A Mean-Field Control Problem of Optimal Portfolio Liquidation with Semimartingale Strategies

Xiaonyu Xia
joint work with Ulrich Horst and Guanxing Fu

Wenzhou University
Overview

1. Portfolio liquidation models with self-exciting order flow
2. Models with càdlàg semimartingale strategies
3. Numerical simulations
4. Conclusion
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Benchmark model: Graewe & Horst ’17

The large investor’s stochastic control problem is given by

\[
\operatorname{ess\ inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E} \left[ \int_0^T \left\{ \eta \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2 \right\} \, ds \right]
\]

subject to the state dynamics

\[
\begin{align*}
X_t &= \mathcal{X} - \int_0^t \xi_s \, ds, \quad t \in [0, T], \\
X_T &= 0, \\
Y_t &= \int_0^t \left\{ -\rho_s Y_s + \gamma \xi_s \right\} \, ds, \quad t \in [0, T].
\end{align*}
\]
Additional feedback effect

- Large selling orders may have an impact on future price dynamics
  - diminish the pool of counterparties and/or generate herding effects where other market participants start selling (or buying) in anticipation of further price decreases (or increases)
  - attract predatory traders that employ front-running strategies (See Brunnermeier & Pedersen '05; Carlin et al. '07; Schied & Schöneborn '09 for an in-depth analysis of predatory trading)
Hawkes process

- Market order dynamics follow Hawkes processes whose base intensities depend on the large investors’ trading activities
  - Hawkes processes: a powerful tool to model self-exciting order flow and its impact on stock price volatility (Bacry et al. ’13, ’15; El Euch et al. ’18; Jaisson and Rosenbaum ’15; Horst and Xu ’19)
  - in the context of liquidation models (Alfonsi & Blanc’16; Amaral & Papanicolaou ’19; Cartea et al. ’18)
Additional transient price impact

- Market order dynamics follows a Hawkes process with exponential kernel
  \[ \zeta_t^{\pm} := \mu_t + \xi_t^{\pm} + \alpha \int_0^t e^{-\beta(t-s)} dN_s^{\pm} \]

- Expected number of (net) sell orders
  \[ \tilde{Z}_t = \mathbb{E}[\tilde{Z}_t^+ - \tilde{Z}_t^-] = \int_0^t \mathbb{E}[\xi_s] ds + \alpha \int_0^t e^{-\beta(t-s)} \tilde{Z}_s ds \quad \underbrace{C_t}_{\text{(mean-reverting if } \alpha < \beta\text{)}} \]

- Expected number of (net) sell child orders follows the dynamics
  \[ dC_t = \left( -(\beta - \alpha)C_t + \alpha(\mathbb{E}[X] - \mathbb{E}[X_t]) \right) dt, \quad C_0 = 0. \]
The mean-field type control problem with absolutely continuous controls

A mean-field type control problem for the large investor:

$$\operatorname{ess \ inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E} \left[ \int_0^T \left\{ \eta_s \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2 \right\} \, ds \right]$$

subject to the state dynamics

\[
\begin{cases}
   dX_t = -\xi_t \, dt, & t \in [0, T], \\
   dY_t = \left( -\rho_t Y_t + \gamma_t (\xi_t + C'_t) \right) \, dt, & t \in [0, T], \\
   dC_t = \left( -\left( \beta - \alpha \right) C_t + \alpha (\mathbb{E}[X] - \mathbb{E}[X_t]) \right) \, dt, & t \in [0, T], \\
   X_0 = \mathcal{X}, \quad X_T = 0, \quad Y_0 = 0, \quad C_0 = 0.
\end{cases}
\]
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Model settings

