BSDEs with weak constraints at stopping time

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Motivation

Financial market :

- $W := (W_t)_{t\in[0,T]}$ a d-dim. Brownian motion defined on $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$.
- Riskless asset $S^0 := (S^0_t)_{t\in[0,T]}$

$$dS^0_t = S^0_t r dt.$$

- Riskly assets $S := (S_t)_{t\in[0,T]}$

$$dS^i_t = S^i_t(\alpha^i_t dt + \sigma^i_t dW_t), \quad i = 1, \cdots, p\,(p \leq d).$$

Motivation

Investment strategy : $(r = 0)$

$(x, (Z_t)_t)$ with associated wealth process $(X_t^{x,Z})_t$ defined as

$$X_t^{x,Z} := x + \sum_{i=1}^{p} \int_0^t Z_u^i \frac{dS_u^i}{S_u^i} = x + \sum_{i=1}^{p} \int_0^t Z_u^i (\sigma_u^i dW_u + \alpha_u^i du), \quad t \in [0, T].$$

$\rightsquigarrow$ Superhedging the American option with payoff ξ :

$$\text{Price of } \xi := Y_0(\xi) := \inf\{x > 0, \exists Z \,;\, X_\tau^{x,Z} \geq \xi_\tau \,, \forall \tau \in \mathcal{T}\}.$$

$$Y_t^Z := x - \int_0^t g(s, Z_s) ds + \int_0^t Z_s dW_s, \; t \in [0, T]$$

$$Y_\tau^Z \geq \xi_\tau, \; \forall \tau \in \mathcal{T} \; \mathbb{P} - a.s.$$

Superhedging price is too high.

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Motivation

In practice, one replaces the too stringent condition

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by a condition of the form

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- $\ell(x) = \mathbf{1}_{x \geq 0} \rightsquigarrow X_\tau^{x,Z} \geq \xi_\tau$ at least with probability m.
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Literature on partial hedging

- Föllmer and Leukert (1999, 2000) ; Treviño Aguilar (2016)
- Bouchard, Elie and Touzi (2010) ; Soner and Touzi (2002,2003)
- Bouchard, Elie and Reveillac (2015) ; Dumitrescu (2016)
- Bouchard, Chassagneux and Bouveret (2016)
- Briand, Elie and Hu (2018)

BSDE with weak constraints

Definition (BSDE with weak constraints)

A pair of predictable processes (Y, Z) $(\in \mathbb{S}_2 \times \mathbb{H}_2)$ is a supersolution of the BSDE with generator $g : \Omega \times [0, T] \times \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ and weak reflections (ψ, μ, τ) if, for any $0 \leq t \leq s \leq T$,

$$Y_t \geq Y_s + \int_t^s g(u, Y_u, Z_u)du - \int_t^s Z_u dW_u \tag{1}$$

$$\mathbb{E}_\tau[\psi(\theta, Y_\theta)] \geq \mu, \quad \text{for all } \theta \geq \tau. \tag{2}$$

- Due to the weak constraints, we do not expect uniqueness of the solution.
- We now introduce the set $\Theta(\tau, \mu)$ of (τ, μ)-initial supersolutions, which is defined as follows :

$$\Theta(\tau, \mu) := \{Y_\tau : \ (Y, Z) \text{ supersolution of the BSDE}\}.$$

Equivalent formulation

- Let $\mathscr{A}_{\tau,\mu}$ denote the set of elements $\alpha \in \mathbf{H}_2$ such that

$$M^{\tau,\mu,\alpha} := \mu + \int_{\tau}^{\tau\vee\cdot} \alpha_s dW_s \quad \text{takes values in } [0,1].$$

- For each $\theta \in \mathcal{T}_\tau$ there exists $\alpha^\theta \in \mathscr{A}_{\mu,\tau}$ such that $\mathbb{E}_\tau[\psi(\theta, Y_\theta)] \geq \mu$ is equivalent to $Y_\theta \geq \psi^{-1}(\theta, M_\theta^{\tau,\mu,\alpha^\theta})$ a.s.

Lemma

The condition

$$\mathbb{E}\left[\psi(\theta, Y_\theta)|\mathcal{F}_\tau\right] \geq \mu, \text{ for all } \theta \geq \tau$$

is equivalent to

the existence of a predictable process α such that $Y_\theta \geq \psi^{-1}(\theta, M_\theta^{\tau,\mu,\alpha})$, for all $\theta \geq \tau$.

