, v), \ x \in \mathbb{R}^n, \tag{2.2}$$

where Hamiltonian function

$$H(x, p, A, u, v) = \frac{1}{2} tr\big((\sigma\sigma^*)(x, u, v) \cdot A\big) + b(x, u, v) \cdot p + f(x, u, v),$$

$(x, p, A, u, v) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}(n) \times U \times V$ and $\mathcal{S}(n)$ denotes the set of $n \times n$ symmetric matrices.

Definition of viscosity solution

Definition 2.1

(i) A viscosity subsolution (resp. supersolution) of (2.1) is a pair (ρ, w), where ρ is a real number and w is continuous on $\mathbb{R}^n$ such that for $x \in \mathbb{R}^n$ and a test function $\varphi \in C_b^3(\mathbb{R}^n)$, we have

$$\rho \leq (\text{resp. } \geq) \inf_{u \in U} \sup_{v \in V} H(x, D\varphi(x), D^2\varphi(x), u, v),$$

whenever $w - \varphi$ has a local maximum (resp. minimum) at x.

(ii) A viscosity solution of (2.1) is a pair (ρ, w) that is both a viscosity subsolution and a viscosity supersolution of (2.1).

Here $C_b^3(\mathbb{R}^n)$ denotes the set of the real-valued functions that are continuously differentiable up to the third order and whose derivatives of order 1 to 3 are bounded.

Existence of the viscosity solution

Theorem 2.1

Let Assumptions (H1)-(H3) hold. Then ergodic HJBI equation (2.1) (resp. (2.2)) has a viscosity solution (ρ, w), where w satisfies the following property: there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n$, it holds

$$|w(x) - w(y)| \leq C|x - y|, \; |w(x)| \leq C|x|.$$

The proof is based on vanishing limit in the discounted payoff case.

The link with the finite horizon HJBI equations

We now turn our attention to the study of the long time behaviour of the solution of the following second order HJBI equation

$$
\begin{cases}
\frac{\partial}{\partial t} V(t,x) = \inf_{u \in U} \sup_{v \in V} H(x, DV(t,x), D^2 V(t,x), u, v), \\
V(0,x) = \Phi(x), \ (t,x) \in [0,T] \times \mathbb{R}^n,
\end{cases}
\tag{2.3}
$$

where Φ is a Lipschitz function on $\mathbb{R}^n$.

Theorem 2.2

Suppose (ρ, w) is a viscosity solution of ergodic HJBI equation (2.1) and denote

$$W(T,x) = V(T,x) - (\rho T + w(x)), \ (T,x) \in [0,\infty) \times \mathbb{R}^n.$$

Then there exists a positive constant C (independent of T) such that

$$-C(1 + |x|) \leq W(T,x) \leq C(1 + |x|).$$

In particular, it holds $\lim_{T \to \infty} \frac{V(T,x)}{T} = \rho$.

Remark

• When the dynamics is non-degenerate and uncontrolled, Alvarez, Bardi (2003, Arch. Ration. Mech. Anal.) obtained a similar convergence result for a singular perturbation problem by using PDE techniques.

• When the invariant measure of the dynamics exists and there is only one player in the system, a similar result is also obtained by Cosso, Fuhrman, Pham (2016, SPA) based on the related stochastic control problem and BSDE theory.

An approximation of possibly degenerate processes

For $r > 0$, we construct a series of linear SDEs as follows

$$\begin{cases} dX_s^{r,x,u,v} = [-\dfrac{K}{2} \cdot (X_s^{r,x,u,v} - X_s^{x,u,v}) + b(X_s^{x,u,v}, u_s, v_s)]ds \\ \qquad\qquad + \sigma(X_s^{x,u,v}, u_s, v_s)dB_s + rIdB_s^1, \ s \geq 0, \\ X_0^{r,x,u,v} = x, \end{cases} \qquad (3.1)$$

where $W = (B, B^1)$ is a new B.M. and I is a $(n \times n)$-dimensional identity matrix.

