# A Mean-Field Control Problem of Optimal Portfolio Liquidation with Semimartingale Strategies

**Xiaonyu Xia**

joint work with Ulrich Horst and Guanxing Fu

Wenzhou University

# Overview

1. Portfolio liquidation models with self-exciting order flow
2. Models with càdlàg semimartingale strategies
3. Numerical simulations
4. Conclusion

# Overview

# Benchmark model: Graewe & Horst '17

The large investor's stochastic control problem is given by

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \{\eta \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2\}\, ds\right]$$

subject to the state dynamics

$$\begin{cases} X_t = \mathcal{X} - \int_0^t \xi_s\, ds, \quad t \in [0, T], \\ X_T = 0, \\ Y_t = \int_0^t \{-\rho_s Y_s + \gamma \xi_s\}\, ds, \quad t \in [0, T]. \end{cases}$$

# Additional feedback effect

- Large selling orders may have an impact on future price dynamics
  - diminish the pool of counterparties and/or generate herding effects where other market participants start selling (or buying) in anticipation of further price decreases (or increases)
  - attract predatory traders that employ front-running strategies (See Brunnermeier & Pedersen '05; Carlin et al. '07; Schied & Schöneborn '09 for an in-depth analysis of predatory trading)

# Hawkes process

- Market order dynamics follow Hawkes processes whose base intensities depend on the large investors' trading activities
  - Hawkes processes:a powerful tool to model self-exciting order flow and its impact on stock price volatility (Bacry et al. '13, '15; El Euch et al. '18; Jaisson and Rosenbaum '15; Horst and Xu '19)
  - in the context of liquidation models (Alfonsi & Blanc'16; Amaral & Papanicolaou '19; Cartea et al. '18)

# Additional transient price impact

- Market order dynamics follows a Hawkes process with exponential kernel

$$\zeta_t^{\pm} := \mu_t + \xi_t^{\pm} + \alpha \int_0^t e^{-\beta(t-s)} dN_s^{\pm}$$

- Expected number of (net) sell orders

$$\bar{Z}_t = \mathbb{E}[\bar{Z}_t^+ - \bar{Z}_t^-] = \int_0^t \mathbb{E}[\xi_s] ds + \underbrace{\alpha \int_0^t e^{-\beta(t-s)} \bar{Z}_s ds}_{C_t}.$$

- Expected number of (net) sell child orders follows the dynamics

$$dC_t = (-(\beta - \alpha) C_t + \alpha(\mathbb{E}[\mathcal{X}] - \mathbb{E}[X_t]))\, dt, \quad C_0 = 0.$$

(mean-reverting if $\alpha < \beta$)

# The mean-field type control problem with absolutely continuous controls

A mean-field type control problem for the large investor:

$$\operatorname*{ess\,inf}_{\xi \in L^2_{\mathcal{F}}(0,T;\mathbb{R})} \mathbb{E}\left[\int_0^T \left\{\eta_s \xi_s^2 + \xi_s Y_s + \lambda_s X_s^2\right\} ds\right]$$

subject to the state dynamics

$$\begin{cases} dX_t = -\xi_t \, dt, \quad t \in [0, T], \\ dY_t = \left(-\rho_t Y_t + \gamma_t(\xi_t + C_t')\right) dt, \quad t \in [0, T], \\ dC_t = \left(-(\beta - \alpha) C_t + \alpha(\mathbb{E}[\mathcal{X}] - \mathbb{E}[X_t])\right) dt, \quad t \in [0, T], \\ X_0 = \mathcal{X}, \; X_T = 0, \; Y_0 = 0, \; C_0 = 0. \end{cases}$$