BSDE with weak constraints

$$Y_t \geq Y_s + \int_t^s g(u, Y_u, Z_u)du - \int_t^s Z_u dW_u \tag{3}$$

$$\mathbb{E}_\tau[\psi(\theta, Y_\theta)] \geq \mu, \quad \text{for all } \theta \geq \tau. \tag{4}$$

Proposition

$(Y, Z) \in \mathbb{S}_2 \times \mathbb{H}_2$ satisfies (3)-(4) if and only if (Y, Z) satisfies (3) and there exists $\alpha \in \mathscr{A}_{\tau,\mu}$ such that $Y_\nu \geq \operatorname*{esssup}_{\theta \in \mathcal{T}_\nu} \mathcal{E}^g_{\nu,\theta}[\psi^{-1}(\theta, M^{\tau,\mu,\alpha}_\theta)]$ a.s. for all $\nu \in \mathcal{T}_\tau$.

We define $\mathcal{Y}(\tau, \mu) := \operatorname*{ess\,inf}_{\alpha \in \mathscr{A}_{\tau,\mu}} \operatorname*{esssup}_{\theta \in \mathcal{T}_\tau} \mathcal{E}^g_{\tau,\theta}[\psi^{-1}(\theta, M^{\tau,\mu,\alpha}_\theta)]$ and we have

Proposition

$$\operatorname{ess\,inf} \Theta(\tau, \mu) = \mathcal{Y}(\tau, \mu) \ a.s.$$

The g-expectation of ξ at stopping time $\tau_1 \leq \tau_2$ is defined as $\mathcal{E}^g_{\tau_1,\tau_2}[\xi] := Y_{\tau_1}$, where

$$Y = \xi + \int_{.\wedge\tau_2}^{\tau_2} g(s, Y_s, Z_s)ds - \int_{.\wedge\tau_2}^{\tau_2} Z_s dW_s.$$

BSDEs with weak constraints

For an initial threshold m_0 and a given admissible control α, we define $M_t^\alpha := M_t^{0,m_0,\alpha}$ and

$$\begin{aligned} \mathcal{Y}_\tau^\alpha &:= \operatorname*{ess\,inf}_{\alpha' \in \mathscr{A}_\tau^\alpha} \operatorname*{esssup}_{\theta \in \mathcal{T}_\tau} \mathcal{E}_{\tau,\theta}^g[\psi^{-1}(\theta, M_\theta^\alpha)] \\ &= \operatorname*{ess\,inf}_{\alpha' \in \mathscr{A}_\tau^\alpha} \mathscr{Y}_{\tau,T}^{g,\alpha}[\psi^{-1}(T, M_T^\alpha)] \end{aligned}$$

Theorem (DPP)

For all $(\tau_1, \tau_2, \alpha) \in \mathcal{T}_0 \times \mathcal{T}_0 \times \mathscr{A}_0$ such that $\tau_2 \in \mathcal{T}_{\tau_1}$ a.s., we have :

$$\mathcal{Y}^\alpha(\tau_1) = \operatorname*{ess\,inf}_{\alpha' \in \mathscr{A}_{\tau_1}^\alpha} \mathscr{Y}_{\tau_1,\tau_2}^{g,\alpha'}[\mathcal{Y}^{\alpha'}(\tau_2)] \text{ a.s.}$$

and consequently $\mathcal{Y}^\alpha$ is a $\mathscr{Y}^{g,\alpha}$-submartingale process.

Definition (*Strong $\mathscr{Y}^{g,\xi}$-submartingale* process)

An optional process $(Y_t) \in \mathbf{S}_2$ satisfying $Y_\sigma \geq \xi_\sigma$ a.s. for all $\sigma \in \mathcal{T}_0$ and such that $\mathbb{E}[\operatorname*{esssup}_{\tau \in \mathcal{T}_0}(Y_\tau)^2] < \infty$ is said to be a strong $\mathscr{Y}^{g,\xi}$-submartingale if $Y_S \leq \mathscr{Y}_{S,\tau}^{g,\xi}(Y_\tau)$ a.s. on $S \leq \tau$, for all $S, \tau \in \mathcal{T}_0$.

Mertens decomposition

Theorem ($\mathscr{Y}^{g,\xi}$-Mertens decomposition)

Let (Y_t) be an optional process and (ξ_t) be a right-continuous left-limited strong semimartingale. The process (Y_t) is a strong $\mathscr{Y}^{g,\xi}$-submartingale process if and only if there exist two non-decreasing right-continuous predictable processes $A, K \in \mathbf{K}_2$ such that $A_0 = 0$ and $K_0 = 0$, a non-decreasing right-continuous adapted purely discontinuous process C' in $\mathbf{S}_2$ with $C'_{0-} = 0$ and a process $Z \in \mathbf{H}_2$ such that a.s. for all $t \in [0, T]$,

$$\begin{cases} Y_t = Y_T + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + A_T - A_t - K_T + K_t - C'_{T-} + C'_{t-}, \\ Y_t \geq \xi_t \text{ a.s. ;} \\ \displaystyle\int_0^T (Y_{s-} - \xi_{s-})dA_s = 0 \text{ a.s.; a.s. for all } \tau \in \mathcal{T}_0; \\ dA_t \perp dK_t. \end{cases} \tag{5}$$

Moreover, this decomposition is unique.