<u>Remark</u>:

• The process $X^{r,x,u,v}$ is non-degenerate due to

$$\langle \sigma^r(\sigma^r)^*(x,u,v)\xi, \xi \rangle \geq r^2|\xi|^2, \text{ for any } \xi \in \mathbb{R}^n, \ \sigma^r(x,u,v) := (\sigma(x,u,v), rI).$$

• There exists a constant C such that for all $t > 0$ and $(x,u,v) \in \mathbb{R}^n \times \mathcal{U} \times \mathcal{V}$,

$$E[|X_t^{r,x,u,v} - X_t^{x,u,v}|^2] \leq Cnr^2.$$

An approximation of possibly degenerate processes

Proposition 3.1

Suppose that the assumptions (H1)-(H3) hold and let $X^{r,x,u,v}$ be the solution of SDE (3.1). Then for any Borel set $D \subseteq \mathbb{R}^n$, it holds, for all $(u,v) \in \mathcal{U} \times \mathcal{V}$,

$$P\{X_s^{r,x,u,v} \in D\} \leq (\frac{K}{2})^{\frac{n}{2}} r^{-n} [1 - e^{-Ks}]^{-\frac{n}{2}} Leb(D),$$

where $Leb(D)$ is the Lebesgue measure of the Borel set D and K is the constant given in Assumption (H3).

<u>Remark</u> For any Borel set $D \subseteq \mathbb{R}^n$, we denote $h(0,x;s,D) = P\{X_s^{r,x,u,v} \in D\}$. Then, Proposition 3.1 says that

$$h(0,x;s,y) \leq (\frac{K}{2})^{\frac{n}{2}} r^{-n} [1 - e^{-Ks}]^{-\frac{n}{2}},$$

which is weaker than the classical Gaussian's type bound (e.g. Aroson (1967)):

$$h(0,x;s,y) \leq M s^{-\frac{n}{2}} \exp\{-\frac{N|x-y|^2}{s}\}. \tag{3.2}$$

An approximation of possibly degenerate processes

• The PDE approach used to obtain (3.2) can not apply to our framework due to the presence of admissible controls (u, v). Indeed, for any given $(u, v) \in \mathcal{U} \times \mathcal{V}$, one may define

$$b^{u,v}(t, x) = b(x, u_t, v_t), \ \sigma^{r,u,v}(t, x) = \sigma^r(x, u_t, v_t),$$

which are obviously not deterministic and then the solution $X^{r,x,u,v}$ is not Markovian.

• We consider an infinite time horizon problem whereas the upper bound (3.2) is only obtained for some finite time horizon $[0, T]$.

Our proof is based on a probabilistic method and Proposition 3.1 can be applied to get the upper bound of the fundamental solution of the related stochastic PDE (since b and σ may be stochastic) with infinite time horizon.

Estimates (I) for upper and lower ergodic values

Theorem 3.1

Let assumptions (H1)-(H3) hold and σ is uniformly bounded. Suppose that w is Lipschitz on $\mathbb{R}^n$, then we have

(i) If (ρ, w) is a viscosity subsolution of ergodic HJBI equation (2.1) then

$$\rho \leq \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} \liminf_{T \to \infty} J(T, x, u, \beta(u)).$$

(ii) If (ρ, w) is a viscosity supersolution of ergodic HJBI equation (2.1) then

$$\rho \geq \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} \limsup_{T \to \infty} J(T, x, u, \beta(u))(= \rho^+(x)).$$

(iii) If (ρ, w) is a viscosity subsolution of ergodic HJBI equation (2.2) then

$$\rho \leq \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} \liminf_{T \to \infty} \frac{1}{T} J(T, x, \alpha(v), v)(= \rho^-(x)).$$

Estimates (I) for upper and lower ergodic value

Theorem 3.1 (continued)

(iv) If (ρ, w) is a viscosity supersolution of ergodic HJBI equation (2.2) then

$$\rho \geq \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} \limsup_{T \to \infty} \frac{1}{T} J(T, x, \alpha(v), v).$$

<u>Remark</u>: The basic framework of the proof is adapted from the work by Świech (1996, JMAA), including sup- and inf-convolution technique to yield the approximation of viscosity solution. However, there are some essential differences:

• The construction of non-degenerate diffusion processes is different.

• The admissible controls are $\mathbb{F}^B$-adapted rather than $\mathbb{F}^{(B,B^1)}$-adapted.