Consider the mean-field control problem with càdlàg semimartingale strategies:

$$\min_{Z \in \mathcal{A}} \mathbb{E} \left[ \int_0^T \left( Y_t - dZ_t + \frac{\gamma_2}{2} d[Z]_t + \sigma_t d[Z, W]_t \right) + \int_0^T \lambda X_t^2 \, dt \right]$$

subject to the state dynamics

$$\begin{aligned}
    dX_s &= -dZ_s \\
    dY_s &= \left( -\rho Y_s + \gamma_1 C'_s \right) ds + \gamma_2 \, dZ_s + \sigma_s \, dW_s \\
    dC_s &= -\left( \beta - \alpha \right) C_s ds + \alpha(\mathbb{E}[x_0] - \mathbb{E}[X_s]) ds \\
    X_0 &= x_0; \ Y_0 = C_0 = 0; \ X_T = 0.
\end{aligned}$$

for $0 \leq s \leq T$ where the set of admissible control is given by

$$\mathcal{A} = \{ Z : Z \text{ is an } \mathbb{F} \text{ semimartingale with } [Z]_s < \infty \text{ for all } s \in [0, T] \}.$$
Existing literature

- Mean field control problems
  - with singular controls: Fu & Horst ’17; Guo et al. ’20; Hafayed ’13; Hu et al. ’17
  - with càdlàg semimartingale strategies: Fu & Horst & X. ’22
- Non-Markovian singular control problems
  - one-dimensional non-Markovian setting: Bank & Karoui ’04; Bank ’05; Bank & Riedel ’01
  - multi-dimensional non-Markovian setting: Ackermann & Kruse & Urusov ’21; Fu & Horst & X. ’22
Heuristic result

The value function depends on the state process only through its distribution and that it is of linear quadratic form driven by three deterministic processes and a BSDE:

\[ V(t, \mathcal{X}) = \text{Var}(\mu)(A_t) + \bar{\mu}^T B_t \bar{\mu} + D_t^T \bar{\mu} + \mathbb{E}[F_t]. \]

where

\[
\begin{align*}
- A_{11} + \gamma_2 A_{21} &= 0 \\
- A_{12} + \gamma_2 A_{22} + \frac{1}{2} &= 0 \\
- A_{13} + \gamma_2 A_{23} &= 0 \\
- B_{11} + \gamma_2 B_{21} &= 0 \\
- B_{12} + \gamma_2 B_{22} + \frac{1}{2} &= 0 \\
- B_{13} + \gamma_2 B_{23} &= 0 \\
- D_1 + \gamma_2 D_2 &= 0.
\end{align*}
\]
The process $A$ satisfies a standard ODE system:

\[
\begin{align*}
\dot{A}_{11,t} &= \left(-\lambda + \frac{(\rho A_{11,t} + \lambda)^2}{\gamma_2 \rho + \lambda}\right) \\
\dot{A}_{13,t} &= \left(\frac{\gamma_1(\beta - \alpha)}{\gamma_2} A_{11,t} + (\beta - \alpha) A_{13,t} - \frac{(\rho A_{11,t} + \lambda)(\gamma_1(\beta - \alpha) - 2 \rho A_{13,t})}{2(\gamma_2 \rho + \lambda)}\right) \\
\dot{A}_{33,t} &= \left(2(\beta - \alpha) A_{33,t} + 2 \frac{\gamma_1(\beta - \alpha)}{\gamma_2} A_{13,t} + \frac{(\gamma_1(\beta - \alpha) - 2 \rho A_{13,t})^2}{4(\gamma_2 \rho + \lambda)}\right)
\end{align*}
\]

\[
A_{11,T} = \frac{\gamma_2}{2}, \quad A_{13,T} = 0, \quad A_{33,T} = 0.
\]
The process $B$ satisfies the fully coupled system of Riccati-type equations:

$$
\begin{align*}
\dot{B}_{11,t} &= \left(2\frac{\gamma_1 \alpha}{\gamma_2} B_{11,t} + 2\alpha B_{13,t} - \lambda + \frac{\left(\cdots B_{11,t} + \cdots B_{13,t} + \cdots\right)^2}{4\gamma_2^2(\gamma_2 \rho - \gamma_1 \alpha + \lambda)} \right) \\
\dot{B}_{33,t} &= \left(2(\beta - \alpha) B_{33,t} + 2\frac{\gamma_1 (\beta - \alpha)}{\gamma_2} B_{13,t} + \frac{\left(\cdots B_{13,t} + \cdots B_{33,t} + \cdots\right)^2}{4\gamma_2^2(\gamma_2 \rho - \gamma_1 \alpha + \lambda)} \right) \\
\dot{B}_{13,t} &= \left\{ \frac{\gamma_1 (\beta - \alpha)}{\gamma_2} B_{11,t} + \alpha B_{33,t} + (\beta - \alpha + \frac{\gamma_1 \alpha}{\gamma_2}) B_{13,t} \\
&\quad + \frac{\left(\cdots B_{11,t} + \cdots B_{13,t} + \cdots\right) \cdot \left(\cdots B_{13,t} + \cdots B_{33,t} + \cdots\right)}{4\gamma_2^2(\gamma_2 \rho - \gamma_1 \alpha + \lambda)} \right\} \\
B_{11,T} &= \frac{\gamma_2}{2}, \quad B_{13,T} = 0, \quad B_{33,T} = 0.
\end{align*}
$$
The vector-valued process $D$ satisfies the coupled linear ODE system:

\[
\begin{align*}
\dot{D}_{1,t} &= \left\{-\frac{2\alpha \gamma_1 E[x_0]}{\gamma_2} B_{11,t} - 2\alpha \gamma_1 E[x_0] B_{13,t} + \frac{\gamma_1 \alpha}{\gamma_2} D_{1,t} + \alpha D_{3,t} \\
&\quad + \left(-2\lambda \gamma_2 + 2(\gamma_1 \alpha - \gamma_2 \rho) B_{11,t} + \gamma_1 \gamma_2 \alpha + \alpha \gamma_2 B_{13,t}\right)\right. \\
&\quad \left. \cdot \frac{-\gamma_1 \gamma_2 \alpha E[x_0] + (\gamma_1 \alpha - \gamma_2 \rho) D_{1,t} + \alpha \gamma_2 D_{3,t}}{2 \gamma_2^2 (\gamma_2 \rho - \gamma_1 \alpha + \lambda)} \right\}
\end{align*}
\]

\[
\begin{align*}
\dot{D}_{3,t} &= \left\{-2\alpha E[x_0] B_{33,t} - \frac{2\gamma_1 \alpha E[x_0]}{\gamma_2} B_{13,t} + \frac{\gamma_1 (\beta - \alpha)}{\gamma_2} D_{1,t} + (\beta - \alpha) D_{3,t} \\
&\quad + \left(2(\gamma_1 \alpha - \gamma_2 \rho) B_{13,t} + 2\gamma_2 \alpha B_{33,t} + \gamma_1 \gamma_2 (\beta - \alpha)\right)\right. \\
&\quad \left. \cdot \frac{-\gamma_1 \gamma_2 \alpha E[x_0] + (\gamma_1 \alpha - \gamma_2 \rho) D_{1,t} + \alpha \gamma_2 D_{3,t}}{2 \gamma_2^2 (\gamma_2 \rho - \gamma_1 \alpha + \lambda)} \right\}
\end{align*}
\]

\[
D_{1,T} = D_{3,T} = 0
\]
The process $F$ satisfies the BSDE because of the randomness of the volatility $\sigma$:

$$-dF_t = \begin{cases} 
\sigma_t^2 \frac{2A_{11} - \gamma_2}{2\gamma_2} + \alpha \gamma_1 \mathbb{E}[x_0] \frac{D_1}{\gamma_2} + \alpha \mathbb{E}[x_0] D_3 \\
- \frac{1}{4(\lambda + \gamma_2 \rho - \alpha \gamma_1)} \left(-\alpha \gamma_1 \mathbb{E}[x_0] + (\gamma_1 \alpha - \gamma_2 \rho) \frac{D_1}{\gamma_2} + \alpha D_3 \right)^2 \\
- Z_t \, dW_t 
\end{cases} \, dt$$

$F_T = 0.$
Assumptions

We assume throughout that the following standing assumption holds.