Mertens decomposition

- Since (Y_t) is a strong $\mathscr{Y}^{g,\xi}$-submartingale and thanks to the characterization of the solution of a reflected BSDE in terms of an optimal stopping problem with g-expectations, we have

$$Y_S \leq \operatorname*{ess\,inf}_{\tau \in \mathcal{T}_S} \operatorname*{esssup}_{S' \in \mathcal{T}_S} \mathcal{E}^g_{S,S'\wedge\tau}(Y_\tau \mathbf{1}_{S' \geq \tau} + \xi_{S'} \mathbf{1}_{S' < \tau}) \text{ a.s.}$$

- (ξ_t) is RCLL , (Y_t) is r.l.s.c. and the Mokobodzki's condition is satisfied, then the process (Y_t) coincides with the first component of the solution of the doubly reflected BSDE associated with obstacles (Y_t) and (ξ_t).
- For the converse implication, the reflected BSDE (5) can be seen as a reflected BSDE associated to the *generalized driver* $g(t,\omega,y,z)dt - dK_t - dC'_{t-}$.
- $$Y_S = \operatorname*{esssup}_{S' \in \mathcal{T}_S} \mathcal{E}^{g-dK-dC'}_{S,S'\wedge\tau} \left[Y_\tau \mathbf{1}_{\tau \leq S'} + \xi_{S'} \mathbf{1}_{S' < \tau} \right] \text{ a.s.}$$

BSDE representation

Theorem (Reflected BSDE representation of the minimal t-values process)

There exists a family $(\mathcal{Z}^\alpha, \mathcal{A}^\alpha, \mathcal{K}^\alpha)_{\alpha\in\mathscr{A}_0}$ such that, for all $\alpha \in \mathscr{A}_0$, we have, for all $0 \le t \le T$,

$$\begin{cases} \mathcal{Y}_t^\alpha = \psi^{-1}(T, M_T^\alpha) + \int_t^T g(s, \mathcal{Y}_s^\alpha, \mathcal{Z}_s^\alpha)ds - \int_t^T \mathcal{Z}_s^\alpha dW_s + \mathcal{K}_t^\alpha - \mathcal{K}_T^\alpha - \mathcal{A}_t^\alpha + \mathcal{A}_T^\alpha; \\ \mathcal{Y}_t^\alpha \ge \psi^{-1}(t, M_t^\alpha) \text{ a.s.}; \\ \int_0^T \left(\mathcal{Y}_{s-}^\alpha - \psi^{-1}(s, M_{s-}^\alpha)\right) d\mathcal{A}_s^\alpha = 0 \quad a.s.; \quad d\mathcal{A}^\alpha \perp d\mathcal{K}^\alpha; \\ \operatorname*{ess\,inf}_{\alpha'\in\mathscr{A}_\tau^\alpha} \mathbb{E}[\int_\tau^T \exp(\delta_s^{\tau,\alpha'})d(A_s^{\alpha'} - \mathcal{A}_s^{\alpha'} + \mathcal{K}_s^{\alpha'})] = 0 \text{ a.s., for all } \tau \in \mathcal{T}_0; \end{cases}$$

where $\delta_s^{t,\alpha} := \int_t^s \{\beta_u^\alpha dW_u + (\lambda_u^\alpha - \frac{(\beta_u^\alpha)^2}{2})du\}$, *with*

$$\lambda_s^\alpha := \frac{g(s, \mathcal{Y}_s^\alpha, \mathcal{Z}_s^\alpha) - g(s, Y_s^\alpha, \mathcal{Z}_s^\alpha)}{\mathcal{Y}_s^\alpha - Y_s^\alpha} \mathbf{1}_{\{\mathcal{Y}_s^\alpha - Y_s^\alpha \neq 0\}};$$

$$\beta_s^\alpha := \frac{g(s, Y_s^\alpha, \mathcal{Z}_s^\alpha) - g(s, Y_s^\alpha, Z_s^\alpha)}{|\mathcal{Z}_s^\alpha - Z_s^\alpha|^2} (\mathcal{Z}_s^\alpha - Z_s^\alpha) \mathbf{1}_{\{\mathcal{Z}_s^\alpha - Z_s^\alpha \neq 0\}},$$

and $(Y^\alpha, Z^\alpha, A^\alpha)$ the solution of the reflected BSDE with driver g and obstacle $\psi^{-1}(\cdot, M_\cdot^\alpha)$.

Questions ?

Thank you for your attention !