# Overview

# Model settings

Consider the mean-field control problem with càdlàg semimartingale strategies:

$$\min_{Z\in\mathscr{A}} \mathbb{E}\left[\int_0^T \left(Y_{t-}\, dZ_t + \frac{\gamma_2}{2}\, d[Z]_t + \sigma_t d[Z,W]_t\right) + \int_0^T \lambda X_t^2\, dt\right]$$

subject to the state dynamics

$$\begin{cases} dX_s = -dZ_s \\ dY_s = \Big(-\rho Y_s + \gamma_1 C_s'\Big)\, ds + \gamma_2\, dZ_s + \sigma_s\, dW_s \\ dC_s = -(\beta-\alpha) C_s ds + \alpha(\mathbb{E}[x_0] - \mathbb{E}[X_s])\, ds \\ X_{0-} = x_0;\ \ Y_{0-} = C_{0-} = 0;\ \ X_T = 0. \end{cases}$$

for $0 \le s \le T$ where the set of admissible control is given by

$$\mathscr{A} = \{Z : Z \text{ is an } \mathbb{F} \text{ semimartingale with } [Z]_s < \infty \text{ for all } s \in [0,T]\}.$$

# Existing literature

- Mean field control problems
  - with singular controls: Fu & Horst '17; Guo et al. '20; Hafayed '13; Hu et al. '17
  - with càdlàg semimartingale strategies: Fu & Horst & X. '22
- Non-Markovian singular control problems
  - one-dimensional non-Markovian setting: Bank & Karoui '04; Bank '05; Bank & Riedel '01
  - multi-dimensional non-Markovian setting: Ackermann & Kruse & Urusov '21; Fu & Horst & X. '22

# Heuristic result

The value function depends on the state process only through its distribution and that it is of linear quadratic form driven by three deterministic processes and a BSDE:

$$V(t,\mathcal{X}) = \mathrm{Var}(\mu)(A_t) + \bar{\mu}^\top B_t \bar{\mu} + D_t^\top \bar{\mu} + \mathbb{E}[F_t].$$

where

$$\begin{cases} -A_{11} + \gamma_2 A_{21} = 0 \\ -A_{12} + \gamma_2 A_{22} + \dfrac{1}{2} = 0 \\ -A_{13} + \gamma_2 A_{23} = 0 \\ -B_{11} + \gamma_2 B_{21} = 0 \\ -B_{12} + \gamma_2 B_{22} + \dfrac{1}{2} = 0 \\ -B_{13} + \gamma_2 B_{23} = 0 \\ -D_1 + \gamma_2 D_2 = 0. \end{cases}$$

The process $A$ satisfies a standard ODE system:

$$\begin{cases} \dot{A}_{11,t} = \left(-\lambda + \dfrac{(\rho A_{11,t} + \lambda)^2}{\gamma_2 \rho + \lambda}\right) \\ \dot{A}_{13,t} = \left(\dfrac{\gamma_1(\beta - \alpha)}{\gamma_2} A_{11,t} + (\beta - \alpha) A_{13,t} - \dfrac{(\rho A_{11,t} + \lambda)\left(\gamma_1(\beta - \alpha) - 2\rho A_{13,t}\right)}{2(\gamma_2 \rho + \lambda)}\right) \\ \dot{A}_{33,t} = \left(2(\beta - \alpha) A_{33,t} + 2\dfrac{\gamma_1(\beta - \alpha)}{\gamma_2} A_{13,t} + \dfrac{\left(\gamma_1(\beta - \alpha) - 2\rho A_{13,t}\right)^2}{4(\gamma_2 \rho + \lambda)}\right) \\ A_{11,T} = \dfrac{\gamma_2}{2}, \quad A_{13,T} = 0, \quad A_{33,T} = 0. \end{cases}$$