• The assumptions of b, σ and f are weaker.

• The well-known estimate of Krylov of the distribution of a stochastic integral, used by Świech (1996, JMAA), can not applied to our ergodic situation (we use Proposition 3.1).

Uniqueness result of the viscosity solution

Corollary 3.1

(i) Let (ρ_1, w_1) and (ρ_2, w_2) be a viscosity subsolution and supersolution of (2.2), respectively. Then it holds that $\rho_1 \leq \rho_2$.

(ii) Let (ρ_1, w_1) and (ρ_2, w_2) be two viscosity solutions of (2.2) and w_1 and w_2 are Lipschitz. Then there exists a constant R such that, if

$$w_1 \equiv w_2, \text{ on } \bar{B}_R(0). \tag{3.3}$$

Then $\rho_1 = \rho_2$, $w_1 \equiv w_2$ on $\mathbb{R}^n$.

<u>Remark</u>: The uniqueness of w does not hold in general. In fact, for all constants C, it is obvious that $(\rho, w + C)$ are the classical solutions of ergodic HJBI (2.2)

$$\rho = \sup_{v \in V} \inf_{u \in U} H(x, Dw(x), D^2 w(x), u, v), \ x \in \mathbb{R}^n,$$

if (ρ, w) is a classical solution.

The existence of a value of the game

Theorem 3.2

Suppose that the conditions (H1)-(H3) are satisfied and σ is uniformly bounded. Then our SDGs with ergodic payoff has a value under the classical Isaacs condition

$$\inf_{u \in U} \sup_{v \in V} H(x, p, A, u, v) = \sup_{v \in V} \inf_{u \in U} H(x, p, A, u, v), \ (x, p, A) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{S}(n),$$

where $\mathcal{S}(n)$ denotes the set of $n \times n$ symmetric matrices.

Corollary 3.2 (DPP)

Suppose that (H1) and (H2) are satisfied and σ is uniformly bounded. If (ρ, w) is a viscosity solution of (2.1), then for any $T > 0$,

$$w(x) = \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} E[\int_0^T f(X_s^{x,u,\beta(u)}, u_s, \beta(u)_s)ds + w(X_T^{x,u,\beta(u)})] - \rho T.$$

Theorem 3.3

Let (H1)-(H3) hold and σ is uniformly bounded. Suppose that w is Lipschitz on $\mathbb{R}^n$, then we have

(i) If (ρ, w) is a viscosity subsolution (resp. supersolution) of (2.1) then

$$\rho \le (\text{resp.} \ge) \lim_{\lambda \to 0} \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} \lambda E[\int_0^\infty e^{-\lambda s} f(X_s^{x,u,\beta(u)}, u_s, \beta(u)_s) ds].$$

(ii) If (ρ, w) is a viscosity subsolution (resp. supersolution) of (2.2) then

$$\rho \le (\text{resp.} \ge) \lim_{\lambda \to 0} \inf_{\alpha \in \mathcal{A}} \sup_{v \in \mathcal{V}} \lambda E[\int_0^\infty e^{-\lambda s} f(X_s^{x,\alpha(v),v}, \alpha(v)_s, v_s) ds].$$

The generalization of Abelian-Tauberian theorem

<u>Remark</u>: Let (ρ, w) be a viscosity solution of (2.1), from Theorem 3.1 and Corollary 3.1 as well as Theorem 3.3 we get the following property

$$(\rho =) \lim_{T \to \infty} \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} \frac{1}{T} E \int_0^T f(X_s^{x,u,\beta(u)}, u_s, \beta(u)_s) ds$$

$$= \lim_{\lambda \to 0^+} \sup_{\beta \in \mathcal{B}} \inf_{u \in \mathcal{U}} \lambda E \int_0^\infty e^{-\lambda s} f(X_s^{x,u,\beta(u)}, u_s, \beta(u)_s) ds.$$

This property can be seen as the generalization of the classical Abelian-Tauberian theorem, stating that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \varphi(t) dt = \lim_{\lambda \to 0^+} \lambda \int_0^\infty \varphi(t) e^{-\lambda t} dt$$

if either one of the two limits exists, to the stochastic differential game cases.