1. The coefficients $\gamma_1$, $\gamma_2$, $\alpha$, $\beta$ and $\lambda$ are nonnegative constants.
2. The coefficients satisfy $\beta - \alpha > 0$ and $\gamma_2 \rho - \gamma_1 \alpha + \lambda > 0$.
3. The initial position $x_0$ is assumed to be an integrable r.v. that is independent of the Brownian motion. The volatility process $\sigma$ is a square integrable progressively measurable process. In particular, $\sigma$ is allowed to be degenerate.
Wellposedness of the Riccati Equation

**Theorem (Fu & Horst & X. ’22)**

*In addition to the standing assumption, we assume $\alpha$ is small enough and $\gamma_2 \rho > 0$. Then the matrix Riccati equation of $B$ admits a unique solution in $L^\infty([0, T]; \mathbb{R}^3) \cap C([0, T]; \mathbb{R}^3)$.*

The system of $B$ can be rewritten in the matrix form as:

$$
\begin{cases}
\dot{P} = \left( P_t N_2 N_0 N_2^\top P_t + N_1 P_t + P_t N_1^\top - M \right) \\
P_T = G,
\end{cases}
$$

where

$$
P = \begin{pmatrix} B_{11} & B_{13} \\ B_{13} & B_{33} \end{pmatrix}, \quad M = -\begin{pmatrix} \frac{\gamma_1^2 \alpha^2 - 4 \lambda \gamma_2 \rho}{4(\gamma_2 \rho - \gamma_1 \alpha + \lambda)} & \frac{\gamma_1 (\beta - \alpha)(\gamma_1 \alpha - 2 \lambda)}{4(\gamma_2 \rho - \gamma_1 \alpha + \lambda)} \\ \frac{\gamma_1 (\beta - \alpha)(\gamma_1 \alpha - 2 \lambda)}{4(\gamma_2 \rho - \gamma_1 \alpha + \lambda)} & \frac{\gamma_1^2 (\beta - \alpha)^2}{4(\gamma_2 \rho - \gamma_1 \alpha + \lambda)} \end{pmatrix},
$$

$$
G = \begin{pmatrix} \gamma_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad N_1, N_2, N_0 = \cdots.
$$
Our ideas

Define

\[ \tilde{\mathcal{P}} = \mathcal{P} + \tilde{\Lambda}, \]

where the matrix \( \tilde{\Lambda} \) is given by

\[ \tilde{\Lambda} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \Lambda \alpha^2 & \Lambda \alpha (\beta - \alpha) \\ \Lambda \alpha (\beta - \alpha) & \Lambda (\beta - \alpha)^2 \end{pmatrix} . \]

The process \( \tilde{\mathcal{P}} \) satisfies the dynamics

\[ \tilde{\mathcal{P}}' = \tilde{\mathcal{P}} \mathcal{N}_2 \mathcal{N}_0 \mathcal{N}_2^\top \tilde{\mathcal{P}} + \tilde{\mathcal{N}}_1 \tilde{\mathcal{P}} + \tilde{\mathcal{P}} \tilde{\mathcal{N}}_1^\top - \tilde{\mathcal{M}}, \]

\[ \tilde{\mathcal{P}}_T = \mathcal{G} + \tilde{\Lambda}, \]

where

\[ \tilde{\mathcal{M}} = -\tilde{\Lambda} \mathcal{N}_2 \mathcal{N}_0 \mathcal{N}_2^\top \tilde{\Lambda} + \mathcal{N}_1 \tilde{\Lambda} + \tilde{\Lambda} \mathcal{N}_1^\top + \mathcal{M} \geq 0. \]
Main results

Theorem (Fu & Horst & X. ’22)

If the standing assumption is satisfied, and if either \( \lambda = 0 \) or \( \lambda > 0 \) and \( \alpha \) is small enough, then the following holds.

i) In terms of the processes \( A, B, D, F \) introduced before, the value function is given by

\[
V(t, \mathcal{X}) = V(t, \mu) = \text{Var}(\mu)(A_t) + \bar{\mu}^T B_t \bar{\mu} + D_t^T \bar{\mu} + \mathbb{E}[F_t].
\]