The process $B$ satisfies the fully coupled system of Riccati-type equations:

$$\begin{cases} \dot{B}_{11,t} = \left( 2\dfrac{\gamma_1\alpha}{\gamma_2} B_{11,t} + 2\alpha B_{13,t} - \lambda + \dfrac{\left( \cdots B_{11,t} + \cdots B_{13,t} + \cdots \right)^2}{4\gamma_2^2(\gamma_2\rho - \gamma_1\alpha + \lambda)} \right) \\ \dot{B}_{33,t} = \left( 2(\beta-\alpha) B_{33,t} + 2\dfrac{\gamma_1(\beta-\alpha)}{\gamma_2} B_{13,t} + \dfrac{\left( \cdots B_{13,t} + \cdots B_{33,t} + \cdots \right)^2}{4\gamma_2^2(\gamma_2\rho - \gamma_1\alpha + \lambda)} \right) \\ \dot{B}_{13,t} = \Bigg\{ \dfrac{\gamma_1(\beta-\alpha)}{\gamma_2} B_{11,t} + \alpha B_{33,t} + (\beta - \alpha + \dfrac{\gamma_1\alpha}{\gamma_2}) B_{13,t} \\ \qquad + \dfrac{\left( \cdots B_{11,t} + \cdots B_{13,t} + \cdots \right) \cdot \left( \cdots B_{13,t} + \cdots B_{33,t} + \cdots \right)}{4\gamma_2^2(\gamma_2\rho - \gamma_1\alpha + \lambda)} \Bigg\} \\ B_{11,T} = \dfrac{\gamma_2}{2}, \quad B_{13,T} = 0, \quad B_{33,T} = 0. \end{cases}$$

The vector-valued process $D$ satisfies the coupled linear ODE system:

$$
\begin{cases}
\dot{D}_{1,t} = \Big\{ -\dfrac{2\gamma_1\alpha\mathbb{E}[x_0]}{\gamma_2} B_{11,t} - 2\alpha\mathbb{E}[x_0]B_{13,t} + \dfrac{\gamma_1\alpha}{\gamma_2} D_{1,t} + \alpha D_{3,t} \\
\qquad + \Big( -2\lambda\gamma_2 + 2(\gamma_1\alpha - \gamma_2\rho)B_{11,t} + \gamma_1\gamma_2\alpha + \alpha\gamma_2 B_{13,t} \Big) \\
\qquad \cdot \dfrac{\Big( -\gamma_1\gamma_2\alpha\mathbb{E}[x_0] + (\gamma_1\alpha - \gamma_2\rho)D_{1,t} + \alpha\gamma_2 D_{3,t} \Big)}{2\gamma_2^2(\gamma_2\rho - \gamma_1\alpha + \lambda)} \Big\} \\
\dot{D}_{3,t} = \Big\{ -2\alpha\mathbb{E}[x_0]B_{33,t} - \dfrac{2\gamma_1\alpha\mathbb{E}[x_0]}{\gamma_2} B_{13,t} + \dfrac{\gamma_1(\beta-\alpha)}{\gamma_2} D_{1,t} + (\beta-\alpha)D_{3,t} \\
\qquad + \Big( 2(\gamma_1\alpha - \gamma_2\rho)B_{13,t} + 2\gamma_2\alpha B_{33,t} + \gamma_1\gamma_2(\beta-\alpha) \Big) \\
\qquad \cdot \dfrac{\Big( -\gamma_1\gamma_2\alpha\mathbb{E}[x_0] + (\gamma_1\alpha - \gamma_2\rho)D_{1,t} + \alpha\gamma_2 D_{3,t} \Big)}{2\gamma_2^2(\gamma_2\rho - \gamma_1\alpha + \lambda)} \Big\} \\
D_{1,T} = D_{3,T} = 0
\end{cases}
$$

The process $F$ satisfies the BSDE because of the randomness of the volatility $\sigma$:

$$\begin{cases} -dF_t = \Big\{ \sigma_t^2 \dfrac{2A_{11} - \gamma_2}{2\gamma_2^2} + \alpha\gamma_1 \mathbb{E}[x_0] \dfrac{D_1}{\gamma_2} + \alpha \mathbb{E}[x_0] D_3 \\ \qquad - \dfrac{1}{4(\lambda + \gamma_2\rho - \alpha\gamma_1)} \left( -\alpha\gamma_1 \mathbb{E}[x_0] + (\gamma_1\alpha - \gamma_2\rho) \dfrac{D_1}{\gamma_2} + \alpha D_3 \right)^2 \Big\} dt \\ \qquad - Z_t \, dW_t \\ F_T = 0. \end{cases}$$

# Assumptions

We assume throughout that the following **standing assumption** holds.