The pollution accumulation and consumption model

We consider that an economy consumes some good and meanwhile generates pollution. Suppose that the stock of pollution is gradually degraded and described by $X_t = \max\{Y_t, 0\}$ at time t, where Y is given by

$$dY_t = [u_t - v_t Y_t]dt + \sigma(Y_t)dB_t, \ Y_0 = x > 0. \tag{4.1}$$

Herein, u: the flow of consumption, $U = [0, \gamma]$; v: the decay rate of pollution, e.g., generated by natural cleaning of pollution via winds, rains, etc., $V = [a, b]$; σ: a bounded Lipschitz function.

The associated long-run average social welfare is given by

$$J(x, u, v) = \liminf_{T \to \infty} \frac{1}{T} E[\int_0^T g(u_t) - f(X_t)dt], \tag{4.2}$$

where $g \in C^2(0, \infty)$: the utility of consumption; $f \in C(0, \infty)$: the disutility of pollution.

The objective and literature review

The objective is to find an optimal consumption rate u^* that maximize the long-run average social welfare $J(x, u, v)$ under the worst-case scenario (or robust control) v^*.

<u>Literature Review</u>:

• When the decay rate v is a constant, such problem has been extensively studied, such as Kawaguchi, Morimoto (2007, J. Econ. Dyn. Control); Nguyen, Yin (2016, SICON).

• When the decay rate v is no longer a constant, we refer to Jasso-Fuentes, López-Barrientos (2014, Int. J. Control) for the related study under the additional condition, such as, *non-degenerate condition* and *Lyapunov stability condition* as well as *feedback control*.

Player 1 (economy): the flow of consumption rate u;

Player 2 ("nature"): the decay rate of pollution v.

The dynamics is given by (4.1) and the ergodic payoff is given by (4.2). The related Hamiltonian function H has the form: For $(y, p, A, u, v) \in \mathbb{R}^3 \times U \times V$,

$$H(y, p, A, u, v) = [pu + g(u)] - ypv + \frac{1}{2}|\sigma(y)|^2 A - f(\frac{y + |y|}{2}).$$

The objective is to find the lower ergodic value, which is equal to the value since the Isaacs condition automatically holds.

Corollary 4.1

(i) The ergodic HJBI equation

$$\rho = \sup_{u \in U} \inf_{v \in V} H(y, Dw(y), D^2 w(y), u, v), \ y \in \mathbb{R}. \tag{4.3}$$

has a viscosity solution (ρ, w). Moreover, ρ is the value for the pollution accumulation problem with long-run average social welfare (4.1) and (4.2).

(ii) The optimal consumption rate and the robust decay rate of pollution is given by the following feedback form, respectively,

$$u^*(y) = \bar{u}(Dw(y)), \ v^*(y) = \bar{v}(y, Dw(y)),$$

where the mappings $\bar{u}$ and $\bar{v}$ are given by

$$\bar{u}(p) = argsup_{u \in U}[pu + g(u)], \ \bar{v}(y, p) = arginf_{v \in V}(-ypv).$$

A simple case: $g(u) = 2u^{\frac{1}{2}}$, $f(x) = d \cdot x$

Let $d > 0$ be some constant. Then

$$\rho = -\frac{d}{a}dist^2(\frac{a}{d}, [0, \sqrt{\gamma}]) + \frac{a}{d}, \ w(x) = -\frac{d}{a}x,$$

is a classical solution of ergodic HJBI equation (4.3). Moreover, the optimal consumption rate $u^* = Proj^2_{[0,\sqrt{\gamma}]}(\frac{a}{d})$ and the robust decay rate of pollution $v^* = a$.

<u>Remark</u>: If the lower bound of the decay rate a is bigger than $d\sqrt{\gamma}$, then

$$\rho = 2\sqrt{\gamma} - \gamma\frac{d}{a}, \ u^* = \gamma,$$

which gives the relation between the value of the long-run average welfare and the robust decay rate. Meanwhile, it shows that the robust lower bound of decay rate is at least $d\sqrt{\gamma}$ if one always wants to pursuit the largest consumption flow γ.

Thank you for Your Attention!