In particular, the value function of the original control problem follows

\[
V(0, \mathcal{X}) = \text{Var}(\mu)(A_0) + \bar{\mu}^T B_0 \bar{\mu} + D_0^T \bar{\mu} + \mathbb{E}[F_0].
\]
Theorem (Fu & Horst & X. ’22)

ii) The optimal strategy jumps only at the beginning and the end of the trading period where the initial and terminal jump is given by

\[ \Delta Z_t = -\frac{l^A_t}{\tilde{a}} (x_t - \bar{\mu}) - \frac{l^B_t}{a} \bar{\mu} - \frac{l^D_t}{a} \quad \text{and} \quad \Delta Z_T = x_T \]

respectively. On the time interval \((t, T)\) the optimal strategy satisfies the dynamics

\[
dZ_s = \left( -\frac{j^A_s}{\tilde{a}} (x_s - \mathbb{E}[x_s]) - \frac{j^B_s}{a} \mathbb{E}[x_s] - \frac{j^D_s}{a} - \frac{j^A_s}{\tilde{a}} \mathcal{H}(x_s - \mathbb{E}[x_s]) - \frac{j^B_s}{a} ((\mathcal{H} + \overline{\mathcal{H}}) \mathbb{E}[x_s] + \mathcal{G}) \right) ds - \frac{j^A_s}{\tilde{a}} \mathcal{D}_s dW_s, \quad s \in (t, T)
\]
Theorem (Fu & Horst & X. ’22)

iii) The optimal state dynamics reads

\[ dX_s = \left( HH X_s + \overline{H \mathbb{E}}[X_s] + G \right) ds + D_s dW_s + K dZ_s, \quad s \in [t, T), \]

where the initial value is given by

\[ X_t = \begin{pmatrix} X_t^- - \Delta Z_t \\ Y_t^- + \gamma_2 \Delta Z_t \\ C_t^- \end{pmatrix}^T. \]
Verification

**Proposition (Fu & Horst & X. '22)**

Let $\gamma_2 \rho - \gamma_1 \alpha + \lambda > 0$. Then the cost functional can be rewritten as

$$J(t, Z) := \mathbb{E}\left[ \int_t^T \frac{1}{\tilde{a}} \left( I_s^A (X_s - \bar{\mu}_s) \right)^2 ds + \int_t^T \frac{1}{a} \left( I_s^B \bar{\mu}_s + I_s^D \right)^2 ds \right]$$

$$+ \text{Var}(\mu_t^-)(A_t) + \bar{\mu}_t^T B_t \bar{\mu}_t^- + D_t^T \bar{\mu}_t^- + \mathbb{E}[F_t].$$

In particular, the cost functional reaches its global minimum if

$$\int_t^T \left( I_s^A (X_s - \bar{\mu}_s) \right)^2 ds = \int_t^T \left( I_s^B \bar{\mu}_s + I_s^D \right)^2 ds = 0, \quad \text{a.s.}$$
The optimal strategy

**Theorem**

The candidate state process given above satisfies

\[ I^A(\tilde{X}_s - \mathbb{E}[\tilde{X}_s]) = 0, \quad I^B\mathbb{E}[\tilde{X}_s] + I^D = 0, \quad s \in [t, T). \]
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Risk parameter dependence

**Figure:** Dependence of the optimal strategy on the risk parameter $\lambda$ for $\lambda = 1.5$ (left) and $\lambda = 0$ (right). Other parameters are chosen as $\rho = 0.7$, $\gamma_1 = 0.1$, $\gamma_2 = 0.5$, $\alpha = 0.5$, $\beta = 1.1$. 
Child order flow parameter dependence

**Figure:** Dependence of the optimal strategy on the impact parameter $\alpha$ for $\alpha = 0$ (left) and $\alpha = 1.8$ (right). Other parameters are chosen as $\rho = 0.4$, $\lambda = 0$, $\gamma_1 = 0.1$, $\gamma_2 = 0.5$, $\beta = 3$. 
Transitient market impact parameter dependence

Figure: Dependence of the optimal strategy on the impact parameter $\gamma_2$ for $\gamma_2 = 2$ (left) and $\gamma_2 = 0.3$ (right). Other parameters are chosen as $\rho = 0.7$, $\lambda = 0$, $\gamma_1 = 0.1$, $\alpha = 0.5$, $\beta = 1.1$. 
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Conclusion

- We considered a mean-field control problem with càdlàg semimartingale strategies arising in portfolio liquidation models with transient market impact and self-exciting order flow.
- We showed that the value function can be described in terms of the solution to a fully coupled system of Riccati equations.
- We obtained that the optimal strategy jumps only at the beginning and the end of the trading period.
The end.

Thank you!