1. The coefficients $\gamma_1$, $\gamma_2$, $\alpha$, $\beta$ and $\lambda$ are nonnegative constants.
2. The coefficients satisfy $\beta - \alpha > 0$ and $\gamma_2\rho - \gamma_1\alpha + \lambda > 0$.
3. The initial position $x_0$ is assumed to be an integrable r.v. that is independent of the Brownian motion. The volatility process $\sigma$ is a square integrable progressively measurable process. In particular, $\sigma$ is allowed to be degenerate.

# Wellposedness of the Riccati Equation

**Theorem (Fu & Horst & X. '22)**

*In addition to the standing assumption, we assume $\alpha$ is small enough and $\gamma_2\rho > 0$. Then the matrix Riccati equation of $B$ admits a unique solution in $L^\infty([0,T];\mathbb{R}^3) \cap C([0,T];\mathbb{R}^3)$.*

The system of $B$ can be rewritten in the matrix form as:

$$\begin{cases} \dot{\mathcal{P}} = \left(\mathcal{P}_t\mathcal{N}_2\mathcal{N}_0\mathcal{N}_2^\top\mathcal{P}_t + \mathcal{N}_1\mathcal{P}_t + \mathcal{P}_t\mathcal{N}_1^\top - \mathcal{M}\right) \\ \mathcal{P}_T = \mathcal{G}, \end{cases}$$

where

$$\mathcal{P} = \begin{pmatrix} B_{11} & B_{13} \\ B_{13} & B_{33} \end{pmatrix}, \quad \mathcal{M} = -\begin{pmatrix} \frac{\gamma_1^2\alpha^2 - 4\lambda\gamma_2\rho}{4(\gamma_2\rho-\gamma_1\alpha+\lambda)} & \frac{\gamma_1(\beta-\alpha)(\gamma_1\alpha-2\lambda)}{4(\gamma_2\rho-\gamma_1\alpha+\lambda)} \\ \frac{\gamma_1(\beta-\alpha)(\gamma_1\alpha-2\lambda)}{4(\gamma_2\rho-\gamma_1\alpha+\lambda)} & \frac{\gamma_1^2(\beta-\alpha)^2}{4(\gamma_2\rho-\gamma_1\alpha+\lambda)} \end{pmatrix},$$

$$\mathcal{G} = \begin{pmatrix} \gamma_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_0 = \cdots.$$

# Our ideas

Define

$$\widetilde{\mathcal{P}} = \mathcal{P} + \widetilde{\Lambda},$$

where the matrix $\widetilde{\Lambda}$ is given by

$$\widetilde{\Lambda} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{pmatrix} = \begin{pmatrix} \Lambda\alpha^2 & \Lambda\alpha(\beta-\alpha) \\ \Lambda\alpha(\beta-\alpha) & \Lambda(\beta-\alpha)^2 \end{pmatrix}.$$

The process $\widetilde{\mathcal{P}}$ satisfies the dynamics

$$\begin{aligned} \widetilde{\mathcal{P}}' &= \widetilde{\mathcal{P}}\mathcal{N}_2\mathcal{N}_0\mathcal{N}_2^\top\widetilde{\mathcal{P}} + \widetilde{\mathcal{N}_1}\widetilde{\mathcal{P}} + \widetilde{\mathcal{P}}\widetilde{\mathcal{N}_1}^\top - \widetilde{\mathcal{M}}, \\ \widetilde{\mathcal{P}}_T &= \mathcal{G} + \widetilde{\Lambda}, \end{aligned}$$

where

$$\widetilde{\mathcal{M}} = -\widetilde{\Lambda}\mathcal{N}_2\mathcal{N}_0\mathcal{N}_2^\top\widetilde{\Lambda} + \mathcal{N}_1\widetilde{\Lambda} + \widetilde{\Lambda}\mathcal{N}_1^\top + \mathcal{M} \geq 0.$$

# Main results

**Theorem (Fu & Horst & X. '22)**

*If the standing assumption is satisfied, and if either $\lambda = 0$ or $\lambda > 0$ and $\alpha$ is small enough, then the following holds.*
*i) In terms of the processes $A, B, D, F$ introduced before, the value function is given by*

$$V(t, \mathcal{X}) = V(t, \mu) = Var(\mu)(A_t) + \bar{\mu}^\top B_t \bar{\mu} + D_t^\top \bar{\mu} + \mathbb{E}[F_t].$$

*In particular, the value function of the original control problem follows*

$$V(0, \mathcal{X}) = \mathrm{Var}(\mu)(A_0) + \bar{\mu}^\top B_0 \bar{\mu} + D_0^\top \bar{\mu} + \mathbb{E}[F_0].$$

**Theorem (Fu & Horst & X. '22)**

*ii) The optimal strategy jumps only at the beginning and the end of the trading period where the initial and terminal jump is given by*

$$\Delta Z_t = -\frac{I_t^A}{\tilde{a}}(\mathcal{X}_{t-} - \overline{\mu}) - \frac{I_t^B}{a}\overline{\mu} - \frac{I_t^D}{a} \quad \text{and} \quad \Delta Z_T = X_{T-}$$

*respectively. On the time interval $(t, T)$ the optimal strategy satisfies the dynamics*

$$\begin{aligned} dZ_s = &\Big( -\frac{\dot{I}_s^A}{\tilde{a}}(\mathcal{X}_s - \mathbb{E}[\mathcal{X}_s]) - \frac{\dot{I}_s^B}{a}\mathbb{E}[\mathcal{X}_s] - \frac{\dot{I}_s^D}{a} - \frac{I_s^A}{\tilde{a}}\mathcal{H}(\mathcal{X}_s - \mathbb{E}[\mathcal{X}_s]) \\ &- \frac{I_s^B}{a}((\mathcal{H} + \overline{\mathcal{H}})\mathbb{E}[\mathcal{X}_s] + \mathcal{G})\Big)\, ds - \frac{I_s^A}{\tilde{a}}\mathcal{D}_s\, dW_s, \qquad s \in (t, T) \end{aligned}$$

**Theorem (Fu & Horst & X. '22)**

*iii) The optimal state dynamics reads*

$$d\mathcal{X}_s = \left(\mathcal{H}\mathcal{X}_s + \overline{\mathcal{H}}\mathbb{E}[\mathcal{X}_s] + \mathcal{G}\right) ds + \mathcal{D}_s \, dW_s + \mathcal{K} \, dZ_s, \qquad s \in [t, T),$$

*where the initial value is given by*

$$\mathcal{X}_t = \begin{pmatrix} X_{t-} - \Delta Z_t & Y_{t-} + \gamma_2 \Delta Z_t & C_{t-} \end{pmatrix}^\top.$$

# Verification

**Proposition (Fu & Horst & X. '22)**

*Let $\gamma_2\rho - \gamma_1\alpha + \lambda > 0$. Then the cost functional can be rewritten as*

$$J(t,Z) := \mathbb{E}\Big[\int_t^T \frac{1}{\tilde{a}}\Big(I_s^A(\mathcal{X}_s - \bar{\mu}_s)\Big)^2 ds + \int_t^T \frac{1}{a}\Big(I_s^B\bar{\mu}_s + I_s^D\Big)^2 ds\Big]$$
$$+ \mathrm{Var}(\mu_{t-})(A_t) + \bar{\mu}_{t-}^\top B_t\bar{\mu}_{t-} + D_t^\top\bar{\mu}_{t-} + \mathbb{E}[F_t].$$

*In particular, the cost functional reaches its global minimum if*

$$\int_t^T \Big(I_s^A(\mathcal{X}_s - \bar{\mu}_s)\Big)^2 ds = \int_t^T \Big(I_s^B\bar{\mu}_s + I_s^D\Big)^2 ds = 0, \quad a.s.$$

# The optimal strategy

**Theorem**

*The candidate state process given above satisfies*

$$I^A(\widetilde{\mathcal{X}}_s - \mathbb{E}[\widetilde{\mathcal{X}}_s]) = 0, \quad I^B\mathbb{E}[\widetilde{\mathcal{X}}_s] + I^D = 0, \qquad s \in [t, T).$$

# Overview

# Risk parameter dependence

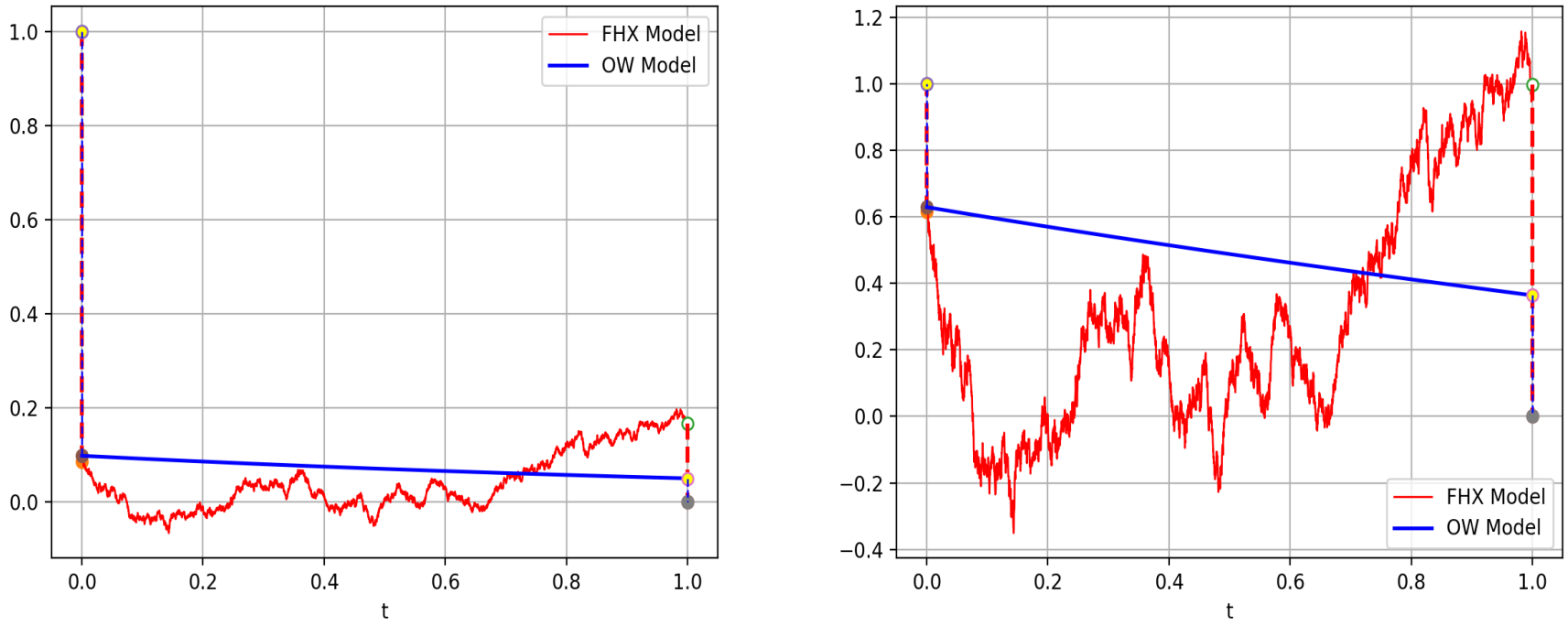


Figure: Dependence of the optimal strategy on the risk parameter $\lambda$ for $\lambda = 1.5$ (left) and $\lambda = 0$(right). Other parameters are chosen as $\rho = 0.7$, $\gamma_1 = 0.1$, $\gamma_2 = 0.5$, $\alpha = 0.5$, $\beta = 1.1$.

# Child order flow parameter dependence

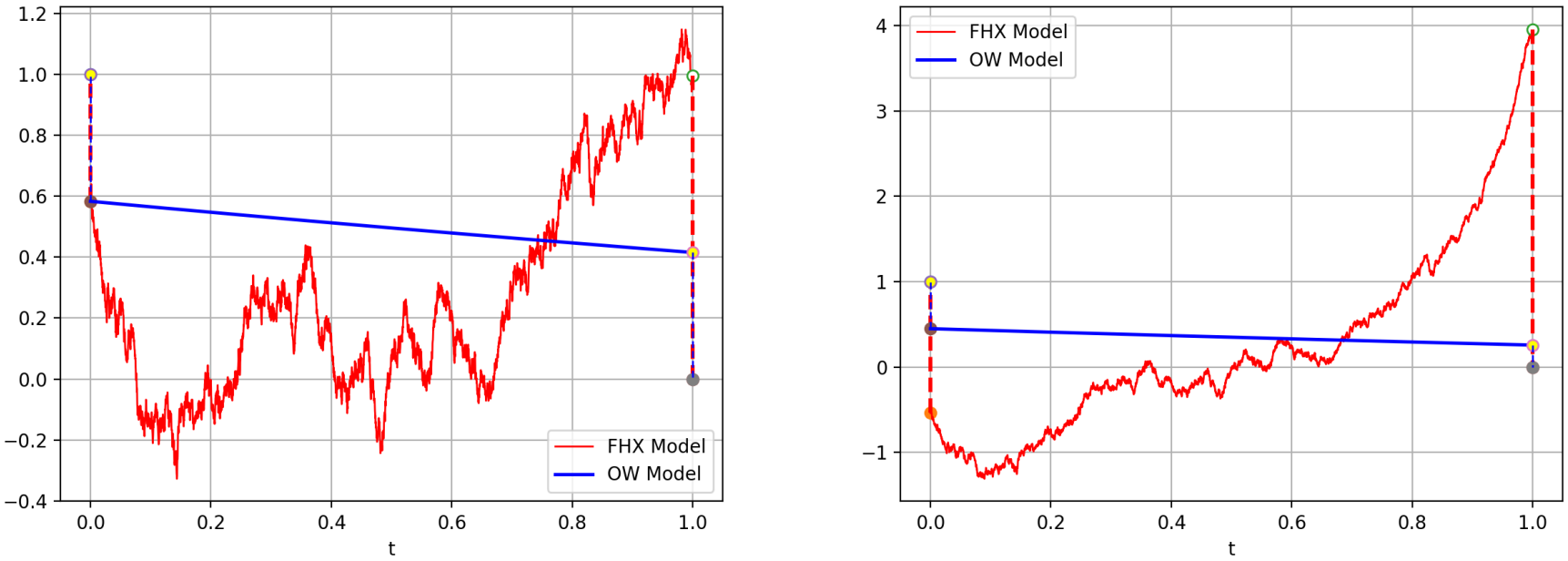


Figure: Dependence of the optimal strategy on the impact parameter $\alpha$ for $\alpha = 0$ (left) and $\alpha = 1.8$ (right). Other parameters are chosen as $\rho = 0.4$, $\lambda = 0$, $\gamma_1 = 0.1$, $\gamma_2 = 0.5$, $\beta = 3$.

# Transient market impact parameter dependence

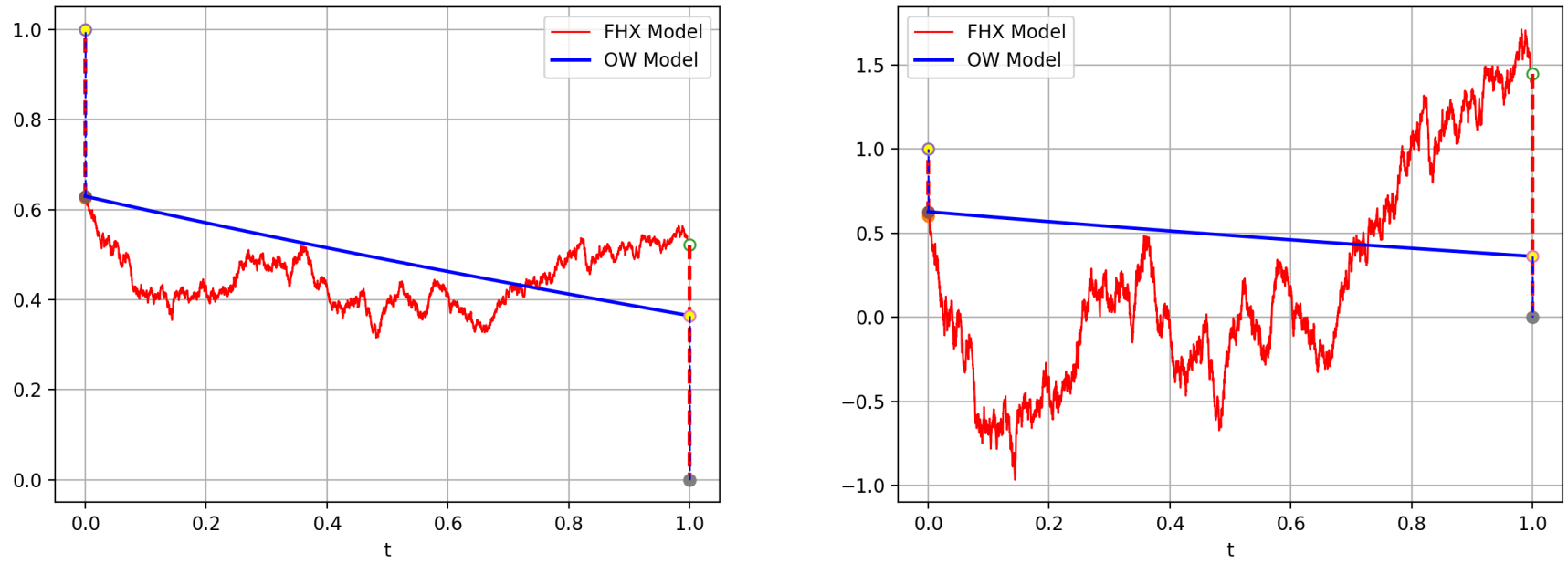


Figure: Dependence of the optimal strategy on the impact parameter $\gamma_2$ for $\gamma_2 = 2$ (left) and $\gamma_2 = 0.3$(right). Other parameters are chosen as $\rho = 0.7$, $\lambda = 0$, $\gamma_1 = 0.1$, $\alpha = 0.5$, $\beta = 1.1$.

# Overview

# Conclusion

- We considered a mean-field control problem with càdlàg semimartingale strategies arising in portfolio liquidation models with transient market impact and self-exciting order flow.
- We showed that the value function can be described in terms of the solution to a fully coupled system of Riccati equations.
- We obtained that the optimal strategy jumps only at the beginning and the end of the trading period.

## The end.

**Thank you!**